Hindawi
Advances in Astronomy
Volume 2020, Article ID 6629183, 12 pages
https://doi.org/10.1155/2020/6629183

Research Article

New Treatment of the Rotary Motion of a Rigid Body with Estimated Natural Frequency

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Received 6 October 2020; Revised 3 November 2020; Accepted 6 November 2020; Published 7 December 2020

Academic Editor: Juan L. G. Guirao

In this paper, the problem of the motion of a rigid body about a fixed point under the action of a Newtonian force field is studied when the natural frequency $\omega = 0.5$. This case of singularity appears in the previous works and deals with different bodies which are classified according to the moments of inertia. Using the large parameter method, the periodic solutions for the equations of motion of this problem are obtained in terms of a large parameter, which will be defined later. The geometric interpretation of the considered motion will be given in terms of Euler's angles. The numerical solutions for the system of equations of motion are obtained by one of the well-known numerical methods. The comparison between the obtained numerical solutions and analytical ones is carried out to show the errors between them and to prove the accuracy of both used techniques. In the end, we obtain the case of the regular precession type as a special case. The stability of the motion is considered by the phase diagram procedures.

1. Introduction

Consider a rigid body of mass M moves in an asymmetric field around a fixed point O [1]. Let us assume that the surface of its ellipsoid of inertia is optional, as well as the mass center. Let the frame OX, OY, and OZ be a fixed system in space, and the frame Ox, Oy, and Oz is the main axes frame for the surface of the ellipsoid of inertia of the body which moves with the it. Initially, we consider the main axis z for the surface of the ellipsoid of inertia that makes an angle $\xi_0 \neq (k\pi/2)$; $k = 0, 1,$ and 2 with the fixed axis Z in space. Let the body spins with small speed angular velocity r_0 about the axis z. Suppose that $p, q,$ and r represent the components of the angular velocity vector of the body about the main axes of the ellipsoid of the inertia surface; γ, γ', and γ'' are the directional cosines vector of the axis Z; g is the acceleration of gravity; $A, B,$ and C are the principal moments of inertia. The point (x_0, y_0, z_0) is the center of mass in the moving coordinate system; $\underline{R}$ is the position vector of the center of attraction 0_1 on the fixed downward coordinate Z axis, and $\underline{\rho}$ is the position vector of the element dm. Let $\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}$, and $\underline{\hat{Z}}$ be the unit vectors in the shown directions (Figure 1). Consider d$\underline{F}$ is the attraction force element due to the attracting center and acted on the element dm at the point $p(x, y, z)$.

2. Formulation of the Problem

Without a loss of generality, we choose the positive direction of both the axis z and the axis x that do not make an obtuse angle ξ_0 with the direction of axis Z. Under the restriction on ξ_0 and the choice of the coordinate system, we get [2]

$$\gamma_0 \geq 0, \quad 0 < \gamma_0'' < 1. \tag{1}$$

The differential equations of motion can be reduced to an autonomous system of two degrees of freedom and one first integral as follows [3]:

$$\begin{aligned} 4\ddot{p}_2 + p_2 &= 4\varepsilon^{-2} F\left(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \varepsilon\right), \\ \ddot{\gamma}_2 + \gamma_2 &= \varepsilon^{-2} \Phi\left(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \varepsilon\right), \end{aligned} \tag{2}$$

$$\gamma_2^2 + \dot{\gamma}_2^2 + 2\varepsilon^{-1}\left(\nu p_2 \gamma_2 + \nu_2 \dot{p}_2 \dot{\gamma}_2 + s_{21}\right) + \varepsilon^{-2}(\ldots) = \gamma_0''^{-2} - 1, \tag{3}$$

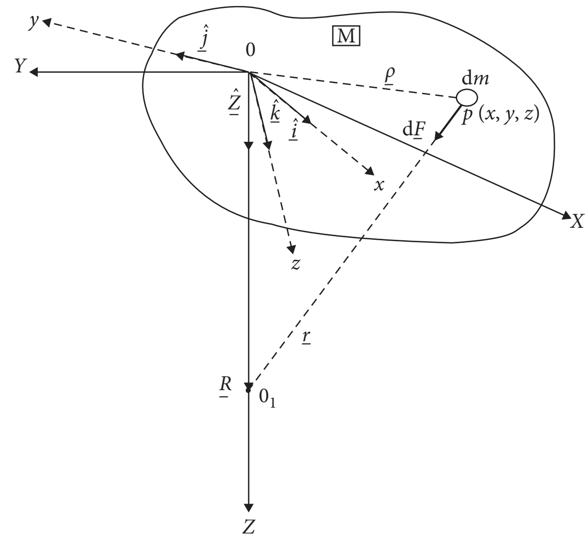

FIGURE 1: Description of motion in terms of moving and fixed frames.

where

$$\begin{aligned}
F = {} & C_1 A_1^{-1} p_2 \dot{p}_2^2 + x_0' \dot{p}_2 \dot{\gamma}_2 - y_0' a^{-1} p_2 \dot{\gamma}_2 \\
& - y_0' A_1^{-1}\left(A_1 + a^{-1}\right)\gamma_2 \dot{p}_2 - z_0' a^{-1} p_2 \\
& - 0.75 \nu e_1 p_2 - 0.25 p_2 s_{11} + A_1 b^{-1} x_0' s_{21} + O\left(\varepsilon^{-1}\right) + \cdots, \\
\Phi = {} & -(1 - C_1) A_1^{-1} p_2 \dot{p}_2 \dot{\gamma}_2 + x_0' \dot{\gamma}_2^2 - y_0' \gamma_2 \dot{\gamma}_2 - z_0' b^{-1} \gamma_2 \\
& + x_0' b^{-1} - A_1^{-2} \gamma_2 \dot{p}_2^2 \\
& + 0.75 \nu \left(e + e_1 \gamma_2\right) - \gamma_2 s_{11} \\
& + (1 + B_1) p_2 s_{21} + O\left(\varepsilon^{-1}\right) + \cdots,
\end{aligned} \tag{4}$$

$$\begin{aligned}
p_2 &= p_1 - \varepsilon^{-1}\left(e + e_1 \gamma_2\right), \\
\gamma_2 &= \gamma_1 - \varepsilon^{-1} \nu p_2, \\
q_1 &= -A_1^{-1} \dot{p}_2 + \varepsilon^{-1} A_1^{-1}\left(y_0' a^{-1} - e_2 \dot{\gamma}_2\right) + \cdots, \\
r_1 &= 1 + 0.5 \varepsilon^{-2} s_{11} + \cdots, \\
\gamma_1' &= \dot{\gamma}_2 + \varepsilon^{-1} \nu_2 \dot{p}_2 + \cdots, \\
\gamma_1'' &= 1 + \varepsilon^{-1} s_{21} + \varepsilon^{-2}\left(s_{22} - 0.5 s_{11}\right) + \cdots,
\end{aligned} \tag{5}$$

$$\begin{aligned}
p_1 &= \frac{p}{c}\sqrt{\gamma_0''}, \\
&\cdot (pq), \\
r_1 &= \frac{r}{r_0}, \\
\gamma_1 &= \frac{\gamma}{\gamma_0''}, \\
&\cdot (\gamma \gamma' \gamma''), \\
\tau &= r_0^{-1} t, \\
&\cdot \left(. \equiv \frac{\mathrm{d}}{\mathrm{d}\tau} \right);
\end{aligned} \tag{6}$$

$$\begin{aligned}
s_{11} &= \frac{a\left(p_{20}^2 - p_2^2\right) + b\left(\dot{p}_{20}^2 - \dot{p}_2^2\right)}{A_1^2 - 2\left[x_0'\left(\gamma_{20} - \gamma_2\right) + y_0'\left(\dot{\gamma}_{20} - \dot{\gamma}_2\right)\right]}, \\
s_{21} &= a\left(p_{20}\gamma_{20} - p_2\gamma_2\right) - b A_1^{-1}\left(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2\right), \\
s_{22} &= a\left[\nu\left(p_{20}^2 - p_2^2\right) + e\left(\gamma_{20} - \gamma_2\right) + e_1\left(\gamma_{20}^2 - \gamma_2^2\right)\right] \\
&\quad + b A_1^{-1}\left[-\nu_2\left(\dot{p}_{20}^2 - \dot{p}_2^2\right) + a^{-1} y_0'\left(\dot{\gamma}_{20} - \dot{\gamma}_2\right) - e_2\left(\dot{\gamma}_{20}^2 - \dot{\gamma}_2^2\right)\right],
\end{aligned} \tag{7}$$

$$\begin{aligned}
A_1 &= \frac{C - B}{A}, \\
&(ABC), \\
a &= \frac{A}{C}, \\
&(ab), \\
c^2 &= \frac{Mgl}{C}, \\
\varepsilon &= \frac{c\sqrt{\gamma_0''}}{r_0}, \\
x_0 &= l x_0', \\
&(xyz), \\
l^2 &= x_0^2 + y_0^2 + z_0^2, \\
4 A_1 B_1 &= -1, \\
eb &= 4 x_0' A_1, \\
3\nu &= 4\left(1 + B_1\right), \\
3 e_1 &= 4 z_0'\left(A_1 b^{-1} - a^{-1}\right), \\
e_2 &= e_1 + a^{-1} z_0', \\
\nu_2 &= \nu - A_1^{-1}.
\end{aligned} \tag{8}$$

The symbols like ABC are abbreviated equations.

3. Construction of Periodic Solutions with Zeros Basic Amplitudes

In this section, we use the suggested method for constructing the aimed solutions for the autonomous system (2). Consider the condition [4]

$$p_2(0,0) = \dot{p}_2(0,0) = \dot{\gamma}_2(0,\varepsilon) = 0. \tag{9}$$

The generating system for (2) is obtained when $\varepsilon \longrightarrow \infty$ as follows:

$$\begin{aligned}
4\ddot{p}_2^{(0)} + p_2^{(0)} &= 0, \\
\ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} &= 0.
\end{aligned} \tag{10}$$

The solutions for system (10) with a period $T_0 = 4\pi$ are

$$\begin{aligned}
p_2^{(0)} &= a_0^* \cos(0.5\tau), \\
\gamma_2^{(0)} &= b_0^* \cos\tau,
\end{aligned} \tag{11}$$

where a_0^* and b_0^* are constants.

Let system (2) has periodic solutions with a period $T_0 + \alpha$ in the form [5]

$$p_2 = a^* \cos\psi + \sum_{n=1}^{N} \varepsilon^{-n} p_n^*(a^*, \psi) + O(\varepsilon^{-N-1}),$$
$$\gamma_2 = b^* \cos\phi + \sum_{n=1}^{N} \varepsilon^{-n} \gamma_n^*(a^*, \phi) + O(\varepsilon^{-N-1}). \quad (12)$$

For these solutions, we let the initial conditions

$$p_2(0,\varepsilon) = a^* = a_0^* + a^*(\varepsilon),$$
$$\gamma_2(0,\varepsilon) = b^* = b_0^* + b^*(\varepsilon), \quad (13)$$
$$\dot{\gamma}_2(0,\varepsilon) = 0.$$

Here, $a^*(\varepsilon), b^*(\varepsilon) \longrightarrow 0$ at $\varepsilon \longrightarrow \infty$. Considering first integral (3) with conditions (13), we get

$$0 < b_0^* = \left(1 - \gamma_0''^2\right)^{1/2} (\gamma_0'')^{-1} < \infty,$$
$$b^*(\varepsilon) = -\varepsilon^{-1}\nu[a_0^* + a^*(\varepsilon)] + \cdots. \quad (14)$$

Let a^*, ψ, and ϕ are changed with time according to

$$\dot{a}^* = \sum_{n=1}^{N} \varepsilon^{-n} A_n^*(a^*) + O(\varepsilon^{-N-1}), \quad (15)$$

$$\dot{\psi} = 0.5 + \sum_{n=1}^{N} \varepsilon^{-n} \psi_n(a^*) + O(\varepsilon^{-N-1}), \quad (16)$$

$$\dot{\phi} = 1 + \sum_{n=1}^{N} \varepsilon^{-n} \phi_n(a^*) + O(\varepsilon^{-N-1}). \quad (17)$$

The following derivatives are obtained:

$$\dot{p}_2 = -0.5a^* \sin\psi + O(\varepsilon^{-1}),$$
$$\dot{\gamma}_2 = -b^* \sin\phi + O(\varepsilon^{-1}),$$
$$\ddot{p}_2 = -0.25a^* \cos\psi + \varepsilon^{-1}\left[0.25\frac{\partial^2 p_1^*}{\partial\psi^2} - a^*\psi_1\cos\psi - A_1^*\sin\psi\right]$$
$$+ \varepsilon^{-2}\left[A_1^*\frac{\partial^2 p_1^*}{\partial a^*\partial\psi} - (A_2^* + 2A_1^*\psi_1)\sin\psi + A_1^*\frac{dA_1^*}{da^*}\cos\psi + 0.25\frac{\partial^2 p_2^*}{\partial\psi^2} + \psi_1\frac{\partial^2 p_1^*}{\partial\psi^2} - a^*\left(\psi_1^2 + 2\psi_2\right)\cos\psi - a^*A_1^*\sin\psi\frac{d\psi_1}{da^*}\right] + O(\varepsilon^{-3}),$$
$$\ddot{\gamma}_2 = -b^*\cos\phi + \varepsilon^{-1}\left[\frac{\partial^2\gamma_1^*}{\partial\phi^2} - 2b^*\phi_1\cos\phi\right]$$
$$+ \varepsilon^{-2}\left[\frac{\partial^2\gamma_2^*}{\partial\phi^2} + 2\phi_1\frac{\partial^2\gamma_1^*}{\partial\phi^2} - b^*\left(\phi_1^2 + 2\phi_2\right)\cos\phi + 2A_1^*\frac{\partial^2\gamma_1^*}{\partial a^*\partial\phi} - b^*A_1^*\frac{d\phi_1}{da^*}\sin\phi\right] + O(\varepsilon^{-3}). \quad (18)$$

Using equations (7), (12), and (18), we get

$$s_{11}^{(0)} = aa_0^{*2}\left(\cos^2\psi_0 - \cos^2\psi\right) - 0.25bA_1^{-2}a_0^{*2}\sin^2\psi$$
$$- 2b_0^*[x_0'(\cos\phi_0 - \cos\phi) + y_0'\sin\phi],$$
$$s_{21}^{(0)} = a_0^* b_0^*\left[a(\cos\psi_0\cos\phi_0 - \cos\psi\cos\phi) + 0.5bA_1^{-1}\sin\psi\sin\phi\right], \quad (19)$$
$$s_{22}^{(0)} = a\left[\nu a_0^{*2}\left(\cos^2\psi_0 - \cos^2\psi\right) + eb_0^*(\cos\phi_0 - \cos\phi) + e_1 b_0^{*2}\left(\cos^2\phi_0 - \cos^2\phi\right)\right]$$
$$+ bA_1^{-1}\left[0.25\nu_2 a_0^{*2}\sin^2\psi + a^{-1}y_0' b_0^*\sin\phi + e_2 b_0^{*2}\sin^2\phi\right],$$

where ψ_0 and ϕ_0 are the initial values of the corresponding functions.

Using (4), (12), (18), and (19), we obtain

$$
\begin{aligned}
F^{(0)} &= 0.25 C_1 A_1^{-1} a_0^{*3} \cos\psi \sin^2\psi + 0.5 a_0^* b_0^* x_0' \sin\psi \sin\phi \\
&\quad + a^{-1} a_0^* b_0^* y_0' \cos\psi \sin\phi + 0.5 A_1^{-1}\left(A_1 + a^{-1}\right) a_0^* b_0^* y_0' \sin\psi \cos\phi \\
&\quad - z_0' a^{-1} a_0^* \cos\psi - 0.75 \nu e_1 a_0^* \cos\psi \\
&\quad - 0.25 a_0^* \cos\psi \left\{ a a_0^{*2}\left(\cos^2\psi_0 - \cos^2\psi\right) - 0.25 b A_1^{-2} a_0^{*2} \sin^2\psi - 2 b_0^* \left[x_0'\left(\cos\phi_0 - \cos\phi\right) + y_0' \sin\phi\right]\right\} \\
&\quad + A_1 b^{-1} x_0' a_0^* b_0^* \left[a\left(\cos\psi_0 \cos\phi_0 - \cos\psi\cos\phi\right) + 0.5 b A_1^{-1} \sin\psi \sin\phi\right], \\
\Phi^{(0)} &= 0.25\left(C_1 - 1\right) A_1^{-1} a_0^{*2} b_0^* \sin 2\psi \sin\phi + 0.5 x_0' b_0^{*2}\left(1 - \cos 2\phi\right) \\
&\quad + 0.5 y_0' b_0^{*2} \sin 2\phi - z_0' b^{-1} b_0^* \cos\phi + x_0' b^{-1} \\
&\quad - 0.125 A_1^{-2} a_0^{*2} b_0^* \left(1 - \cos 2\psi\right)\cos\phi + 0.75 \nu e + 0.75 \nu e_1 b_0^* \cos\phi \\
&\quad - a a_0^{*2} b_0^* \cos^2\psi_0 \cos\phi + 0.5 a a_0^{*2} b_0^* \left(1 + \cos 2\psi\right)\cos\phi \\
&\quad + 0.125 b A_1^{-2} a_0^{*2} b_0^* \left(1 - \cos 2\psi\right)\cos\phi + 2 x_0' b_0^{*2} \cos\phi_0 \cos\phi \\
&\quad - x_0' b_0^{*2}\left(1 + \cos 2\phi\right) + y_0' b_0^{*2} \sin 2\phi \\
&\quad + a_0^{*2} b_0^* \left(1 + B_1\right)\left[0.5 b A_1^{-1} \sin\psi \sin\phi + a\left(\cos\psi_0 \cos\phi_0 - \cos\psi\cos\phi\right)\right]\cos\psi.
\end{aligned} \tag{20}
$$

Substituting from (12), (18), and (20) into (2) and equating coefficients of ε^{-1} in both sides, we get

$$
\begin{aligned}
\frac{\partial^2 p_1^*}{\partial\psi^2} + p_1^* &= 4 a_0^* \psi_1 \cos\psi + 4 A_1^* \sin\psi, \\
\frac{\partial^2 \gamma_1^*}{\partial\phi^2} + \gamma_1^* &= 2 b_0^* \phi_1 \cos\phi, \\
\frac{\partial^2 p_2^*}{\partial\psi^2} + p_2^* &= 4 A_2^* \sin\psi + a_0^* \left[4\psi_2 + 0.25 C_1 A_1^{-1} a_0^{*2} - 3.25 a a_0^{*2} - 4 z_0' a^{-1} - 3 \nu e_1 + 0.125 b A_1^{-2} a_0^{*2} + 2 x_0' b_0^* \cos\phi_0\right]\cos\psi \\
&\quad + 0.25 a_0^{*3}\left(a - C_1 A_1^{-1} - 0.25 b A_1^{-2}\right)\cos 3\psi + 4 a a_0^* x_0' A_1 b^{-1} b_0^* \cos\psi_0 \cos\phi_0 \\
&\quad + x_0' a_0^* b_0^* \left(1 - 2 a A_1 b^{-1}\right)\cos\left(\phi - \psi\right) - x_0' a_0^* b_0^* \left(3 + 2 a A_1 b^{-1}\right)\cos\left(\phi + \psi\right) \\
&\quad + y_0' a_0^* b_0^* \left(2 a^{-1} - A_1^{-1} a^{-1}\right)\sin\left(\phi - \psi\right) + y_0' a_0^* b_0^* \left(2 + 2 a^{-1} + A_1^{-1} a^{-1}\right)\sin\left(\phi + \psi\right), \\
\frac{\partial^2 \gamma_2^*}{\partial\phi^2} + \gamma_2^* &= \left[2\phi_2 - z_0' b^{-1} + 0.125 A_1^{-2} a_0^{*2}\left(b - 1\right) + 0.75 \nu e_1 - a a_0^{*2} \cos^2\psi_0 - 0.5 a B_1 a_0^{*2} + 2 x_0' b_0^* \cos\phi_0\right] b_0^* \cos\phi \\
&\quad - 0.5 x_0' b_0^{*2} + x_0' b^{-1} + 0.75 \nu e + \left(1 + B_1\right) a a_0^{*2} b_0^* \cos\psi_0 \cos\phi_0 \cos\psi - 0.67 x_0' b_0^{*2} \cos 2\phi + 1.5 y_0' b_0^{*2} \sin 2\phi \\
&\quad + 0.5 a_0^{*2} \left\{\left[0.25 A_1^{-2}\left(1 - b\right) - a B_1 + A_1^{-1} b_0^* \left(b - 1\right)\right]\cos\left(2\psi - \phi\right)\right. \\
&\quad \left. + \left[0.25 A_1^{-2}\left(1 - b\right) - a B_1 - A_1^{-1} b_0^* \left(b - 1\right)\right]\cos\left(2\psi + \phi\right)\right\}.
\end{aligned} \tag{21}
$$

Canceling singular terms from (21) as in [6], we get

$$\psi_1 = A_1^* = \phi_1 = A_2^* = 0,$$
$$\psi_2 = \left[-0.06C_1A_1^{-1}a_0^{*2} + 0.81aa_0^{*2} + z_0'a^{-1} + 0.75\nu e_1 - 0.02bA_1^{-2}a_0^{*2} - 0.5x_0'b_0^*\cos\phi_0\right],$$
$$\phi_2 = 0.5\left[z_0'b^{-1} - 0.125A_1^{-2}a_0^{*2}(b-1) - 0.75\nu e_1 + aa_0^{*2}\left(0.5B_1 + \cos^2\psi_0\right) - 2x_0'b_0^*\cos\phi_0\right]. \tag{22}$$

Substituting from (22) into (15)–(17) and integrating, we obtain

$$a^* = a_0^*\ \text{(arbitrary const.)},$$
$$\psi = 0.5\tau + 0.5\varepsilon^{-2}\left[-0.125C_1A_1^{-1}a_0^{*2} - 0.375aa_0^{*2} + 2aa_0^{*2} + 2z_0'a^{-1} + 1.5\nu e_1 - 0.31bA_1^{-2}a_0^{*2} - x_0'b_0^*\cos\phi_0\right]\tau,$$
$$\phi = \tau + 0.5\varepsilon^{-2}\left[z_0'b^{-1} - 0.125A_1^{-2}a_0^{*2}(b-1) - 0.75\nu e_1 + aa_0^{*2}(1+0.5B_1) - 2x_0'b_0^*\right]\tau. \tag{23}$$

From the previous results, we get

$$\psi(0) = \psi_0 = 0,$$
$$\phi(0) = \phi_0 = 0. \tag{24}$$

From (13) and (23), we obtain a^* from the order greater than $O(\varepsilon^{-2})$.

The periodic solutions p_2 and γ_2 are obtained by substituting (22) and (23) into (21) and using (12) and (14). Finally, the periodic solutions $p_1, q_1, r_1, \gamma_1, \gamma_1'$, and γ_1'' are obtained from (5), (19), (23), and (24).

4. Construction of Periodic Solutions with Nonzeros Basic Amplitudes

We use the large parameter method [7] for constructing the periodic solutions with nonzeros basic amplitudes for system (2) when A<B<C or A>B>C. Consider generating system (10) has periodic solutions with a period $T_0 = 2\pi n$ as follows:

$$p_2^{(0)}(\tau) = E\cos(0.5\tau - \mu),$$
$$\gamma_2^{(0)}(\tau) = M_3\cos\tau, \tag{25}$$

where $E = \sqrt{M_1^2 + M_2^2}$, $\mu = \tan^{-1}(M_2/M_1)$, and M_1, M_2, and M_3 are constants.

Let system (2) has periodic solutions with a period $T_0 + \alpha$ that reduces to generating solutions (21) when $\varepsilon \longrightarrow \infty$, where α is a function of ε such that $\alpha(\infty) = 0$. Consider the following initial conditions:

$$p_2(0,\varepsilon) = \widetilde{M}_1,$$
$$\dot{p}_2(0,\varepsilon) = 00.5\widetilde{M}_2,$$
$$\gamma_2(0,\varepsilon) = \widetilde{M}_3,$$
$$\dot{\gamma}_2(0,\varepsilon) = 0. \tag{26}$$

The notation $\sim$ denotes the following substitution:

$$M_i \longrightarrow \widetilde{M}_i = M_i + \beta_i, \quad i = 1,2,3, \tag{27}$$

where $\beta_1, 0.5\beta_2$, and β_3 represent the deviations of the initial values of the required solutions from their values of the generating ones M_1, M_2, and M_3, respectively. These deviations are functions of ε and vanish when $\varepsilon \longrightarrow \infty$. Now, we construct the required solutions in the following forms [8]:

$$p_2 = \widetilde{E}\cos(\psi - \mu) + \sum_{n=1}^{N}\varepsilon^{-n}p_n^*(\widetilde{E},\psi) + O\left(\varepsilon^{-N-1}\right),$$
$$\gamma_2 = \widetilde{M}_3\cos\phi + \sum_{n=1}^{N}\varepsilon^{-n}\gamma_n^*(\widetilde{E},\phi) + O\left(\varepsilon^{-N-1}\right), \tag{28}$$

where p_n^* and γ_n^* are periodic functions in ψ and ϕ, respectively. The quantity $\widetilde{M}_3$ is determined from the first integral (3). Let $\widetilde{E}, \psi$, and ϕ are changed with time according to

$$\frac{d\widetilde{E}}{d\tau} = \sum_{n=1}^{N}\varepsilon^{-n}E_n(\widetilde{E}) + O\left(\varepsilon^{-N-1}\right), \tag{29}$$

$$\frac{d\psi}{d\tau} = 0.5 + \sum_{n=1}^{N}\varepsilon^{-n}\psi_n(\widetilde{E}) + O\left(\varepsilon^{-N-1}\right), \tag{30}$$

$$\frac{d\phi}{d\tau}=1+\sum_{n=1}^{N}\varepsilon^{-n}\phi_n(\tilde{E})+O\left(\varepsilon^{-N-1}\right). \tag{31}$$

Substituting initial conditions (26) into integral (3), when $\tau=0$, we deduce that

$$0<M_3=\frac{\sqrt{1-\gamma_0''^2}}{\gamma_0''}<\infty, \qquad \beta_3=-\varepsilon^{-1}\nu\tilde{M}_1+\cdots. \tag{32}$$

The derivatives become

$$\begin{aligned}
\dot{p}_2&=\frac{d\tilde{E}}{d\tau}\frac{\partial p_2}{\partial\tilde{E}}+\frac{d\psi}{d\tau}\frac{\partial p_2}{\partial\psi},\\
\dot{\gamma}_2&=\frac{d\tilde{E}}{d\tau}\frac{\partial \gamma_2}{\partial\tilde{E}}+\frac{d\phi}{d\tau}\frac{\partial \gamma_2}{\partial\phi},\\
\ddot{p}_2&=\left(\frac{d\tilde{E}}{d\tau}\right)^2\frac{\partial^2 p_2}{\partial\tilde{E}^2}+\frac{d^2\tilde{E}}{d\tau^2}\frac{\partial p_2}{\partial\tilde{E}}+2\frac{d\tilde{E}}{d\tau}\frac{d\psi}{d\tau}\frac{\partial^2 p_2}{\partial\tilde{E}\partial\psi}+\left(\frac{d\psi}{d\tau}\right)^2\frac{\partial^2 p_2}{\partial\psi^2}+\frac{d^2\psi}{d\tau^2}\frac{\partial p_2}{\partial\psi},\\
\ddot{\gamma}_2&=\left(\frac{d\tilde{E}}{d\tau}\right)^2\frac{\partial^2 \gamma_2}{\partial\tilde{E}^2}+\frac{d^2\tilde{E}}{d\tau^2}\frac{\partial \gamma_2}{\partial\tilde{E}}+2\frac{d\tilde{E}}{d\tau}\frac{d\phi}{d\tau}\frac{\partial^2 \gamma_2}{\partial\tilde{E}\partial\phi}+\left(\frac{d\phi}{d\tau}\right)^2\frac{\partial^2 \gamma_2}{\partial\phi^2}+\frac{d^2\phi}{d\tau^2}\frac{\partial \gamma_2}{\partial\phi}.
\end{aligned} \tag{33}$$

Using equations (7), (28), and (33), we get

$$\begin{aligned}
s_{11}^{(0)}&=E^2\left[\left(a\cos^2\mu-0.5\right)+0.25bA_1^{-2}\left(\sin^2\mu-0.5\right)+0.5\left(0.25bA_1^{-2}-a\right)\cos(\tau-2\mu)\right]-2M_3\left[x_0'(1-\cos\tau)+y_0'\sin\tau\right],\\
s_{21}^{(0)}&=M_3E\left[a\cos\mu+0.5\left(0.5bA_1^{-1}-a\right)\cos(0.5\tau+\mu)-0.5\left(0.5bA_1^{-1}+a\right)\cos(1.5\tau-\mu)\right],\\
s_{22}^{(0)}&=E^2\left[\nu a\left(\cos^2\mu-0.5\right)-0.25bA_1^{-1}\nu_2\left(\sin^2\mu-0.5\right)-0.5\left(\nu a+0.25bA_1^{-1}\nu_2\right)\cos(\tau-2\mu)\right]\\
&\quad+0.5M_3^2\left(e_1a+bA_1^{-1}e_2\right)(1-\cos2\tau)+M_3\left[ae(1-\cos\tau)+by_0'a^{-1}A_1^{-1}\sin\tau\right].
\end{aligned} \tag{34}$$

Using (4), (28), (33), and (34), we obtain

$$\begin{aligned}
F^{(0)}&=0.25C_1A_1^{-1}E^3\cos(0.5\tau-\mu)\sin^2(0.5\tau-\mu)+EM_3\sin\tau\left[0.5x_0'\sin(0.5\tau-\mu)+y_0'a^{-1}\cos(0.5\tau-\mu)\right]\\
&\quad+0.5y_0'A_1^{-1}\left(A_1+a^{-1}\right)M_3E\cos\tau\sin(0.5\tau-\mu)-E\left(z_0'a^{-1}+0.75\nu e_1\right)\cos(0.5\tau-\mu)\\
&\quad-0.25E\cos(0.5\tau-\mu)\left\{E^2\left[a\cos^2\mu-0.5+0.25bA_1^{-2}\left(\sin^2\mu-0.5\right)+0.5\left(0.25bA_1^{-2}-a\right)\cos(\tau-2\mu)\right]\right.\\
&\quad\left.-2M_3\left[x_0'(1-\cos\tau)+y_0'\sin\tau\right]\right\}\\
&\quad+A_1b^{-1}x_0'M_3E\left[a\cos\mu+0.5\left(0.5bA_1^{-1}-a\right)\cos(0.5\tau+\mu)-0.5\left(0.5bA_1^{-1}+a\right)\cos(1.5\tau-\mu)\right],\\
\Phi^{(0)}&=b^{-1}x_0'-0.5M_3^2x_0'+0.75\nu e-\left\{z_0'b^{-1}M_3+0.125A_1^{-2}M_3E^2-0.75\nu e_1M_3\right.\\
&\quad\left.+M_3E^2\left[a\left(\cos^2\mu-0.5\right)+0.25bA_1^{-2}\left(\sin^2\mu-0.5\right)\right]-2M_3^2x_0'+0.5a(1+B_1)M_3E^2\right\}\cos\tau\\
&\quad-1.5M_3^2\left(x_0'\cos2\tau-y_0'\sin2\tau\right)+(1+B_1)M_3E^2a\cos\mu(\cos0.5\tau\cos\mu+\sin0.5\tau\sin\mu)\\
&\quad+0.25E^2M_3\left[0.5(1-C_1)A_1^{-1}+0.25A_1^{-2}(1-b)+a-(1+B_1)\left(0.5A_1^{-1}+a\right)\right]cos2(\mu-\tau)\\
&\quad+0.25M_3E^2\times\left[0.5A_1^{-1}(C_1-1)+0.25A_1^{-2}(1-b)+a+(1+B_1)\left(0.5bA_1^{-1}-a\right)\right]\cos2\mu.
\end{aligned} \tag{35}$$

Substituting from (28), (33), and (35) into initial system (2) and equating coefficients of ε^{-1} and ε^{-2} in both sides, we obtain the following:

Coefficients of ε^{-1}:

$$\frac{\partial^2 p_1^*}{\partial\psi^2} + p_1^* = 4\left(E\psi_1\cos\mu - E_1\sin\mu\right)\cos\psi + 4\left(E\psi_1\sin\mu + E_1\cos\mu\right)\sin\psi, \tag{36}$$

$$\frac{\partial^2 \gamma_1^*}{\partial\phi^2} + \gamma_1^* = 2\phi_1 M_3\cos\phi.$$

We neglect the singular terms [4] to get

$$\begin{aligned} E\psi_1\cos\mu - E_1\sin\mu &= 0,\\ E\psi_1\sin\mu + E_1\cos\mu &= 0, \end{aligned} \tag{37}$$

$$\phi_1 = 0, \tag{38}$$

such that determinant (37) becomes

$$\Delta = E\begin{vmatrix}\cos\mu & -\sin\mu\\ \sin\mu & \cos\mu\end{vmatrix} = E\left(\cos^2\mu + \sin^2\mu\right) = E \neq 0. \tag{39}$$

For this case, the solution of (37) becomes

$$\psi_1 = E_1 = 0. \tag{40}$$

The particular solutions for (36) become

$$p_1^* = \gamma_1^* = 0. \tag{41}$$

Coefficients of ε^{-2}:

$$\begin{aligned}\frac{d^2 p_2^*}{d\tau^2} + 0.25p_2^* = {}& \left[E_2\cos\mu + E\psi_2\sin\mu\right]\sin 0.5\tau\\ & -\Big\{E_2\sin\mu - \Big[E\psi_2 + 0.06C_1A_1^{-1}E^3 - z_0'a^{-1}E - 0.75\nu e_1E - 0.25E^3a\left(\cos^2\mu - 0.5\right) - 0.06E^3bA_1^{-2}\left(\sin^2\mu - 0.5\right)\\ & - 0.06E^3\left(0.25bA_1^{-2} - a\right) + 0.5M_3x_0'E\Big]\cos\mu\Big\}\cos 0.5\tau\\ & + A_1b^{-1}x_0'M_3Ea\cos\mu - 0.06E^3\left(C_1A_1^{-1} + 0.25bA_1^{-2} - a\right)\left(\cos 3\mu\cos 1.5\tau + \sin 3\mu\sin 1.5\tau\right)\\ & + M_3E\Big\{\left[0.5A_1b^{-1}\left(0.5bA_1^{-1} - a\right)\right]x_0'\cos\mu + 0.5\left[a^{-1} - 0.5A_1^{-1}\left(A_1 + a^{-1}\right) + 0.5\right]y_0'\sin\mu\Big\}\cos 0.5\tau\\ & - 0.5M_3E\Big\{\left[A_1b^{-1}\left(0.5bA_1^{-1} - a\right)\right]x_0'\sin\mu - \left[a^{-1} - \left(1 + A_1^{-1}a^{-1}\right) + 0.5\right]y_0'\cos\mu\Big\}\sin 0.5\tau\\ & - 0.5M_3E\Big\{\left[0.25 + \left(0.5 + A_1b^{-1}a\right)\right]x_0'\cos\mu + \left[a^{-1} + \left(1 + A_1^{-1}a^{-1}\right) + 0.5\right]y_0'\sin\mu\Big\}\cos 1.5\tau\\ & - 0.5M_3E\Big\{1.5A_1b^{-1}ax_0'\sin\mu - \left[a^{-1} + \left(1 + A_1^{-1}a^{-1}\right) + 0.5\right]y_0'\cos\mu\Big\}\sin 1.5\tau,\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{d^2\gamma_2^*}{d\tau^2} + \gamma_2^* = {}& x_0'\left(b^{-1} - 0.5M_3^2\right) + 0.75\nu e\\ & + M_3\Big\{2\phi_2 - z_0'b^{-1} - 0.125A_1^{-2}E^2 + 0.75\nu e_1 - E^2\left[a\left(\cos^2\mu - 0.5\right) + 0.25bA_1^{-2}\left(\sin^2\mu - 0.5\right)\right]\\ & + 2M_3x_0' - 0.5aE^2\left(1 + B_1\right)\Big\}\cos\tau\\ & + 1.5M_3^2\left(y_0'\sin 2\tau - x_0'\cos 2\tau\right) + \left(1 + B_1\right)M_3E^2a\left(\cos\mu\cos 0.5\tau + \sin\mu\sin 0.5\tau\right)\cos\mu\\ & + 0.125M_3E^2\left[\left(1 - C_1\right)A_1^{-1} + 0.5A_1^{-2} - \left(0.5bA_1^{-2} - 2a\right) - \left(1 + B_1\right)\left(bA_1^{-1} + 2a\right)\right]\left(\cos 2\mu\cos 2\tau + \sin 2\mu\sin 2\tau\right)\\ & + 0.125M_3E^2\left[\left(C_1 - 1\right)A_1^{-1} + 0.5A_1^{-2} - \left(0.5bA_1^{-2} - 2a\right) + \left(1 + B_1\right)\left(bA_1^{-1} - 2a\right)\right]\cos 2\mu.\end{aligned} \tag{43}$$

Neglecting singular terms from (42) and (43) yields [4]

$$\begin{aligned} E_2 &= 0.125E\sin 2\mu\left[0.25C_1A_1^{-1}E^2 - 4z_0'a^{-1} - 3\nu e_1 - E^2a\left(\cos^2\mu - 0.5\right) - 0.25E^2bA_1^{-2}\left(\sin^2\mu - 0.5\right) - 0.25E^2\left(0.25bA_1^{-2} - a\right) + 2M_3x_0'\right],\\ \psi_2 &= 0.25\cos^2\mu\left[-0.25C_1A_1^{-1}E^2 + 4z_0'a^{-1} + 3\nu e_1 + E^2a\left(\cos^2\mu - 0.5\right) + 0.25bE^2A_1^{-2}\left(\sin^2\mu - 0.5\right) + 0.25E^2\left(0.25bA_1^{-2} - a\right) - 2M_3x_0'\right],\\ \phi_2 &= 0.5\left\{z_0'b^{-1} + 0.125A_1^{-2}E^2 - 0.75\nu e_1 + E^2\left[a\left(\cos^2\mu - 0.5\right) + 0.25bA_1^{-2}\left(\sin^2\mu - 0.5\right)\right] - 2M_3x_0' + 0.5a\left(1 + B_1\right)E^2\right\}.\end{aligned} \tag{44}$$

Substituting from (38), (40), and (44) into (29) and (30) and integrating, we get

$$\begin{aligned}2\widetilde{E} &= 2E - \varepsilon^{-2}E\sin 2\mu\big[-0.25C_1A_1^{-1}E^2 + 4z_0'a^{-1} + 3\nu e_1 + E^2a\big(\cos^2\mu - 0.5\big)\\&\quad + 0.25bE^2A_1^{-2}\big(\sin^2\mu - 0.5\big) + 0.25E^2\big(0.25bA_1^{-2} - a\big) - 2M_3x_0'\big]\tau + \cdots,\\2\psi &= \tau + 0.5\varepsilon^{-2}\big[-0.25C_1A_1^{-1}E^2 + 4z_0'a^{-1} + 3\nu e_1 + E^2a\big(\cos^2\mu - 0.5\big)\\&\quad + 0.25b\cdot E^2A_1^{-2}\big(\sin^2\mu - 0.5\big) + 0.25E^2\big(0.25bA_1^{-2} - a\big) - 2M_3x_0'\big]\cos^2\mu\tau + \cdots,\\\phi &= \tau + 0.25\varepsilon^{-2}\big\{2z_0'b^{-1} + 0.25A_1^{-2}E^2 - 1.5\nu e_1 + E^2\big[2a\big(\cos^2\mu - 0.5\big) + 0.5bA_1^{-2}\big(\sin^2\mu - 0.5\big)\big]\\&\quad - 4M_3x_0' + a(1 + B_1)E^2\big\}\tau + \cdots.\end{aligned} \tag{45}$$

Substituting (44) into (42) and (43) and solving the resulted equations, we get p_2^* and γ_2^*. The periodic solutions p_2 and γ_2 are constructed using (28), (32), (41), and (45). Using (5) and (34), we get the first terms of the required solutions as follows:

$$\begin{aligned}p_1 &= M_1\cos 0.5\tau + M_2\sin 0.5\tau - \varepsilon^{-1}\left(\frac{x_0'}{bB_1} - e_1M_3\cos\tau\right) + \cdots,\\q_1 &= 0.5A_1^{-1}\left(M_1\sin 0.5\tau - M_2\cos 0.5\tau\right) + \varepsilon^{-1}\left(\frac{y_0'}{aA_1} + e_2A_1^{-1}M_3\sin\tau\right) + \cdots,\\r_1 &= 1 + 0.25\varepsilon^{-2}\big\{2aM_1^2 - E^2 + 0.5bA_1^{-2}\big(M_2^2 - 0.5E^2\big) + \big(0.25bA_1^{-2} - a\big)\big[\big(M_1^2 - M_2^2\big)\cos\tau + 2M_1M_2\sin\tau\big]\\&\quad - 4M_3\left[x_0'(1 - \cos\tau) + y_0'\sin\tau\right]\big\} + \cdots,\\\gamma_1 &= M_3\cos\tau + \varepsilon^{-1}\nu\left(-M_1\cos\tau + M_1\cos 0.5\tau + M_2\sin 0.5\tau\right) + \cdots,\\\gamma_1' &= -M_3\sin\tau + \varepsilon^{-1}\left[\nu M_1\sin\tau + 0.5\nu_2\left(-M_1\sin 0.5\tau + M_2\cos 0.5\tau\right)\right] + \cdots,\\\gamma_1'' &= 1 + \varepsilon^{-1}M_3E\big[a\cos\mu + 0.5\big(b\omega A_1^{-1} - a\big)\cos(0.5\tau - \mu) - 0.25\big(bA_1^{-1} + 2a\big)\cos(1.5\tau - \mu)\big]\\&\quad + \varepsilon^{-2}\left\{M_3(1 - a)^{-1}x_0' + \frac{0.5M_3^2z_0'(a - b)}{(a + b - 1)} + M_3(1 - b)^{-1}y_0'\sin\tau - M_3(1 - a)^{-1}x_0'\cos\tau - \frac{0.5M_3^2z_0'(a - b)\cos 2\tau}{(a + b - 1)}\right.\\&\quad + E^2\big[\nu a\big(\cos^2\mu - 0.5\big) - 0.25bA_1^{-1}\nu_2\big(\sin^2\mu - 0.5\big) - 0.125\big(4\nu a + bA_1^{-1}\nu_2\big)\cos 2(0.5\tau - \mu)\big]\\&\quad \left. - 0.5E^2\big[\big(a\cos^2\mu - 0.5\big) + 0.25bA_1^{-2}\big(\sin^2\mu - 0.5\big) + 0.125\big(bA_1^{-2} - 4\big)\cos 2(0.5\tau - \mu)\big]\right\} + \cdots.\end{aligned} \tag{46}$$

The correction of the period is

$$\alpha(\varepsilon) = \varepsilon^{-2}\pi n\big\{2M_3x_0' - 2z_0' - 0.125A_1^{-2}E^2 - E^2\big[a\big(\cos^2\mu - 0.5\big) + 0.25bA_1^{-2}\big(\sin^2\mu - 0.5\big)\big] - 0.5aE^2(1 + B_1)\big\} + \cdots. \tag{47}$$

5. Geometric Interpretation of Motion

In this section, we describe the body motion using Euler's angles $\xi, \zeta,$ and η which come from the obtained solutions (Figure 2). Replacing the time t by $t + t_0$ where t_0 is an arbitrary interval, the periodic solutions remain periodic since the initial system is autonomous [9]. For this case, we obtain from (32),

$$\eta_0 = 0.5\pi + r_0^{-1}t_0 + \cdots, \tag{48}$$

$$\xi_0 = \tan^{-1}M_3, \tag{49}$$

where η_0 are ξ_0 are arbitrary initial angles.

Making use of (46) and (49) when $\tau = r_0^{-1}t$, we find Euler's angles as follows:

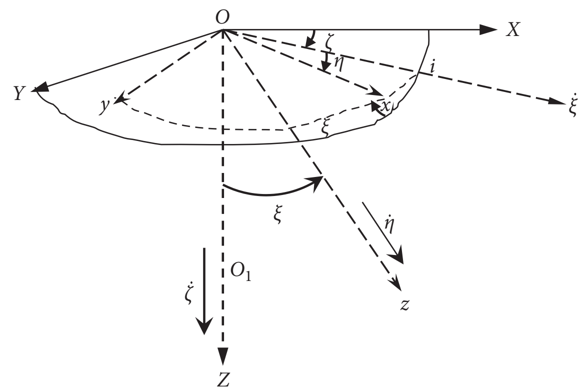

Figure 2: The rotational planes in terms of Euler's angles.

$$\begin{aligned}
\xi &= \xi_0 - \varepsilon^{-1} E[\xi_1(t+t_0) - \xi_1(t_0)] - \varepsilon^{-2}[\xi_2(t+t_0) - \xi_2(t_0)] + \cdots,\\
\zeta &= \zeta_0 + 0.5Mg\ell C^{-1} r_0 \cos^2\xi_0 Q_{10} t + 0.5\varepsilon^{-1}\sec\xi_0[\zeta_1(t+t_0) - \zeta_1(t_0)] + 0.5\varepsilon^{-2}\cos\xi_0[\zeta_2(t+t_0) - \zeta_2(t_0)] + \cdots,\\
\eta &= \eta_0 + \left(r_0^{-1} - 0.5Mg\ell C^{-1} r_0 \cos^3\xi_0 h_{10}\right)t - 0.5\varepsilon^{-1}\cot\xi_0[\eta_1(t+t_0) - \eta_1(t_0)] - 0.5\varepsilon^{-2}\cos^2\xi_0[\eta_2(t+t_0) - \eta_2(t_0)] + \cdots,
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\xi_1(t) &= 0.5\left(0.5bA_1^{-1} - a\right)\cos\left(\frac{t}{2r_0} + \mu\right) - 0.5\left(0.5bA_1^{-1} + a\right)\cos\left(\frac{3t}{2r_0} - \mu\right),\\
\xi_2(t) &= y_0' a^{-1} A_1^{-1}\sin\frac{t}{r_0} + b^{-1}B_1^{-1}x_0'\cos\frac{t}{r_0} - 0.5\tan\xi_0 z_0'\left(\frac{a-b}{a+b-1}\right)\cos 2\frac{t}{r_0}\\
&\quad - 0.5E^2\cot\xi_0\left[a(\nu - 0.5) + 0.25bA_1^{-1}\left(\nu_2 + 0.5A_1^{-1}\right)\right]\cos\left(\frac{t}{r_0} - 2\mu\right),\\
\zeta_1(t) &= \eta_1(t) = 0.67\left(1 + 0.5A_1^{-1}\right)\left(M_1\sin\frac{3t}{2r_0} - M_2\cos\frac{3t}{2r_0}\right) + \left(2 - A_1^{-1}\right)\left(M_2\cos\frac{t}{2r_0} + M_1\sin\frac{t}{2r_0}\right),\\
\zeta_2(t) &= (Q_{11} + Q_{13} + Q_{16})\sin\frac{t}{r_0} - (Q_{11}' + Q_{13}' - Q_{16}')\cos\frac{t}{r_0}\\
&\quad + 0.5\left(Q_{12}\sin\frac{2t}{r_0} - Q_{12}'\cos\frac{2t}{r_0}\right) + 2\left(Q_{14}\sin\frac{t}{2r_0} + Q_{14}'\cos\frac{t}{2r_0}\right) + 0.67\left(Q_{15}\sin\frac{3t}{2r_0} - Q_{15}'\cos\frac{3t}{2r_0}\right),\\
\eta_2(t) &= h_{11}\sin\frac{t}{r_0} - h_{11}'\cos\frac{t}{r_0} + 0.5\left(h_{12}\sin\frac{2t}{r_0} - h_{12}'\cos\frac{2t}{r_0}\right) + \left(h_{13}\sin\frac{t}{r_0} - h_{13}'\cos\frac{t}{r_0}\right) + 2\left(h_{14}\sin\frac{t}{2r_0} + h_{14}'\cos\frac{t}{2r_0}\right)\\
&\quad + 0.67\left(h_{15}\sin\frac{3t}{2r_0} - h_{15}'\cos\frac{3t}{2r_0}\right) + \left(h_{16}\sin\frac{t}{r_0} + h_{16}'\cos\frac{t}{r_0}\right) + 0.34\left(h_{17}\sin\frac{3t}{r_0} - h_{17}'\cos\frac{3t}{r_0}\right).
\end{aligned} \tag{51}$$

Table 1: The analytical solutions p_2, γ_2, and their derivatives.

t	p_2	γ_2	$\dot{p}_2$	$\dot{\gamma}_2$
0	1.5	11.06602	0.8164966	$8.60977E-05$
10	2.018443	8.279188	0.6022027	−7.342405
20	2.361099	1.322297	0.3354627	−10.98657
30	2.498126	−6.300518	0.03950729	−9.096887
40	2.417591	−10.74971	−0.259889	−2.625185
50	2.126507	−9.784271	−0.5366514	5.168785
60	1.650225	−3.890493	−0.7666767	10.35926
70	1.030224	3.962964	−0.9299318	10.33185
80	0.3205004	9.82036	−1.012199	5.10033
90	−0.4171353	10.73134	−1.006313	−2.700216
100	−1.118443	6.237032	−0.9127875	−9.140733
110	−1.722344	−1.398863	−0.7397668	−10.97719
120	−2.176246	−8.330236	−0.5023198	−7.284575
130	−2.440619	−11.06585	−0.2211253	0.07719428
140	−2.492437	−8.227805	0.07932674	7.400064
150	−2.327188	−1.245598	0.3728705	10.9956
160	−1.959263	6.363904	0.6339409	9.052767
170	−1.420706	10.76787	0.839801	2.550172
180	−0.7584189	9.748108	0.9725226	−5.236892
190	−0.0300812	3.818282	1.020547	−10.38613
200	0.7008771	−4.034786	0.9796914	−10.30393
210	1.370795	−9.855532	0.8535145	−5.031701
220	1.921331	−10.71208	0.6530044	2.774997
230	2.304537	−6.172976	0.3956243	9.183975
240	2.48704	1.475497	0.1037893	10.9671
250	2.452946	8.380901	−0.1970856	7.226189
260	2.205224	11.06502	−0.4807954	−0.1544821
270	1.765449	8.175891	−0.7226327	−7.457364
280	1.171921	1.168729	−0.9015357	−11.00408
290	0.4763283	−6.427075	−1.001924	−9.008176
300	−0.2607465	−10.78558	−1.015054	−2.474989

Table 2: The numerical solutions p_2, γ_2, and their derivatives.

t	p_2	γ_2	$\dot{p}_2$	$\dot{\gamma}_2$
0	1.5	11.06602	0.8164966	$8.60977E-05$
10	2.018441	8.279626	0.6022035	−7.341455
20	2.361096	1.324263	0.3354649	−10.9858
30	2.498125	−6.297489	0.03951085	−9.098105
40	2.417592	−10.74771	−0.2598841	−2.629244
50	2.126513	−9.78573	−0.5366458	5.163152
60	1.650237	−3.896276	−0.7666712	10.3552
70	1.030242	3.954753	−0.9299275	10.33263
80	0.3205241	9.813907	−1.012197	5.107155
90	−0.4171081	10.73098	−1.006314	−2.68967
100	−1.118414	6.244486	−0.9127917	−9.131646
110	−1.722318	−1.38609	−0.739775	−10.97505
120	−2.176226	−8.318236	−0.5023316	−7.291935
130	−2.440607	−11.06137	−0.2211403	0.06275124
140	−2.492436	−8.234457	0.07930994	7.385382
150	−2.3272	−1.261325	0.372853	10.9887
160	−1.959289	6.346504	0.6339244	9.058519
170	−1.420744	10.75812	0.8397874	2.567106
180	−0.7584675	9.752295	0.9725135	−5.216549
190	−0.03013661	3.835833	1.020544	−10.37296
200	0.7008187	−4.011765	0.9796954	−10.30583
210	1.37074	−9.83878	0.8535257	−5.049277

Table 2: Continued.

t	p_2	γ_2	$\dot{p}_2$	$\dot{\gamma}_2$
220	1.921283	−10.71119	0.6530228	2.749476
230	2.304503	−6.190003	0.3956484	9.163264
240	2.487025	1.44783	0.1038172	10.96256
250	2.452952	8.356145	−0.1970554	7.24163
260	2.205252	11.05635	−0.4807664	−0.1255394
270	1.765499	8.189053	−0.7226074	−7.429035
280	1.171988	1.198392	−0.9015169	−10.99124
290	0.4764102	−6.395274	−1.001913	−9.018668
300	−0.2606581	−10.76818	−1.015054	−2.504935

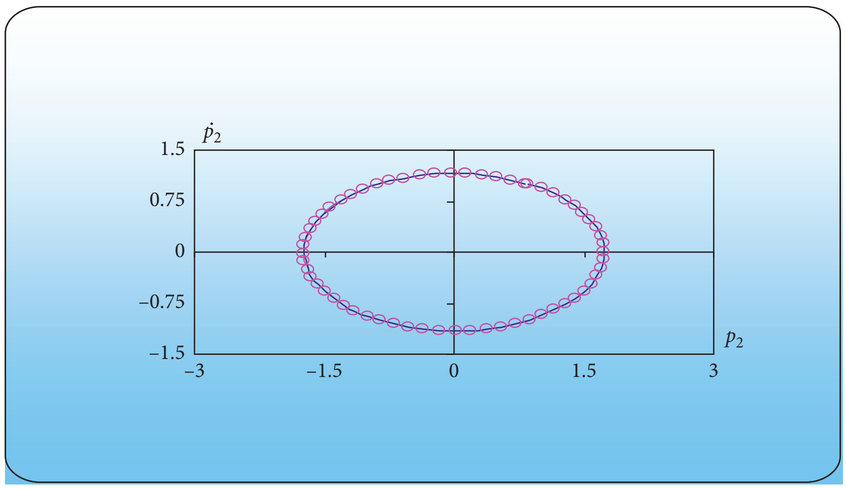

Figure 3: The stability of the analytical and numerical solutions $\dot{p}_2$andp_2.

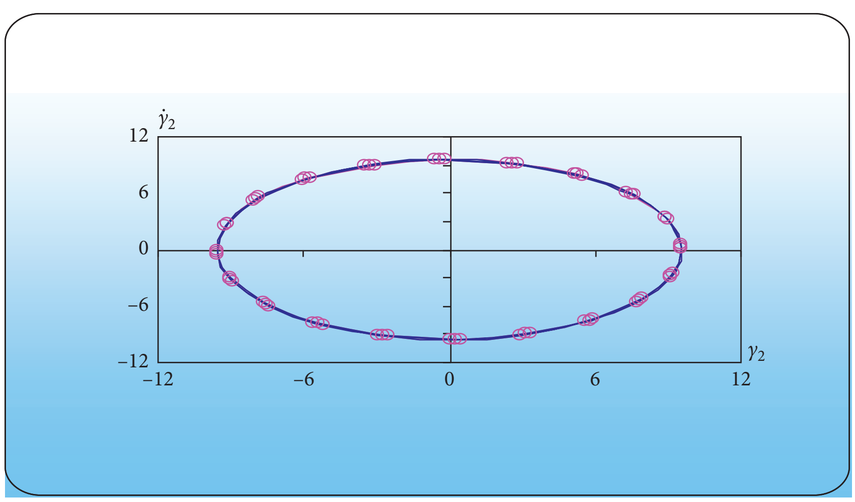

Figure 4: The stability of the analytical and numerical solutions $\dot{\gamma}_2$ andγ_2.

6. The Numerical Solutions

In this section, we assume numerical values data for the parameters of a rigid body, and we achieve a computer program to solve the quasilinear system using the fourth order Runge–Kutta method [7]. We make another program to represent the analytical solutions numerically in a period t between 0 and 300 (Table 1). We use the initial values from Table 1 for obtaining the numerical solutions represented in Table 2. The comparison between the obtained numerical solutions and analytical ones is presented to know the difference between them. The numerical and analytical solutions are in good agreement with others which proves the accuracy of used methods and obtained results.

7. Conclusion

The solutions (46) and the correction of the period (47) are obtained using the large parameter method, which had never been used for solving this kind of problem in the presence of the new assumptions for motion (the weak oscillations of the body about the minor or the major axis of the ellipsoid of inertia instead of the strong oscillations in the previous works). The advantage of this method is that the energy motion of the body is assumed to be sufficiently small instead of sufficiently large with other techniques [10–12]. Also, the obtained solutions treat a singular situation for the natural frequency which was excluded from previous works [13, 14].

Equations (50) and (51) describe the rotation of the body at any time and show that this motion depends on four arbitrary constants $\xi_0, \zeta_0, \eta_0, \text{and} r_0$, such that r_0 is sufficiently small. The obtained solutions give special cases of motions when ($M_1 = M_2 = 0$) and when $M_1 = 0$, $M_2 \neq 0$, or $M_2 = 0$, $M_1 \neq 0$. Also, the obtained solutions give many gyroscopic motions, which depend on the values of the moments of inertia and the initial position of the body center of gravity. In the end, we obtain the case of regular precession [10] as a special case.

The analytical solutions (46) are represented indefinite intervals of time through computer programs (Table 1). The numerical solutions are obtained using the fourth order Runge–Kutta method in terms of another program (Table 2). Tables 1 and 2 give in detail the obtained results of both the analytical solutions and numerical ones. These results show that the analytical solutions are in full agreement with the numerical ones which proves the accuracy of the considered techniques and results. This case of study is considered as a general case of such ones studied in [5]. The stability phase diagrams of the solutions p_2 and γ_2 are given (Figures 3 and 4). From these diagrams, we note that the stability for both the analytical and the numerical solutions in full agreement. This gives the validity of the obtained solutions and the considered procedures. The considered procedures and results are very useful for the general reader's concern with the new applications dealing with the use of functionally graded materials in such structures based on the recent works [15].

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. E. Udwadia and R. E. Kalaba, *Analytical Dynamics: A New Approach*, Cambridge University Press, Cambridge, UK, 2007.

[2] T. S. Amer, A. I. Ismail, and W. S. Amer, "Application of the Krylov-Bogoliubov-Mitropolski technique for a rotating heavy solid under the influence of a gyrostatic moment," *Journal of Aerospace Engineering*, vol. 25, no. 3, pp. 421–430, 2012.

[3] A. I. Ismail, "On the motion of a rigid body in a Newtonian field of force exerted by three attractive centers," *Journal of Aerospace Engineering*, vol. 21, no. 1, pp. 67–77, 2011.

[4] A. H. Nayfeh, *Introduction to Perturbation Technique*, Wiley, Hoboken, NJ, USA, 2011.

[5] A. I. Ismail, "Solving a problem of rotary motion for a heavy solid using the large parameter method," *Advances in Astronomy*, vol. 2020, Article ID 2764867, 7 pages, 2020.

[6] S. V. Ershkov and D. Leshchenko, "On a new type of solving procedure for Euler-Poisson equations (rigid body rotation over the fixed point)," *Acta Mechanica*, vol. 230, no. 3, pp. 871–883, 2019.

[7] C. T. Wu, L. Wang, B. Bonello, L. Ling, N. Ma, and M. A. Schweitzer, "Advanced Mesh-based and particle-based numerical methods for engineering and applied mathematics problems," *Mathematical Problems in Engineering*, vol. 2017, Article ID 1273017, 2 pages, 2017.

[8] A. I. Ismail and T. S. Amer, "A necessary and sufficient condition for solving a rigid body problem," *Technische Mechanik*, vol. 31, no. 1-2, pp. 50–57, 2011.

[9] L. D. Akulenko, Y. S. Zinkevich, T. A. Kozachenko, and D. D. Leshchenko, "The evolution of the motions of a rigid body close to the Lagrange case under the action of an unsteady torque," *Journal of Applied Mathematics and Mechanics*, vol. 81, no. 2, pp. 79–84, 2017.

[10] A. Sanders Jan, V. Ferdinand, and M. James, "Averaging methods in nonlinear dynamical systems," *Applied Mathematical Sciences*, vol. 59, no. 2, p. 434, 2007.

[11] F. L. Chernousko, L. D. Akulenko, and D. D. Leshchenko, *Evolution of Motions of a Rigid Body about Its Center of Mass*, Springer International Publishing, Berlin, Germany, 2017.

[12] A. I. Ismail and F. D. El-Haiby, "Torque free axi-symmetric gyros with changing moments of inertia," *Applied Mathematics*, vol. 7, no. 16, p. 1934, 2016.

[13] T. S. Amer, "On the dynamical motion of a gyro in the presence of external forces," *Advances in Mechanical Engineering*, vol. 9, no. 2, pp. 1–13, 2017.

[14] T. S. Amer and I. M. Abady, "On the motion of a gyro in the presence of a Newtonian force field and applied moments," *Mathematics and Mechanics of Solids*, vol. 23, no. 9, pp. 1263–1273, 2017.

[15] N. Bendenia, M. Zidour, and A. A. Bousahla, "Deflections, stresses and free vibration studies of FG-CNT reinforced sandwich plates resting on Pasternak elastic foundation," *Computers and Concrete*, vol. 26, no. 3, pp. 213–226, 2020.