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*Research Article*

# A Quantized Hill's Dynamical System

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In this paper, we present a modified version of Hill's dynamical system that is called the quantized Hill's three-body problem in the sense that the equations of motion for the classical Hill's problem are now derived under the effects of quantum corrections. To do so, the position variables and the parameters that correspond to the quantum corrections of the respective quantized three-body problem are scaled appropriately, and then by taking the limit when the parameter of mass ratio tends to zero, we obtain the relevant equations of motion for the spatial quantized Hill's problem. Furthermore, the Hamiltonian formula and related equations of motion are also derived.

## 1. Introduction

In the analysis of dynamical systems that deal with celestial objects, the restricted three-body problem plays a fundamental role. The problem gains its importance from the variety of its modifications that approximate different real systems [1, 2]. In addition, it can be applied and used in both stellar and planetary dynamics as well as in space missions [3, 4]. Also, this problem can be effectively used to determine the possibility of sub-Jovian and terrestrial planets [5, 6]. Due to its extensive astronomical applications, considerable variants have been proposed in order to study a test particle in solar and planetary systems [7–9]. For example, the modification in which the more massive body is a source of radiation and the smaller one is either oblate or triaxial body motivates us to apply this perturbed version in our solar system. This modification may be more realistic than the classical one since in the solar system, the Sun is radiating and some planets are not spherical but sufficiently oblate (or triaxial) bodies [10–12].

On the other hand, the classical restricted problem has some simpler modified versions, instead of the aforementioned complex perturbed models, such as Sitnikov and Robe problems (see, e.g., [13, 14] and references therein). However, the simplest of its versions is Hill's problem, which can be treated as a perturbed two-body problem. In particular, this problem is considered to be a limiting case of the restricted three-body problem when the parameter mass $\mu$ tends to zero, and it may be used to study the satellite motion around a planet [15]. A considerable study on the circular Hill's problem has been established by Hénon [16, 17], where he determined the main families of periodic orbits, revealed the phase space by means of surface of sections of the Poincaré map, and found the stability regions in the parameters' plane. Recently, for the same problem, Lara et al. in [18] have employed a normalization approach in complex variables to compute a single perturbation solution with enough accuracy. The solution captures the main four families of periodic orbits for the Hill problem originated from the libration points. They have also extended the solution validity to energy values. In addition, Nishimura et al. in [19] used the same model to study spacecraft orbital motion. They studied 3-dimensional distant retrograde orbits and also found a sufficient condition for a closed orbit to be unstable. To do so, the authors transformed the relative equations of motion into a time-independent form by using

Fourier series. Also recently, Kalantonis in [20] have studied the families of spatial periodic orbits bifurcating from the vertical self-resonant periodic orbits of the basic families of simple planar periodic orbits.

In the framework of Hill's problem, Markellos et al. in [21, 22] have proposed the models in which the radiation or oblateness of the primaries are also considered. The proposed models were used in order to find estimates for the maximum possible distance of Hill stable direct orbits around the small primary and to estimate the maximum sizes of accretion disks in binary stars. For Hill's problem where the larger primary is a source of radiation, Kalantonis et al. in [23] considered homoclinic connections at both the Lyapunov planar periodic orbits and collinear equilibrium points. Markakis et al. in [24] proposed a respective Hill model by combining the radiation pressure and oblateness effects. They found approximate expressions for the locations of equilibrium points and explored their linear stability. Also, by applying singular perturbations methods, they determined approximate expressions for the Lyapunov orbits emanating from the collinear points in both the coplanar and spatial cases. For the same problem, Perdiou et al. [25] studied the network evolution of the basic families, determined their stability as well as the stability regions of retrograde satellites in the plane of initial conditions by means of appropriate Poincaré surface of sections. The elliptical Hill's problem is constructed by using the same assumptions of circular Hill's problem, but assume that the planet moves on an elliptic orbit around the Sun. This model was obtained by Ichtiaroglou in [26] in which few families of periodic orbits were computed. A further study was addressed by Ichtiaroglou and Voyatzis in [27] where they explored the stability of periodic orbits. Also, for the elliptic case, Voyatzis et al. in [28] computed a large set of families of periodic orbits together with their linear stability and classify them according to their resonance condition.

In this paper, we derive a quantum version for Hill's problem that comes from the quantized three-body problem introduced by Alshaery and Abouelmagd [29]. This special variant is called the quantized Hill's problem, and its equations of motion are obtained in a similar way as Hill's problem is derived from the classical restricted three-body problem; however, the calculations are not direct and further assumptions are considered, which incorporate the perturbations of quantum corrections. In particular, our work presents the equations of motions in both the planar and spatial cases as well as the Hamiltonian formula with the related equations. More precisely, our paper is organized as follows. First, in Section 2, the quantized three-body problem is discussed. In Section 3, the Hill version of the latter is systematically derived, while the relevant Hamiltonian approach is obtained in Section 4. Finally, in Section 5, we summarize our work and conclude.

## 2. Quantized Three-Body Problem

We will accept the notations and nomenclature of the spatial quantized restricted three-body problem, which is studied by Alshaery and Abouelmagd in [29]. In this context, the equations of a test particle within the frame of the quantized model are given by

$$\begin{aligned} \ddot{\xi}_1 - 2n\dot{\eta}_1 &= \frac{\partial}{\partial \xi_1}\Gamma_1(\xi_1,\eta_1,\zeta_1),\\ \ddot{\eta}_1 + 2n\dot{\xi}_1 &= \frac{\partial}{\partial \eta_1}\Gamma_1(\xi_1,\eta_1,\zeta_1),\\ \ddot{\zeta}_1 &= \frac{\partial}{\partial \zeta_1}\Gamma_1(\xi_1,\eta_1,\zeta_1), \end{aligned} \tag{1}$$

where $\Gamma_1$ is the effective potential, and it is read as

$$\begin{aligned} \Gamma_1(\xi_1,\eta_1,\zeta_1) = {}& \frac{1}{2}n^2\left(\xi_1^2+\eta_1^2\right)\\ &+(1-\mu)\Gamma_{11}(\varrho_1)+\mu\Gamma_{21}(\varrho_2), \end{aligned} \tag{2}$$

$$\Gamma_{i1}(\varrho_i) = \frac{1}{\varrho_i}\left(1+\frac{Q_{i1}}{\varrho_i}+\frac{Q_{i2}}{\varrho_i^2}\right), \quad i=1,2. \tag{3}$$

The derivatives of this potential with respect to $\xi_1$, $\eta_1$, and $\zeta_1$ are

$$\begin{aligned} \frac{\partial}{\partial \xi_1}\Gamma_1 &= (x+\mu)\varphi_1(\varrho_1)+(x+\mu-1)\varphi_2(\varrho_2),\\ \frac{\partial}{\partial \eta_1}\Gamma_1 &= y\left[\varphi_1(\varrho_1)+\varphi_2(\varrho_2)\right],\\ \frac{\partial}{\partial \zeta_1}\Gamma_1 &= z\left[\varphi_1(\varrho_1)+\varphi_2(\varrho_2)-n^2\right], \end{aligned} \tag{4}$$

where

$$\begin{aligned} \varrho_1^2 &= (\xi_1+\mu)^2+\eta_1^2+\zeta_1^2,\\ \varrho_2^2 &= (\xi_1+\mu-1)^2+\eta_1^2+\zeta_1^2,\\ \varrho_3^2 &= \xi_1^2+\eta_1^2+\zeta_1^2, \end{aligned} \tag{5}$$

are the distances of the massless body from the two primaries, while the mean motion $n$ is given by

$$n^2 = 1+2Q_1+3Q_2. \tag{6}$$

Relations (1)–(6) represent the equations of motion of a circular restricted three-body problem in a synodic reference frame (see [29] for details).

We would like to refer here that $\varphi_1(\varrho_1)$ and $\varphi_2(\varrho_2)$ are explicit functions in the distances $\varrho_1$ and $\varrho_2$, which are read as

$$\begin{aligned} \varphi_1(\varrho_1) &= (1-\mu)\left[n^2-\frac{1}{\varrho_1^3}-\frac{2Q_{11}}{\varrho_1^4}-\frac{3Q_{12}}{\varrho_1^5}\right],\\ \varphi_2(\varrho_2) &= \mu\left[n^2-\frac{1}{\varrho_2^3}-\frac{2Q_{21}}{\varrho_2^4}-\frac{3Q_{22}}{\varrho_2^5}\right], \end{aligned} \tag{7}$$

where

$$Q_1 = n_1\left(R_{m_1} + R_{m_2}\right),$$
$$Q_2 = n_2\left(l_p\right)^2, \tag{8}$$

$$Q_{11} = k_1\left(R_{m_1} + R_m\right),$$
$$Q_{12} = Q_{22} = k_2\left(l_p\right)^2, \tag{9}$$
$$Q_{21} = k_3\left(R_{m_2} + R_m\right).$$

The quantities $Q_1$, $Q_2$, $Q_{11}$, $Q_{12}$, $Q_{21}$, and $Q_{22}$, which are identified by equations 8 and (9), utilize quantum corrections. In particular, the first two quantities are due to the mean motion, while the last four are corrections of the potential, which the perturbed test particle motion governs. It is clear that the classical (unperturbed) motion can be obtained when each quantity is assigned to zero. Details for the estimated quantum corrections from the basic quantum principles as well as their clear quantum mechanical meaning were introduced in [30–32]. Also, $R_{m_1}$, $R_{m_2}$, and $R_m$ are the gravitational radii of the primaries $m_1$ and $m_2$ and the massless body $m$, respectively, while $l_P$ is the Planck length. Moreover, the numbers $n_1$, $n_2$, $k_1$, $k_2$, and $k_3$ can be estimated from the analysis of Feynman diagrams. Generally, the values of these numbers depend on different definitions; therefore, they are different in both sign and magnitude [30, 31]. In fact, the constants $Q_1$, $Q_{11}$, and $Q_{21}$ measure the size of the relativistic effect or post-Newtonian approximation, while $Q_2$, $Q_{12}$, and $Q_{22}$ measure control quantum correction contributions. However, all these effects tend to zero in the case of large distances.

Utilizing equations (2), (4), and (7), we obtain from (1) the equation of the Jacobi integral in the form

$$2\Gamma_1\left(\xi_1,\eta_1,\zeta_1\right) - \left(\dot{\xi}_1^2 + \dot{\eta}_1^2 + \dot{\zeta}_1^2\right) = \mathscr{C}, \tag{10}$$

where $\mathscr{C}$ is the constant of integration or alternatively the well-known Jacobi constant, which can be utilized for the study of the invariant manifolds of the considered system.

## 3. Hill Version of Quantized Three-Body Problem

In order to find the quantized Hill version, we follow the transformation of Szebehely in [15]. First, we subject the equations of motion of the quantized restricted three-body problem given by system (1) and the related relations to a translation along the $\xi_1$-axis, so we let in this way the coordinates center moves to the mass center of the smaller primary. Therefore, the relations between the old $(\xi_1,\eta_1,\zeta_1)$ and new $(\xi,\eta,\zeta)$ coordinates are given by

$$\xi_1 = \xi - \mu + 1,$$
$$\eta_1 = \eta, \tag{11}$$
$$\zeta_1 = \zeta.$$

Utilizing these relations in equations (1), (2), and (5), we get

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial}{\partial\xi}\Gamma(\xi,\eta,\zeta),$$
$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial}{\partial\eta}\Gamma(\xi,\eta,\zeta), \tag{12}$$
$$\ddot{\zeta} = \frac{\partial}{\partial\zeta}\Gamma(\xi,\eta,\zeta),$$

where

$$\Gamma(\xi,\eta,\zeta) = \frac{1}{2}n^2\left[(\xi-\mu+1)^2 + \eta^2\right] + (1-\mu)\Gamma_{11}(\rho_1) + \mu\Gamma_{21}(\rho_2), \tag{13}$$

while the relevant distances are now

$$\rho_1^2 = (1+\xi)^2 + \eta^2 + \zeta^2,$$
$$\rho_2^2 = \xi^2 + \eta^2 + \zeta^2. \tag{14}$$

We now follow again Szebehely in [15] to scale the variables by introducing

$$\xi = \mu^{1/3}x,$$
$$\eta = \mu^{1/3}y, \tag{15}$$
$$\zeta = \mu^{1/3}z.$$

The aforementioned scale preserves the magnitude of Coriolis and centrifugal terms in the same order in the previous equations. Substituting (15) into equations (12)–(14), we obtain

$$\ddot{x} - 2n\dot{y} = \mu^{-2/3}\frac{\partial}{\partial x}\Omega(x,y,z),$$
$$\ddot{y} + 2n\dot{x} = \mu^{-2/3}\frac{\partial}{\partial y}\Omega(x,y,z), \tag{16}$$
$$\ddot{z} = \mu^{-2/3}\frac{\partial}{\partial z}\Omega(x,y,z),$$

where

$$\Omega(x,y,z) = \frac{1}{2}n^2\left[\left(\mu^{1/3}x - \mu + 1\right)^2 + \mu^{2/3}y^2\right] + (1-\mu)\Gamma_{11}(r_1) + \mu\Gamma_{21}(r_2), \tag{17}$$

and the scaled distances are

$$r_1^2 = 1 + 2\mu^{1/3}x + \mu^{2/3}r^2,$$
$$r_2^2 = \mu^{2/3}r^2, \tag{18}$$
$$r^2 = x^2 + y^2 + z^2.$$

We utilize equations (16) and (17) and let the mass parameter $\mu$ tends to zero; the existing this limit leads to Hill's equations. Thus, equation (16) can now be rewritten as

$$\ddot{x} = L_{10} + L_{11} + L_{12} + L_{13},$$
$$\ddot{y} = L_{20} + L_{21} + L_{22} + L_{23}, \tag{19}$$
$$\ddot{z} = L_{31} + L_{32} + L_{33},$$

where the limits $L_{ij}$, $i = 1, 2, 3$ and $j = 0, 1, 2, 3$, are given by

$$\begin{aligned} L_{10} &= \lim_{\mu\longrightarrow 0}\left[2n\dot{y} + n^2 x\right], \\ L_{11} &= \lim_{\mu\longrightarrow 0} \frac{1-\mu}{\mu^{1/3}}\kappa(r_1, x), \\ L_{12} &= -\lim_{\mu\longrightarrow 0} \frac{1-\mu}{\mu^{1/3}}\alpha_\mu(r_1)\left(1+\mu^{1/3}x\right), \\ L_{13} &= -\lim_{\mu\longrightarrow 0} \gamma_\mu(r)x, \end{aligned} \tag{20}$$

with

$$\begin{aligned} L_{20} &= -\lim_{\mu\longrightarrow 0}\left[2n\dot{x} - n^2 y\right], \\ L_{21} &= -\lim_{\mu\longrightarrow 0} \lambda(r_1, r), \\ L_{22} &= -\lim_{\mu\longrightarrow 0} (1-\mu)\alpha_\mu(r_1)y, \\ L_{23} &= -\lim_{\mu\longrightarrow 0} \beta_\mu(r)y, \end{aligned} \tag{21}$$

$$\begin{aligned} L_{31} &= -\lim_{\mu\longrightarrow 0} \lambda(r_1, r)z, \\ L_{32} &= -\lim_{\mu\longrightarrow 0} (1-\mu)\alpha_\mu(r_1)z, \\ L_{33} &= -\lim_{\mu\longrightarrow 0} \beta_\mu(r)z, \end{aligned} \tag{22}$$

while we have abbreviated

$$\begin{aligned} \kappa(r_1, x) &= n^2 - \frac{1+\mu^{1/3}x}{r_1^3}, \\ \lambda(r_1, r) &= \frac{1-\mu}{r_1^3} + \frac{1}{r^3}, \\ \alpha_\mu(r_1) &= \frac{2Q_{11}}{r_1^4} + \frac{3Q_{12}}{r_1^5}, \\ \beta_\mu(r) &= \frac{2Q_{21}}{\mu^{1/3}r^4} + \frac{3Q_{22}}{\mu^{2/3}r^5}, \\ \gamma_\mu(r) &= \frac{1}{r^3} + \beta_\mu(r). \end{aligned} \tag{23}$$

From equation (18), we observe that $r_1 \rightarrow 1$ and $r_2 \rightarrow 0$ when $\mu \rightarrow 0$. Furthermore, $r_2$ is of order $\mathcal{O}(\mu^{1/3})$, thus $(1/r_1)^k \rightarrow 1$ and $(1/r_2)^k$ is undefined, where $k$ is a positive integer. We also remark here that, if the parameters of quantum corrections are ignored, i.e., $Q_1 = Q_2 = 0$ and $Q_{11} = Q_{12} = Q_{21} = Q_{22} = 0$, then the limits (20)–(22) converge, hence exist, and system (19) is reduced to the classical planar and spatial Hill three-body systems [15, 33]. In the case where these parameters are not equal to zero, we get some divergent limits, such as $L_{11}, L_{12}, L_{13}, L_{23}$, and $L_{33}$. The limits that diverge correspond to the involved functions, which depend on the parameter $\mu$, possessing singularities and in particular, they are not defined at $\mu = 0$. The first two limits have a pole of order one due to some particular terms with coefficients $1/\mu^{1/3}$, while the last three limits have poles of order one and two due to some specific terms with coefficients $1/\mu^{1/3}$ and $1/\mu^{2/3}$, respectively.

On the other hand, the parameters of quantum corrections are very small and can be scaled by the factors $\mu^{1/3}$ and $\mu^{2/3}$. In this sense, we can scale $R_{m_1}$, $R_{m_2}$, $R_m$, and $l_p$. Specifically, this scale may be considered by taking $Q_1 = \mu^{1/3}\alpha_1$, $Q_2 = \mu^{2/3}\alpha_2$, $Q_{11} = \mu^{1/3}\alpha_{11}$, $Q_{21} = \mu^{1/3}\alpha_{21}$, $Q_{12} = \mu^{2/3}\alpha_{12}$, and $Q_{22} = \mu^{2/3}\alpha_{22}$. After the previous discussion and by utilizing these scaled quantum corrections in equations (20)–(22), the singularities at the poles $1/\mu^{1/3}$ and $1/\mu^{2/3}$ can be removed, so

$$\begin{aligned} L_{10} &= 2n\dot{y} + n^2 x, \\ L_{11} &= 2x + 2\alpha_1, \\ L_{12} &= -2\alpha_{11}, \\ L_{13} &= -f_r(r)x, \end{aligned} \tag{24}$$

$$\begin{aligned} L_{20} &= -2\dot{x} + y, \\ L_{21} &= -h_r(r), \\ L_{22} &= 0, \\ L_{23} &= -g_r(r)y, \end{aligned} \tag{25}$$

$$\begin{aligned} L_{31} &= -h_r(r)z, \\ L_{32} &= 0, \\ L_{33} &= -g_r(r)z, \end{aligned} \tag{26}$$

where

$$\begin{aligned} f_r(r) &= \frac{1}{r^3}\left(1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right), \\ g_r(r) &= \frac{1}{r^4}\left(2\alpha_{21} + \frac{3\alpha_{22}}{r}\right), \\ h_r(r) &= 1 + \frac{1}{r^3}. \end{aligned} \tag{27}$$

Substituting equations (24)–(26) into system (19), we get

$$\begin{aligned} \ddot{x} - 2\dot{y} &= 3x + 2\left(\alpha_1 - \alpha_{11}\right) - f_r(r)x, \\ \ddot{y} + 2\dot{x} &= -f_r(r)y, \\ \ddot{z} &= -z - f_r(r)z. \end{aligned} \tag{28}$$

The last system represents the spatial quantized Hill problem; it is the limiting case, which is acquired from the spatial quantized restricted three-body problem, which was firstly presented and studied by Alshaery and Abouelmagd [29]. System (28) can be rewritten in a similar manner as that of the restricted three-body problem, i.e.,

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \mathcal{U}_x(x, y, z), \\ \ddot{y} + 2\dot{x} &= \mathcal{U}_y(x, y, z), \\ \ddot{z} &= \mathcal{U}_z(x, y, z), \end{aligned} \tag{29}$$

where

$$\mathscr{U}(x,y,z) = \frac{1}{2}\left[3x^2 + 4(\alpha_1 - \alpha_{11})x - z^2\right] + w(r). \quad (30)$$

Utilizing equations (29) and (30), the Jacobi integral is read as $\mathscr{C}_q = 2\mathscr{U} - \mathscr{V}^2$, where explicitly it has the form

$$\begin{aligned}\mathscr{C}_q &= \left[3x^2 + 4(\alpha_1 - \alpha_{11})x - z^2\right] \\ &\quad + 2w(r) - \mathscr{V}^2,\end{aligned} \quad (31)$$

with $\mathscr{V}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, while we have abbreviated

$$w(r) = \frac{1}{r}\left(1 + \frac{\alpha_{21}}{r} + \frac{\alpha_{22}}{r^2}\right). \quad (32)$$

## 4. Hamiltonian Approach

In Hamiltonian approach, a dynamical system is described by a set of canonical coordinates $(\mathbf{q}, \mathbf{p})$ where they correspond to the $n$-dimensional vectors $\mathbf{q} = (q_1, q_2, q_3, \ldots, q_n)$ and $\mathbf{p} = (p_1, p_2, p_3, \ldots, p_n)$, while each of the aforementioned components is indexed to the frame of reference of the respective system. The components $q_i$, $i = 1, 2, 3, \ldots, n$, are known as the generalized coordinates and are selected in order to remove the constraints or to take into consideration the symmetry characteristics of the system, while $p_i$ are their conjugate momenta. In classical mechanics, the evolution of time is determined by calculating the total force, which is exerted on each one of the involved bodies, so the time evolutions of both positions and velocities are obtained by using Newton's second law. However, in Hamiltonian mechanics, the evolution of time is computed by finding the relevant Hamiltonian function in the generalized coordinates and then using it into the corresponding Hamilton's equations.

The latter method is tantamount to that which is used to the Lagrangian approach. In fact, for the same generalized momenta, both methods result to the same equations, but the main reason to use Hamiltonian approach, instead that of the Lagrangian, comes from the symplectic features of a Hamiltonian system. In fact, Hamilton's equations comprised $2n$ first-order differential equations, while Lagrange's equations are constituted by $n$ second-order differential equations. Although Hamilton's equations do not generally reduce the difficulty of determining analytical solutions, they may offer some crucial theoretical results due to the fact that the coordinates and momenta are independent variables with nearly symmetric roles. In addition, if the system possesses a kind of symmetry for which a coordinate does not appear in the Hamiltonian, the relevant momentum is conserved and the corresponding coordinate can be neglected by the other equations. This results in reducing the number of coordinates of the dynamical system from $n$ to $n-1$, while in the framework of the Lagrangian approach, all the generalized velocities remain in the Lagrangian; therefore, we still have to deal with a system of equations in $n$ coordinates, which must be handled. Both approaches are of fundamental importance in the study of classical mechanics as well as for formulations of quantum mechanics; however, the Hamiltonian approach provides deeper insights in several fundamental features of some astronomical dynamical systems, such as planetary orbits in celestial mechanics [34, 35].

In our system, the time evolution is defined by the following Hamilton's equations:

$$\begin{aligned}\dot{q}_\chi &= \frac{\partial \mathscr{H}}{\partial p_\chi}, \\ \dot{p}_\chi &= -\frac{\partial \mathscr{H}}{\partial q_\chi},\end{aligned} \quad (33)$$

where $\chi = (x, y, z)$ and the Hamiltonian $\mathscr{H}$ is given by

$$\begin{aligned}\mathscr{H} &= \frac{1}{2}P^2 + \left(q_y p_x - q_x p_y\right) + \Upsilon(q) \\ &\quad - \frac{1}{2}\left[2q_x^2 - q_y^2 - q_z^2 + 4(\alpha_1 - \alpha_{11})q_x\right],\end{aligned} \quad (34)$$

with

$$\begin{aligned}P^2 &= p_x^2 + p_y^2 + p_z^2, \\ \Upsilon(q) &= -\frac{1}{q^3}\left(1 + \frac{\alpha_{21}}{q} + \frac{\alpha_{22}}{q^2}\right).\end{aligned} \quad (35)$$

Utilizing equations (33) and (34), the equations of motion of the spatial quantized Hill model with Hamiltonian formula can be written as

$$\begin{aligned}\dot{q}_x &= p_x + q_y, \\ \dot{q}_y &= p_y - q_x, \\ \dot{q}_z &= p_z, \\ \dot{p}_x &= p_y + 2\left[q_x + (\alpha_1 - \alpha_{11})\right] - f_q(q)q_x \\ \dot{p}_y &= -p_x - q_y - f_q(q)q_y, \\ \dot{p}_z &= -q_z - f_q(q)q_z,\end{aligned} \quad (36)$$

where $q^2 = q_x^2 + q_y^2 + q_z^2$. Furthermore the Hamiltonian can be rewritten as

$$\mathscr{H} = \mathscr{H}_0 + \sigma_1 \mathscr{H}_1 + \sigma_2 \mathscr{H}_2, \quad (37)$$

where

$$\begin{aligned}\mathscr{H}_0 &= \frac{1}{2}P^2 + \left(q_y p_x - q_x p_y\right) \\ &\quad - \frac{1}{2}\left(3q_x^2 - q^2\right) - \frac{1}{r}, \\ \mathscr{H}_1 &= -2\beta_{11}q_x - \frac{\beta_{12}}{r^2}, \\ \mathscr{H}_2 &= -\frac{\beta_{22}}{r^3}.\end{aligned} \quad (38)$$

Here, $\sigma_1\beta_{11} = \alpha_1 - \alpha_{11}$, $\sigma_1\beta_{12} = \alpha_{21}$, and $\sigma_2\beta_{22} = \alpha_{22}$. Within the frame of quantum corrections, $\sigma_1$ is of order $\mathcal{O}(1/c^2)$ and $\sigma_2$ is of order $\mathcal{O}(1/c^3)$, where, as usual, $c$ denotes the speed of light. Thereby, the second and third terms in equation (37) are of order $\mathcal{O}(1/c^2)$ and $\mathcal{O}(1/c^3)$, respectively. Thus, the perturbed Hamiltonian (37) can be reduced classical to one of the unperturbed Hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{39}$$

The obtained relation (39) for Hamiltonian can be used to extend the results of the classical Hill's problem to the quantized problem. Since $\alpha_1$ and $\alpha_{11}$ are of order $\mathcal{O}(1/c^2)$, thus $\alpha_1 - \alpha_{11} \approx 0$ and the term with coefficient can be ignored.

## 5. Conclusions

A dynamical system was introduced, which is a modification of Hill's problem and is called quantized Hill's problem. In order to derive the relevant equations of motion in the three-dimensional space, we started from the corresponding spatial quantized restricted three-body problem, and after scaling the parameters and the position variables, a limiting process for the mass parameter was applied. In particular, the technique used here was similar to the way where the classical Hill's model is obtained from the circular restricted three-body problem, i.e., we translated the origin of the synodic coordinates to the center of the primary mass body and scaled the variables by the factor of the cubic root for the mass parameter of the restricted problem. However, the calculations were not direct, and further assumptions had to be considered to remove singularities appeared in limits calculations. In particular, the parameters corresponding to quantum corrections were also scaled by the factors $\mu^{1/3}$ and $\mu^{2/3}$, making thus the involved limits to be convergent and resulting in this way to the pertinent equations of motion for the spatial quantized Hill's dynamical system.

The Hamiltonian formula of the proposed system, which includes the perturbations of quantum corrections, and the related equations of motion were also given. This formula is of fundamental importance since it may reduce the problem from $n$ to $n-1$ coordinates if a dynamical system has symmetry. Particularly, in the case of symmetry where a coordinate does not appear in the Hamiltonian, the respective momentum is conserved; therefore, it can be neglected by the other set of equations.

We remark that the derived dynamical system can be reduced to the classical Hill's problem when the parameters of quantum corrections set to be equal to zero. Also, the respective planar motion can be easily obtained from the three-dimensional version of the model presented in this work by considering in all calculations $z = 0$. In a future correspondence, we intend to study the basic dynamical features of the proposed model such as the equilibrium points and periodic orbits which are of crucial importance in the study of any dynamical system since they may characterize the behaviour of nearby orbits.

## Data Availability

All data used to support the findings of this work are incorporated within the manuscript.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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