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Review Article

Mappings and Connectedness on Hesitant Fuzzy Soft Multispaces

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In this paper, we introduce the concept of mapping on hesitant fuzzy soft multisets and present some results for this type of mappings. The notions of inverse image and identity mapping are defined, and their basic properties are investigated. Hence, kinds of mappings and the composition of two hesitant fuzzy soft multimapping with the same dimension are presented. The concept of hesitant fuzzy soft multitopology is defined, and certain types of hesitant fuzzy soft multimapping such as continuity, open, closed, and homeomorphism are presented in detail. Also, their properties and results are studied. In addition, the concept of hesitant fuzzy soft multiconnected spaces is introduced.

1. Introduction

Since the introduction of fuzzy sets by Zadeh [1], several extensions of this concept have been introduced. The most agreed one may be Atanassov's intuitionistic fuzzy set (briefly, IFS or A-IFS) [2]. IFSs have the benefit that allows the user to model some uncertainties on the membership function of the elements. That is, fuzzy sets require a membership degree for each element in the universe set, whereas an IFS permits us to include some hesitation on this value. This is modeled with two functions that define an interval. Type 2 fuzzy sets [3, 4] are a generalization of the former, where the membership of a given element is presented as a fuzzy set. Other generalizations, such as type n fuzzy sets exist (see [3] for details about type n fuzzy sets). Dubois and Prade [3] report that Mizumoto and Tanaka [4] were the first to study type 2 fuzzy sets. Fuzzy multisets are another generalization of fuzzy sets. They are based on multisets (elements can be repeated in a multiset, for short, mset). In fuzzy multisets, the membership can be partial (instead of Boolean as for standard multisets). Tokat and Osmanoglu [5] introduced the concept of a soft mset (F, E) as $F\colon E \longrightarrow P^*(U)$, where E is a set of parameters and $P^*(U)$ is a power set of an mset U. In this paper, we adopt the notion of a soft mset in [5], since this way is better than the other [6, 7]. In 2013, Tokat et al. [7] introduced the concept of soft msets as a combination between soft sets and msets. Furthermore, in [7], the soft multitopology and its basic properties were given. Moreover, the soft multiconnectedness was studied in [5]. Additionally, the soft multicompactness on soft multitopological spaces was presented in [8]. In 2015, El-Sheikh et al. [9] introduced the concept of semicompact soft multispaces and the concept of soft multi-Lindelöf spaces. Some other results and properties about soft multisets are presented in [10–12]. The concept of a generalized open soft mset is introduced in soft multitopological spaces, and its properties are presented in [10]. Several authors [13–15] discussed the concept of multisets, its generalizations, and its applications. In 2020, Hashmi et al. [16] introduced the notion of an m-polar neutrosophic set and m-polar neutrosophic topology and their applications to multicriteria decision-making (MCDM) in medical diagnosis and clustering analysis. They introduced a novel approach to census process by using Pythagorean m-polar fuzzy Dombis aggregation operators. Riaz and Hashmi [17] introduced the notion of linear Diophantine fuzzy set (LDFS) and its applications towards MCDM problem. Linear Diophantine fuzzy set (LDFS) is superior to IFSs, PFSs, and q-ROFSs. Riaz and Tehrim [18] introduced the concept of bipolar fuzzy soft mappings with application to bipolar disorders. Tehrim and Riaz [19] presented a novel extension of the TOPSIS method with bipolar neutrosophic soft topology and its applications to multicriteria group decision-making (MCGDM). Riaz et al. [20] presented the multiattribute group decision-making (MAGDM) methods to a hesitant fuzzy soft set. Moreover, Riaz et al. [21] developed the topological structure on a soft rough set by using pairwise soft rough approximations. The multicriteria group decision-making methods are introduced by using N-soft set and N-soft topology to deal with uncertainties in the real-world problems [22].

Recently, the concept of hesitant fuzzy sets was introduced firstly in 2010 by Torra [23] which permits the membership to have a set of possible values and presents some of its basic operations in expressing uncertainty and vagueness. Torra et al. [24] established the similarities and differences with the hesitant fuzzy sets and the previous generalization of fuzzy sets such as intuitionistic fuzzy sets, type 2 fuzzy sets, and type n fuzzy sets. Therefore, other authors [25, 26] introduced the concept of hesitant fuzzy soft sets, and they presented some of the applications in decision-making problems. In 2015, Dey and Pal [27] presented the concept of hesitant multifuzzy soft topological space. In 2019, Kandil et al. [28] introduced some important and basic issues of hesitant fuzzy soft multisets and studied some of its structural properties such as the neighborhood hesitant fuzzy soft multisets, interior hesitant fuzzy soft multisets, hesitant fuzzy soft multitopological spaces, and hesitant fuzzy soft multibasis. Finally, they showed how to apply the concept of hesitant fuzzy soft multisets in decision-making problems.

The main goal of this paper is to introduce the definition of mapping on hesitant fuzzy soft multisets and present some results for this form of mappings. The notions of inverse image and identity mapping are introduced, and their basic properties are investigated in detail. The types of mappings are also given on hesitant fuzzy soft multisets, and their properties are established. Therefore, the composition of two hesitant fuzzy soft multimapping with the same dimension is presented. In addition, the concepts of hesitant fuzzy soft multitopologies and hesitant fuzzy soft multisubspaces are introduced. Some types of hesitant fuzzy soft multimapping such as continuity, open, closed, and homeomorphism are presented in detail. Also, their properties and results are investigated. Finally, the concept of hesitant fuzzy soft multiconnected space is introduced.

2. Preliminaries

The aim of this section is to present the basic concepts and properties of multisets, soft multisets, hesitant fuzzy sets, and hesitant fuzzy soft multisets.

Definition 1 (see [29]). An mset X drawn from the set U is represented by a count function C_X defined as $C_X\colon U \longrightarrow N$, where N represents the set of nonnegative integers.

Here, $C_X(x)$ is the number of occurrences of the element x in the mset X. The mset X is drawn from the set $U = \{x_1, x_2, \ldots, x_n\}$, and it is written as $X = \{(m_1/x_1), (m_2/x_2), \ldots, (m_n/x_n)\}$, where m_i is the number of occurrences of the element x_i, $i = 1, 2, 3, \ldots, n$ in the mset X.

Definition 2 (see [29]). A domain U is defined as a set of elements from which msets are constructed. The mset space $[U]^w$ is the set of all msets whose elements are in U such that no element in the mset occurs more than w times.

The mset space $[U]^\infty$ is the set of all msets over a domain U such that there is no limit on the number of occurrences of an element in an mset. If $U = \{x_1, x_2, \ldots, x_k\}$, then $[U]^w = \{(m_1/x_1), (m_2/x_2), \ldots, (m_k/x_k)\}\colon m_i \in \{0, 1, 2, \ldots, w\}$, $i = 1, 2, \ldots, k\}$.

Definition 3 (see [29]). Let X be an mset drawn from the set U. If $C_X(x) = 0$ for all $x \in U$, then X is called an empty mset and denoted by ϕ, i.e., $\phi(x) = 0$ for all $x \in U$.

Definition 4 (see [5]). Let X be a universal multiset, E be a set of parameters, and $A \subseteq E$. Then, an ordered pair (F, A) is called a soft mset, where F is a mapping given by $F\colon A \longrightarrow P^*(X)$; $P^*(X)$ is the power set of an mset X. For all $e \in A$, $F(e)$ mset is represented by count function $C_{F(e)}\colon X^* \longrightarrow N$, where N represents the set of nonnegative integers and X^* represents the support set of X.

Definition 5 (see [5]). Let (F, A) and (G, B) be two soft msets over X. Then,

(1) (F, A) is said to be a sub-soft mset of (G, B) and denoted by $(F, A) \mathrel{\widetilde{\subseteq}} (G, B)$ if

  (i) $A \subseteq B$
  (ii) $C_{F(e)}(x) \le C_{G(e)}(x)$, for all $x \in X^*$, $e \in A$

(2) Two soft msets (F, A) and (G, B) over X are equal if (F, A) is a sub-soft mset of (G, B) and (G, B) is a sub-soft mset of (F, A).

(3) The union of two soft msets (F, A) and (G, B) over X is the soft mset (H, C), where $C = A \cup B$ and $C_{H(e)}(x) = \max\{C_{F(e)}(x), C_{G(e)}(x)\}$, for all $e \in A \cup B$, $x \in X^*$. It is denoted by $(F, A) \mathrel{\widetilde{\cup}} (G, B)$.

(4) The intersection of two soft msets (F, A) and (G, B) over X is the soft mset (H, C), where $C = A \cap B$ and$C_{H(e)}(x) = \min\{C_{F(e)}(x), C_{G(e)}(x)\}$, for all $e \in A \cap B$, $x \in X^*$. It is denoted by $(F, A) \mathrel{\widetilde{\cap}} (G, B)$.

(5) A soft mset (F, A) over X is said to be a null soft mset and denoted by $\widetilde{\phi}$ if for all $e \in A$, $F(e) = \phi$.

(6) A soft mset (F, A) over X is said to be an absolute soft mset and denoted by $\widetilde{A}$ if for all $e \in A$, $F(e) = X$.

Definition 6 (see [5]). The complement of a soft mset (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c\colon A \longrightarrow P^*(X)$ is a mapping given by $F^c(e) = (X/F(e))$ for all $e \in A$, where $C_{F^c(e)}(x) = C_X(x) - C_{F(e)}(x)$, for all $x \in X^*$.

Definition 7 (see [5]). Let X be a universal mset and E be a set of parameters. Then, the collection of all soft msets over X with parameters from E is called a soft multiclass and is denoted by $SMS(X)_E$.

Definition 8 (see [23]). Let U be a reference set, then a hesitant fuzzy (briefly, an HF) set is defined in terms of a function h from U into the power set of $[0, 1]$.

Definition 9 (see [23]). Let h, h_1, and h_2 be hesitant fuzzy sets over a set U. Then, the following operations are defined:

(1) Full set $\widehat{1}$: $h(x) = \{1\}$ for all $x \in U$
(2) Null set $\widehat{0}$: $h(x) = \{0\}$ for all $x \in U$
(3) Lower bound: $h^-(x) = \min h(x)$
(4) Upper bound: $h^+(x) = \max h(x)$
(5) α-Upper bound: $h^+_\alpha(x) = \{\gamma \in h(x)\colon \gamma \geq \alpha\}$
(6) α-Lower bound: $h^-_\alpha(x) = \{\gamma \in h(x)\colon \gamma \leq \alpha\}$
(7) Complement: $h^c(x) = \{1 - \gamma\colon \gamma \in h(x)\}$
(8) Union: $(h_1 \cup h_2)(x) = \{\gamma \in (h_1(x) \cup h_2(x))\colon \gamma \geq \max(h^-_1(x), h^-_2(x))\}$ or, equivalently, $(h_1 \cup h_2)(x) = (h_1(x) \cup h_2(x))^+_\alpha$ for $\alpha = \max(h^-_1(x), h^-_2(x))$
(9) Intersection: $(h_1 \cap h_2)(x) = \{\gamma \in (h_1(x) \cup h_2(x))\colon \gamma \leq \min(h^+_1(x), h^+_2(x))\}$ or, equivalently, $(h_1 \cap h_2)(x) = (h_1(x) \cup h_2(x))^-_\alpha$ for $\alpha = \min(h^+_1(x), h^+_2(x))$

Definition 10 (see [28]). A hesitant fuzzy multiset of dimension k (briefly, HF^kM set) on a nonempty mset X is denoted by $A = \{< (m/x), h_A(x) >\colon x \in^m X\}$ and is defined in terms of $h_A(x)$ when applied to X, and $h_A(x)$ is a set of some distinct values in $[0, 1]$ sorting into increasing order, indicating the possible membership degrees of the elements $x \in^m X$ to the multiset A.

Definition 11 (see [28]). Let A and B be two HF^kM sets on a nonempty mset X. A is called a hesitant fuzzy submset of B if $h^i_A(x) \leq h^i_B(x)$ for each $x \in^m X, i = 1, 2, ..., k$ and denoted by $A \sqsubseteq B$.

Definition 12 (see [28]). A pair $(\widetilde{F}, E)$ is a hesitant fuzzy soft mset of dimension k if $\widetilde{F}$ is a mapping from E to $HF^kM(X)$, where $HF^kM(X)$ is the set of all hesitant fuzzy msets of dimension k defined over an mset X and $\widetilde{F}(e) \in HF^kM(X)$ $\forall e \in E$, i.e., $\widetilde{F}(e) = \{< (m/x), h_{\widetilde{F}(e)}(x) > | x \in^m X\}$ for all $e \in E$, and $h_{\widetilde{F}(e)}$ is the membership function of $\widetilde{F}(e)$.

Definition 13 (see [28]). An HF^kSM set $(\widetilde{F}, E)$ over $(\widetilde{X}, E)$ is said to be

(1) A relative null HF^kSM set and is denoted by $\widetilde{0}_{\widetilde{X}_E}$, if $h_{\widetilde{F}(e)}(x) = \{0, 0, \overset{k-\text{times}}{\ldots}, 0\}$ for all $x \in^m X$, $e \in E$
(2) A relative absolute HF^kSM set and is denoted by $\widetilde{1}_{\widetilde{X}_E}$, if $h_{\widetilde{F}(e)}(x) = \{1, 1, \overset{k-\text{times}}{\ldots}, 1\}$ for all $x \in^m X$, $e \in E$

Definition 14 (see [28]). Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two hesitant fuzzy soft multisets of dimension k, then $(\widetilde{F}, A)$ is called a hesitant fuzzy soft multi-subset (briefly, HF^kSM) of $(\widetilde{G}, B)$ of dimension k if

(1) $A \subseteq B$
(2) $\widetilde{F}(e)$ is a hesitant fuzzy submset of $\widetilde{G}(e)$, for every $e \in A$, i.e., $h^i_{\widetilde{F}(e)}(x) \leq h^i_{\widetilde{G}(e)}(x)$ for all $e \in A$, $x \in^m X$, $i \in \{1, 2, \ldots, k\}$

Hence, this relationship is denoted by $(\widetilde{F}, A) \sqsubseteq (\widetilde{G}, B)$, and $(\widetilde{G}, B)$ is called an HF^kSM superset of $(\widetilde{F}, A)$.

Definition 15 (see [28]). Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space and $(\widetilde{F}, A)$, $(\widetilde{G}, B)$ be two HF^kSM sets over a hesitant fuzzy soft mset $(\widetilde{X}, E)$ (for short, $\widetilde{X}_E$). A hesitant fuzzy soft mset $(\widetilde{F}, A)$ is called neighborhood of $(\widetilde{G}, B)$ if there exists an open hesitant fuzzy soft mset $(\widetilde{O}, C)$ such that $(\widetilde{G}, B) \sqsubseteq (\widetilde{O}, C) \sqsubseteq (\widetilde{F}, A)$.

Definition 16 (see [28]). Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space and $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two HF^kSM sets over $(\widetilde{X}, E)$ such that $(\widetilde{G}, B) \sqsubseteq (\widetilde{F}, A)$. Then, $(\widetilde{G}, B)$ is called an interior hesitant fuzzy soft mset of $(\widetilde{F}, A)$ if $(\widetilde{F}, A)$ is a neighborhood of $(\widetilde{G}, B)$. Additionally, the union of all interior hesitant fuzzy soft mset of $(\widetilde{F}, A)$ is called the interior of $(\widetilde{F}, A)$, and it is denoted by $(\widetilde{F}, A)^o$.

Theorem 1 (see [28]). *Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space and $(\widetilde{F}, A)$, $(\widetilde{G}, B)$ be two HF^kSM sets over $(\widetilde{X}, E)$, then*

(1) $(\widetilde{F}, A)^o$ is the largest open hesitant fuzzy soft mset contained in $(\widetilde{F}, A)$
(2) $(\widetilde{F}, A)$ is an open hesitant fuzzy soft mset if and only if $(\widetilde{F}, A)^o = (\widetilde{F}, A)$
(3) $((\widetilde{F}, A)^o)^o = (\widetilde{F}, A)^o$
(4) If $(\widetilde{F}, A) \sqsubseteq (\widetilde{G}, B)$, then $(\widetilde{F}, A)^o \sqsubseteq (\widetilde{G}, B)^o$

3. Mappings in Hesitant Fuzzy Soft Multisets

The purpose of this section is to present a concept of mapping in hesitant fuzzy soft multisets. The main properties of the current branch are studied, and some results of this type of sets are established. Also, the concept of inverse mapping in hesitant fuzzy soft multisets is defined. Therefore, the composition of two hesitant fuzzy soft multi-mappings is introduced. Finally, some examples are used to explain the current definitions in a friendly way.

It should be noted that, in this section, let U be a universal set, E be a set of parameters, and X be a multiset over U. The union and intersection of hesitant fuzzy sets are defined by Torra [23], but these definitions did not preserve the dimension, so we introduce the following definitions.

Definition 17. Union of two HF^kSM sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over $(\widetilde{X}, E)$ is the HF^kSM set $(\widetilde{H}, C)$, where $C = A \cup B$, for all $e \in C$,

$$\widetilde{H}_C(e)=\begin{cases}\widetilde{F}_A(e), & \text{if } e\in A-B,\\ \widetilde{G}_B(e), & \text{if } e\in B-A,\\ \widetilde{F}_A(e)\widetilde{\cup}\widetilde{G}_B(e), & \text{if } e\in A\cap B,\end{cases} \quad (1)$$

where $\widetilde{F}_A(e)\widetilde{\cup}\widetilde{G}_B(e)=\{<(m/x),\{h^1_{\widetilde{F}_A(e)}(x)\vee h^1_{\widetilde{G}_B(e)}(x),h^2_{\widetilde{F}_A(e)}(x)\vee h^2_{\widetilde{G}_B(e)}(x),\ldots,h^k_{\widetilde{F}_A(e)}(x)\vee h^k_{\widetilde{G}_B(e)}(x)\}>: x\in^m X, e\in A\cap B\}$. It is written as $(\widetilde{F},A)\widetilde{\sqcup}(\widetilde{G},B)=(\widetilde{H},C)$.

Example 1. Let $U=\{a,b,c,d\}$, $E=\{e_1,e_2,e_3\}$, and $X=\{(2/a),(3/b),(1/c),(4/d)\}$. The hesitant fuzzy soft msets of dimension 3, $(\widetilde{F},E),(\widetilde{G},E)$, are defined as $\widetilde{F}(e_1)=\{<(2/a),\{0.1,0.3,0.3\}>,<(3/b),\{0.1,0.4,0.6\}>,<(1/c),\{0.8,0.8,0.8\}>,<(4/d),\{0.7,0.8,0.9\}>\}$, $\widetilde{F}(e_2)=\{<(2/a),\{0.2,0.3,0.4\}>,<(3/b),\{0.2,0.4,0.6\}>,<(1/c),\{0.4,0.5,0.6\}>,<(4/d),\{0.5,0.7,0.9\}>\}$, $\widetilde{F}(e_3)=\{<(2/a),\{0.4,0.6,0.8\}>,<(3/b),\{0.1,0.2,0.5\}>,<(1/c),\{0.1,0.3,0.5\}>,<(4/d),\{0,0.5,1\}>\}$. $\widetilde{G}(e_1)=\{<(2/a),\{0.1,0.2,0.3\}>,<(3/b),\{0,0.2,0.7\}>,<(1/c),\{0.4,0.9,1\}>,<(4/d)\ \{0.5,0.6,0.7\}>\}$, $\widetilde{G}(e_2)=\{<(2/a),\{0.1,0.2,0.3\}>,<(3/b),\{0.2,0.3,0.5\}>,<1/c,\{0.1,0.2,0.3\}>,<(4/d),\{0.3,0.5,0.7\}>\}$, $\widetilde{G}(e_3)=\{<(2/a),\{0.4,0.5,0.6\}>,<(3/b),\{0.1,0.2,0.3\}>,<(1/c),\{0.1,0.2,0.4\}>,<(4/d),\{0,0.2,0.4\}>\}$. Hence, $(\widetilde{H},E)=(\widetilde{F},E)\widetilde{\sqcup}(\widetilde{G},E)$ such that $\widetilde{H}(e_1)=\{<(2/a),\{0.1,0.3,0.3\}>,<(3/b),\{0.1,0.4,0.7\}>,<(1/c),\{0.8,0.9,1\}>,<(4/d),\{0.7,0.8,0.9\}>\}$, $\widetilde{H}(e_2)=\{<(2/a),\{0.2,0.3,0.4\}>,<(3/b),\{0.2,0.4,0.6\}>,<(1/c),\{0.4,0.5,0.6\}>,<(4/d),\{0.5,0.7,0.9\}>\}$, $\widetilde{H}(e_3)=\{<(2/a),\{0.4,0.6,0.8\}>,<(3/b),\{0.1,0.2,0.5\}>,<(1/c),\{0.1,0.3,0.5\}>,<(4/d),\{0,0.5,1\}>\}$.

Definition 18. Intersection of two HF^kSM sets $(\widetilde{F},A)$ and $(\widetilde{G},B)$ over $(\widetilde{X},E)$ is the HF^kSM set $(\widetilde{H},C)$, where $C=A\cap B$, for all $e\in C$,

$$\widetilde{H}_C(e)=\widetilde{F}_A(e)\widetilde{\cap}\widetilde{G}_B(e)=\left\{<(m/x),\left\{h^1_{\widetilde{F}_A(e)}(x)\wedge h^1_{\widetilde{G}_B(e)}(x),h^2_{\widetilde{F}_A(e)}(x)\wedge h^2_{\widetilde{G}_B(e)}(x),\ldots,h^k_{\widetilde{F}_A(e)}(x)\wedge h^k_{\widetilde{G}_B(e)}(x)\right\}>: x\in^m X, e\in C\right\}. \quad (2)$$

It is written as $(\widetilde{F},A)\widetilde{\sqcap}(\widetilde{G},B)=(\widetilde{H},C)$.

Example 2. Let $U=\{a,b,c,d\}$, $E=\{e_1,e_2,e_3\}$ and $X=\{(2/a),(3/b),(1/c),(4/d)\}$. The hesitant fuzzy soft msets of dimension 3, $(\widetilde{F},E),(\widetilde{G},E)$, are defined as $\widetilde{F}(e_1)=\{<(2/a),\{0.1,0.3,0.3\}>,<(3/b),\{0.1,0.4,0.6\}>,<(1/c),\{0.8,0.8,0.8\}>,<(4/d),\{0.7,0.8,0.9\}>\}$, $\widetilde{F}(e_2)=\{<(2/a),\{0.2,0.3,0.4\}>,<(3/b),\{0.2,0.4,0.6\}>,<(1/c),\{0.4,0.5,0.6\}>,<(4/d),\{0.5,0.7,0.9\}>\}$, $\widetilde{F}(e_3)=\{<(2/a),\{0.4,0.6,0.8\}>,<(3/b),\{0.1,0.2,0.5\}>,<(1/c),\{0.1,0.3,0.5\}>,<(4/d),\{0,0.5,1\}>\}$. $\widetilde{G}(e_1)=\{<(2/a),\{0.1,0.2,0.3\}>,<(3/b),\{0,0.2,0.7\}>,<(1/c),\{0.4,0.9,1\}>,<(4/d),\{0.5,0.6,0.7\}>\}$, $\widetilde{G}(e_2)=\{<(2/a),\{0.1,0.2,0.3\}>,<(3/b),\{0.2,0.3,0.5\}>,<(1/c),\{0.1,0.2,0.3\}>,<(4/d),\{0.3,0.5,0.7\}>\}$, $\widetilde{G}(e_3)=\{<(2/a),\{0.4,0.5,0.6\}>,<(3/b),\{0.1,0.2,0.3\}>,<(1/c),\{0.1,0.2,0.4\}>,<(4/d),\{0,0.2,0.4\}>\}$. Hence, $(\widetilde{H},E)=(\widetilde{F},E)\widetilde{\sqcap}(\widetilde{G},E)$ such that $\widetilde{H}(e_1)=\{<(2/a),\{0.1,0.2,0.3\}>,<(3/b),\{0,0.2,0.6\}>,<(1/c),\{0.4,0.8,0.8\}>,<(4/d),\{0.5,0.6,0.7\}>\}$, $\widetilde{H}(e_2)=\{<(2/a),\{0.1,0.2,0.3\}>,<(3/b),\{0.2,0.3,0.5\}>,<(1/c),\{0.1,0.2,0.3\}>,<(4/d),\{0.3,0.5,0.7\}>\}$, $\widetilde{H}(e_3)=\{<2/a,\{0.4,0.5,0.6\}>,<(3/b),\{0.1,0.2,0.3\}>,<(1/c),\{0.1,0.2,0.4\}>,<(4/d),\{0,0.2,0.4\}>\}$.

Theorem 2 *Let $(\widetilde{F},E)$, $(\widetilde{G},E)$ and $(\widetilde{H},E)$ be three elements in $HF^kSM(\widetilde{X}_E)$. Then,*

(1) $(\widetilde{F},E)\widetilde{\sqcup}(\widetilde{F},E)=(\widetilde{F},E)$
(2) $(\widetilde{F},E)\widetilde{\sqcap}(\widetilde{F},E)=(\widetilde{F},E)$
(3) $(\widetilde{F},E)\widetilde{\sqcup}\widetilde{0}_E=(\widetilde{F},E)$
(4) $(\widetilde{F},E)\widetilde{\sqcap}\widetilde{0}_E=\widetilde{0}_E$
(5) $(\widetilde{F},E)\widetilde{\sqcup}\widetilde{1}_E=\widetilde{1}_E$
(6) $(\widetilde{F},E)\widetilde{\sqcap}\widetilde{1}_E=(\widetilde{F},E)$
(7) $(\widetilde{F},E)\widetilde{\sqcup}(\widetilde{G},E)=(\widetilde{G},E)\widetilde{\sqcup}(\widetilde{F},E)$
(8) $(\widetilde{F},E)\widetilde{\sqcap}(\widetilde{G},E)=(\widetilde{G},E)\widetilde{\sqcap}(\widetilde{F},E)$
(9) $((\widetilde{F},E)\widetilde{\sqcup}(\widetilde{G},E))\widetilde{\sqcup}(\widetilde{H},E)=(\widetilde{F},E)\widetilde{\sqcup}((\widetilde{G},E)\widetilde{\sqcup}(\widetilde{H},E))$
(10) $((\widetilde{F},E)\widetilde{\sqcap}(\widetilde{G},E))\widetilde{\sqcap}(\widetilde{H},E)=(\widetilde{F},E)\widetilde{\sqcap}((\widetilde{G},E)\widetilde{\sqcap}(\widetilde{H},E))$
(11) $((\widetilde{F},E)\widetilde{\sqcup}(\widetilde{G},E))^c=(\widetilde{F},E)^c\widetilde{\sqcap}(\widetilde{G},E)^c$
(12) $((\widetilde{F},E)\widetilde{\sqcap}(\widetilde{G},E))^c=(\widetilde{F},E)^c\widetilde{\sqcup}(\widetilde{G},E)^c$
(13) $(\widetilde{F},E)\widetilde{\sqsubseteq}(\widetilde{G},E)$ *if and only if* $(\widetilde{F},E)\widetilde{\sqcup}(\widetilde{G},E)=(\widetilde{G},E)$
(14) $(\widetilde{F},E)\widetilde{\sqsubseteq}(\widetilde{G},E)$ *if and only if* $(\widetilde{F},E)\sqcap(\widetilde{G},E)=(\widetilde{F},E)$
(15) If $(\widetilde{F},E)\widetilde{\sqcap}(\widetilde{G},E)=\widetilde{0}_E$*, then* $(\widetilde{F},E)\widetilde{\sqsubseteq}(\widetilde{G},E)^c$
(16) $(\widetilde{F},E)\widetilde{\sqsubseteq}(\widetilde{G},E)$ *if and only if* $(\widetilde{G},E)^c\widetilde{\sqsubseteq}(\widetilde{F},E)^c$

Proof. Straightforward. □

Example 3. From Example 2, the complements of $(\widetilde{F},E),(\widetilde{G},E)$ are defined as $\widetilde{F}^c(e_1)=\{<(2/a),\{0.7,0.7,0.9\}>,<(3/b),\{0.4,0.6,0.9\}>,<(1/c),\{0.2,0.2,0.2\}>,<(4/d),\{0.1,0.2,0.3\}>\}$, $\widetilde{F}^c(e_2)=\{<(2/a),\{0.6,0.7,0.8\}>,<(3/b),\{0.4,0.6,0.8\}>,<(1/c),\{0.4,0.5,0.6\}>,<(4/d),\{0.1,0.3,0.5\}>\}$, $\widetilde{F}^c(e_3)=\{<(2/a),\{0.2,0.4,0.6\}>,<(3/b),\{0.5,0.8,0.9\}>,<(1/c),\{0.5,0.7,0.9\}>,<(4/d),\{0,0.5,1\}>\}$. $\widetilde{G}^c(e_1)=\{<(2/a),\{0.7,0.8,0.9\}>,<(3/b),\{0.3,0.8,1\}>,<(1/c),\{0,0.1,0.6\}>,<(4/d),\{0.3,0.4,0.5\}>\}$, $\widetilde{G}^c(e_2)=\{<(2/a),\{0.7,0.8,0.9\}>,<(3/b),\{0.5,0.7,0.8\}>,<(1/c),\{0.7,0.8,0.9\}>,<(4/d),\{0.3,0.5,0.7\}>\}$, $\widetilde{G}^c(e_3)=\{<(2/a),\{0.4,0.5,0.6\}>,<(3/b),\{0.7,0.8,0.9\}>,<(1/c),\{0.6,0.8,0.9\}>,<(4/d),\{0.6,0.8,1\}>\}$. Then, $(\widetilde{F},E)^c\widetilde{\sqcup}(\widetilde{G},E)^c:=(\widetilde{M},E)$ such that $\widetilde{M}(e_1)=\{<(2/a),\{0.7,0.8,0.9\}>,<(3/b),\{0.4,0.8,1\}>,<$

(1/c), {0.2, 0.2, 0.6} > , < (4/d), {0.3, 0.4, 0.5} > }, $\widetilde{M}(e_2)$ = { < (2/a), {0.7, 0.8, 0.9} > , < (3/b), {0.5, 0.7, 0.8} > , < (1/c), {0.7, 0.8, 0.9} > , < (4/d), {0.3, 0.5, 0.7} > }, $\widetilde{M}(e_3)$ = { < (2/a), {0.4, 0.5, 0.6} > , < (3/b), {0.7, 0.8, 0.9} > , < (1/c), {0.6, 0.8, 0.9} > , < (4/d), {0.6, 0.8, 1} > }.

Also, $(\widetilde{H}, E) = (\widetilde{F}, E)\widetilde{\sqcap}(\widetilde{G}, E)$ such that $\widetilde{H}(e_1)$ = { < (2/a), {0.1, 0.2, 0.3} > , < (3/b), {0, 0.2, 0.6} > , < (1/c), {0.4, 0.8, 0.8} > , < (4/d), {0.5, 0.6, 0.7} > }, $\widetilde{H}(e_2)$ = { < (2/a), {0.1, 0.2, 0.3} > , < (3/b), {0.2, 0.3, 0.5} > , < (1/c), {0.1, 0.2, 0.3} > , < (4/d), {0.3, 0.5, 0.7} > }, $\widetilde{H}(e_3)$ = { < (2/a), {0.4, 0.5, 0.6} > , < (3/b), {0.1, 0.2, 0.3} > , < (1/c), {0.1, 0.2, 0.4} > , < (4/d), {0, 0.2, 0.4} > }. Then, the complement of $(\widetilde{H}, E)$ is $\widetilde{H}^c(e_1)$ = { < (2/a), {0.7, 0.8, 0.9} > , < (3/b), {0.4, 0.8, 1} > , < (1/c), {0.2, 0.2, 0.6} > , < (4/d), {0.3, 0.4, 0.5} > }, $\widetilde{H}^c(e_2)$ = { < (2/a), {0.7, 0.8, 0.9} > , < (3/b), {0.5, 0.7, 0.8} > , < (1/c), {0.7, 0.8, 0.9} > , < (4/d), {0.3, 0.5, 0.7} > }, $\widetilde{H}^c(e_3)$ = { < (2/a), {0.4, 0.5, 0.6} > , < (3/b), {0.7, 0.8, 0.9} > , < (1/c), {0.6, 0.8, 0.9} > , < (4/d), {0.6, 0.8, 1} > }. Hence, $((\widetilde{F}, E)\widetilde{\sqcap}(\widetilde{G}, E))^c = (\widetilde{F}, E)^c\widetilde{\sqcup}(\widetilde{G}, E)^c$.

Definition 19. Let $HF^kSM(\widetilde{X}_E)$ and $HF^kSM(\widetilde{Y}_{E'})$ be two families of hesitant fuzzy soft msets over msets X and Y with dimension k and sets of parameters E and $E\prime$, respectively. Let $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E\prime$ be two mappings. Now, a mapping $f = (u, p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ is defined as follows: for a hesitant fuzzy soft mset $(\widetilde{F}, A)$ in $HF^kSM(\widetilde{X}_E)$, $f((\widetilde{F}, A))$ is a hesitant fuzzy soft mset in $HF^kSM(\widetilde{Y}_{E'})$ obtained as follows: for $e' \in p(E) \subseteq E'$ and $y \in Y^*$,

$$h_{f((\widetilde{F},A))(e')}(y) = \begin{cases} \bigvee_{x\in u^{-1}(y)} \left(\widetilde{\cup}_{n\in p^{-1}(e\prime)\cap A}\widetilde{F}_A(n)\right)(h(x)), & \text{if } u^{-1}(y)^1\phi,\ p^{-1}(e')\cap A \neq \phi, \\ \{0, 0, \overset{k-\text{times}}{\ldots}, 0\}, & \text{if otherwise}. \end{cases} \tag{3}$$

Hence, $f((\widetilde{F}, A))$ is called an image hesitant fuzzy soft mset with dimension k of a hesitant fuzzy soft mset $(\widetilde{F}, A)$.

Example 4. Let $X = \{(2/a), (3/b), (1/c), (4/d)\}$ and $Y = \{(3/x), (1/y), (2/z)\}$ be two msets, $E = \{e_1, e_2, e_3, e_4\}$ and $E\prime = \{e_1', e_2', e_3'\}$. Also, let $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E\prime$ be two mappings defined as $u(a) = y$, $u(b) = z$, $u(c) = x$, and $u(d) = y$ and $p(e_1) = e_3'$, $p(e_2) = e_3'$, $p(e_3) = e_1'$, and $p(e_4) = e_2'$. Choose a hesitant fuzzy soft mset $(\widetilde{F}, A)$ in $HF^3SM(\widetilde{X}_E)$ such as $(\widetilde{F}, A) = \{(e_1, \{ < (2/a), \{0.1, 0.3, 0.5\} > , < (3/b), \{0, 0, 0\} > , < (1/c), \{0, 0, 0\} > , < (4/d), \{0, 0, 0\} > \}), (e_2, \{ < (2/a), \{0, 0, 0\} > , < (3/b), \{0, 0, 0\} > , < (1/c), \{0.2, 0.3, 0.4\} > , < (4/d), \{0.3, 0.7, 0.9\} > \})\}$. Then, the hesitant fuzzy soft mset image of $(\widetilde{F}, A)$ under $f = (u, p)\colon HF^3SM(\widetilde{X}_E) \longrightarrow HF^3SM(\widetilde{Y}_{E\prime})$ is obtained as $h_{f((\widetilde{F},A))(e_1')}(x) = \{0, 0, 0\}$ as $p^{-1}(e_1') \cap A = \phi$. Similarly, $h_{f((\widetilde{F},A))(e_1')}(y) = h_{f((\widetilde{F},A))(e_1')}(z) = \{0, 0, 0\}$.

$$\begin{aligned}
h_{f((\widetilde{F},A))(e_3')}(x) &= \bigvee_{q\in u^{-1}(x)} \left(\widetilde{\cup}_{n\in\{\{e_1,e_2\}}\widetilde{F}_A(n)\right)(h(q)) \\
&= \bigvee_{q\in\{c\}} (\widetilde{F}_A(e_1)\widetilde{\cup}\widetilde{F}_A(e_2))(h(q)) \\
&= \bigvee_{q\in\{c\}} (\{<(2/a), \{0.1, 0.3, 0.5\}>, <(3/b), \{0, 0, 0\}>, <(1/c), \{0.2, 0.3, 0.4\}>, <(4/d), \{0.3, 0.7, 0.9\}>\})(h(q)) \\
&= \{0.2, 0.3, 0.4\}, \\
h_{f((\widetilde{F},A))(e_3')}(y) &= \bigvee_{q\in u^{-1}(y)} \left(\widetilde{\cup}_{n\in\{e_1,e_2\}}\widetilde{F}_A(n)\right)(h(q)) \\
&= \bigvee_{q\in\{a,d\}} (\widetilde{F}_A(e_1)\widetilde{\cup}\widetilde{F}_A(e_2))(h(q)) \\
&= \bigvee_{q\in\{a,d\}} (\{<(2/a), \{0.1, 0.3, 0.5\}>, <(3/b), \{0, 0, 0\}>, <(1/c), \{0.2, 0.3, 0.4\}>, <(4/d), \{0.3, 0.7, 0.9\}>\})(h(q)) \\
&= \{0.1, 0.3, 0.5\}\vee\{0.3, 0.7, 0.9\} \\
&= \{0.3, 0.7, 0.9\}, \\
h_{f((\widetilde{F},A))(e_3')}(z) &= \bigvee_{q\in u^{-1}(z)} \left(\widetilde{\cup}_{n\in\{e_1,e_2\}}\widetilde{F}_A(n)\right)(h(q)) \\
&= \bigvee_{q\in\{b\}} (\widetilde{F}_A(e_1)\widetilde{\cup}\widetilde{F}_A(e_2))(h(q)) \\
&= \bigvee_{q\in\{b\}} (\{<(2/a), \{0.1, 0.3, 0.5\}>, <(3/b), \{0, 0, 0\}>, <(1/c), \{0.2, 0.3, 0.4\}>, <(4/d), \{0.3, 0.7, 0.9\}>\})(h(q)) \\
&= \{0, 0, 0\}.
\end{aligned} \tag{4}$$

Then, $(f(((\widetilde{F},tA)),p(E))=\{(e_1',\{<(3/x),\{0,0,0\}>,<(1/y),\{0,0,0\}>,<(2/z),\{0,0,0\}>\}),\ (e_3',\{<(3/x),\{0.2,0.3,0.4\}>,<(1/y),\{0.3,0.7,0.9\}>,<(2/z),\{0,0,0\}>\})\}$.

Definition 20. Let $f=(u,p)$: $HF^kSM(\widetilde{X}_E)\longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be a mapping such that u: $X^*\longrightarrow Y^*$ and p: $E\longrightarrow E'$ be two mappings. If $(\widetilde{H},B)$ is a hesitant fuzzy soft mset in $HF^kSM(\widetilde{Y}_{E'})$, then the inverse image of $(\widetilde{H},B)$ is a hesitant fuzzy soft mset in $HF^kSM(\widetilde{X}_E)$, denoted by $f^{-1}((\widetilde{H},B))$, defined as follows: for $e\in p^{-1}(B)\subseteq E$ and $x\in X^*$,

$$h_{f^{-1}((\widetilde{H},B))(e)}(x)=\begin{cases}\widetilde{H}_B((p(e))(h(u(x))), & \text{if } p(e)\in B,\\ \{0,0,\overset{k-\text{times}}{\ldots},0\}, & \text{if otherwise.}\end{cases}\qquad(5)$$

Example 5. From Example 4, let $(\widetilde{H},B)=\{\{(e_1',\{<(3/x),\{0.1,0.5,0.7\}>,<(1/y),\{0,0.2,0.4\}>,<(2/z),\{0.2,0.3,0.4\}>\}),(e_2',\{<(3/x),\{0.2,0.4,0.6\}>,<(1/y),\{0.1,0.2,0.3\}>,<(2/z),\{0,0,0.2\}>\})\}$. Since $p(e_1)=p(e_2)=e_3'\in B$, then $h_{f^{-1}((\widetilde{H},B))(e_1)}(a)=h_{f^{-1}((\widetilde{H},B))(e_1)}(b)=h_{f^{-1}((\widetilde{H},B))(e_1)}(c)=h_{f^{-1}((\widetilde{H},B))(e_1)}(d)=\{0,0,0\}$ and $h_{f^{-1}((\widetilde{H},B))(e_2)}(a)=h_{f^{-1}((\widetilde{H},B))(e_2)}(b)=h_{f^{-1}((\widetilde{H},B))(e_2)}(c)=h_{f^{-1}((\widetilde{H},B))(e_2)}(d)=\{0,0,0\}$. Since $p(e_3)=p(e_4)=e_1'$, then

$$\begin{aligned}h_{f^{-1}((\widetilde{H},B))(e_3)}(a)&=h_{f^{-1}((\widetilde{H},B))(e_4)}(a)=\{0,0.2,0.4\},\\ h_{f^{-1}((\widetilde{H},B))(e_3)}(b)&=h_{f^{-1}((\widetilde{H},B))(e_4)}(b)=\{0.2,0.3,0.4\},\\ h_{f^{-1}((\widetilde{H},B))(e_3)}(c)&=h_{f^{-1}((\widetilde{H},B))(e_4)}(c)=\{0.1,0.5,0.7\},\\ h_{f^{-1}((\widetilde{H},B))(e_3)}(d)&=h_{f^{-1}((\widetilde{H},B))(e_4)}(d)=\{0,0.2,0.4\}.\end{aligned}\qquad(6)$$

Hence, the inverse image of $(\widetilde{H},B)$ is $f^{-1}((\widetilde{H},B))=\{(e_1,\{<(2/a),\{0,0,0\}>,<(3/b),\{0,0,0\}>,<(1/c),\{0,0,0\}>,<(4/d),\{0,0,0\}>\}),\ (e_2,\{<(2/a),\{0,0,0\}>,<(3/b),\{0,0,0\}>,<(1/c),\{0,0,0\}>,<(4/d),\{0,0,0\}>\}),\ (e_3,\{<(2/a),\{0,0.2,0.4\}><(3/b),\{0.2,0.3,0.4\}>,<(1/c),\{0.1,0.5,0.7\}>,<(4/d),\{0,0.2,0.4\}>\}),\ (e_4,\{<(2/a),\{0,0.2,0.4\}>,<(3/b),\{0.2,0.3,0.4\}>,<(1/c),\{0.1,0.5,0.7\}>,<(4/d),\{0,0.2,0.4\}>\})\}$.

Definition 21. Let $f=(u,p)$: $HF^kSM(\widetilde{X}_E)\longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be a mapping such that u: $X^*\longrightarrow Y^*$ and p: $E\longrightarrow E'$ be two mappings. Let $(\widetilde{F},A)$ and $(\widetilde{G},B)$ be two hesitant fuzzy soft msets in $HF^kSM(\widetilde{X}_E)$. For $e'\in E'$, $y\in Y^*$: the union and intersection of two images $f((\widetilde{F},A))$ and $f((\widetilde{G},B))$ in $HF^kSM(\widetilde{Y}_{E'})$ are defined as

$$\begin{aligned}h_{[f((\widetilde{F},A))\widetilde{\sqcup}f((\widetilde{G},B))](e')}(y)&=h_{f((\widetilde{F},A))(e')}(y)\vee h_{f((\widetilde{G},B))(e')}(y),\\ h_{[f((\widetilde{F},A))\widetilde{\sqcap}f((\widetilde{G},B))](e')}(y)&=h_{f((\widetilde{F},A))(e')}(y)\wedge h_{f((\widetilde{G},B))(e')}(y).\end{aligned}\qquad(7)$$

Definition 22. Let $f=(u,p)$: $HF^kSM(\widetilde{X}_E)\longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be a mapping such that u: $X^*\longrightarrow Y^*$ and p: $E\longrightarrow E'$ be two mappings. Let $(\widetilde{F},A)$ and $(\widetilde{G},B)$ be two hesitant fuzzy soft msets in $HF^kSM(\widetilde{Y}_{E'})$. For $e\in E$, $x\in X^*$: the union and intersection of two inverse images $f^{-1}((\widetilde{F},A))$ and $f^{-1}((\widetilde{G},B))$ in $HF^kSM(\widetilde{X}_E)$ are defined as

$$\begin{aligned}h_{\left[f^{-1}((\widetilde{F},A))\widetilde{\sqcup}f^{-1}((\widetilde{G},B))\right](e)}(x)&=h_{f^{-1}((\widetilde{F},A))(e)}(x)\vee h_{f^{-1}((\widetilde{G},B))(e)}(x),\\ h_{\left[f^{-1}((\widetilde{F},A))\widetilde{\sqcap}f^{-1}((\widetilde{G},B))\right](e)}(x)&=h_{f^{-1}((\widetilde{F},A))(e)}(x)\wedge h_{f^{-1}((\widetilde{G},B))(e)}(x).\end{aligned}\qquad(8)$$

Theorem 3. *Let $f=(u,p)$: $HF^kSM(\widetilde{X}_E)\longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be a mapping such that u: $X^*\longrightarrow Y^*$ and p: $E\longrightarrow E'$ be two mappings. If $(\widetilde{F},A)$, $(\widetilde{G},B)$ are two hesitant fuzzy soft msets in $HF^kS(\widetilde{X}_E)$ and $(\widetilde{F}_i,A_i)$ is a family of hesitant fuzzy soft msets in $HF^kSM(\widetilde{X}_E)$, then*

(1) $f(\widetilde{0}_{\widetilde{X}_E})=\widetilde{0}_{\widetilde{Y}_{p(E)}}$.
(2) $f(\widetilde{1}_{\widetilde{X}_E})\widetilde{\sqsubseteq}\widetilde{1}_{\widetilde{Y}_{E'}}$.
(3) $f((\widetilde{F},A)\widetilde{\sqcup}(\widetilde{G},B))=f((\widetilde{F},A))\widetilde{\sqcup}f((\widetilde{G},B))$. In general, $f(\widetilde{\sqcup}_i(\widetilde{F}_i,A_i))=\widetilde{\sqcup}_if((\widetilde{F}_i,A_i))$.
(4) $f((\widetilde{F},A)\widetilde{\sqcap}(\widetilde{G},B))\widetilde{\sqsubseteq}f((\widetilde{F},A))\widetilde{\sqcap}f((\widetilde{G},B))$. In general, $f(\widetilde{\sqcap}_i(\widetilde{F}_i,A_i))\widetilde{\sqsubseteq}\widetilde{\sqcap}_if((\widetilde{F}_i,A_i))_i$.
(5) If $(\widetilde{F},A)\widetilde{\sqsubseteq}(\widetilde{G},B)$, then $f((\widetilde{F},A))\widetilde{\sqsubseteq}f((\widetilde{G},B))$.

Proof. The proof of parts 1and 2 are obvious.

(3) For $e'\in p(E)\subseteq E'$, $y\in Y^*$, let $f((\widetilde{F},A)\widetilde{\sqcup}(\widetilde{G},B))=f((\widetilde{H},C))$, where $C=A\cup B$, then

$$h_{[f((\widetilde{H},C))](e')}(y)=\begin{cases}\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\bigcup}_{n\in p^{-1}(e')\cap C}\widetilde{H}_C(n)\right)(h(x)), & \text{if } u^{-1}(y)\neq\phi,\ p^{-1}(e')\cap C\neq\phi,\\ \{0,0,\overset{k-\text{times}}{\ldots},0\}, & \text{if otherwise.}\end{cases}\qquad(9)$$

and

$$h_{[f(\widetilde{H},C)](e')}(y)=\begin{cases}\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\bigcup}_{n\in p^{-1}(e')\cap(A\cup B)}\widetilde{H}_C(n)\right)(h(x)), & \text{if } u^{-1}(y)\neq\phi,\ (p^{-1}(e')\cap A)\cup(p^{-1}(e')\cap B)\neq\phi,\\ \{0,0,\overset{k-\text{times}}{\ldots},0\}, & \text{if otherwise.}\end{cases}\qquad(10)$$

Considering only the nontrivial case, we have

$$h_{[f((\widetilde{H},A\cup B))](e_\prime)}(y)=\vee_{x\in u^{-1}(y)}\left(\widetilde{\cup}\begin{cases}\widetilde{F}_A(n), & \text{if } n\in (A-B)\cap p^{-1}(e'),\\ \widetilde{G}_B(n), & \text{if } n\in (B-A)\cap p^{-1}(e'),\\ \widetilde{F}_A(n)\,\widetilde{\cup}\,\widetilde{G}_B(n), & \text{if } n\in A\cap B\cap p^{-1}(e').\end{cases}\right)(h(x)). \tag{11}$$

By Definition 21, we have

$$\begin{aligned} h_{[((\widetilde{F},A))\widetilde{\sqcup}f((\widetilde{G},B))](e_\prime)}(y) &= h_{f((\widetilde{F},A))(e_\prime)}(y)\vee h_{f((\widetilde{G},B))(e_\prime)}(y)\\ &=\left(\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{k\in p^{-1}(e')\cap A}\widetilde{F}_A(k)\right)(h(x))\right)\vee\left(\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{m\in p^{-1}(e')\cap B}\widetilde{G}_B(m)\right)\right)(h(x))\\ &=\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}\begin{cases}\widetilde{F}_A(n), & \text{if } n\in (A-B)\cap p^{-1}(e'),\\ \widetilde{G}_B(n), & \text{if } n\in (B-A)\cap p^{-1}(e'),\\ \widetilde{F}_A(n)\,\widetilde{\cup}\,\widetilde{G}_B(n), & \text{if } n\in A\cap B\cap p^{-1}(e'),\end{cases}\right)(h(x)). \end{aligned} \tag{12}$$

Hence, the proof is complete.

(4) Let $(\widetilde{F},A)\,\widetilde{\sqcap}\,(\widetilde{G},B)=(\widetilde{H},C)$, where $C=A\cap B$. For $e'\in p(E)\subseteq E'$, $y\in Y^*$, and by using Definition 19 and considering only the nontrivial case, we have

$$\begin{aligned} h_{[f((\widetilde{F},A)\widetilde{\sqcap}(\widetilde{G},B))](e_\prime)}(y) &= h_{f((\widetilde{H},C))(e_\prime)}(y)\\ &=\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e_\prime)\cap C}\widetilde{H}_C(n)\right)(h(x))\\ &=\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e_\prime)\cap A\cap B}\left(\widetilde{F}_A(n)\,\widetilde{\cap}\,\widetilde{G}_B(n)\right)\right)(h(x))\\ &\,\widetilde{\leq}\left(\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e_\prime)\cap A}\widetilde{F}_A(n)(h(x))\right)\right)\wedge\left(\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e_\prime)\cap B}\widetilde{G}_B(n)\right)(h(x))\right)\\ &= h_{f((\widetilde{F},A))(e_\prime)}(y)\wedge h_{f((\widetilde{G},B))(e_\prime)}(y). \end{aligned} \tag{13}$$

Hence, $f((\widetilde{F},A)\widetilde{\sqcap}(\widetilde{G},B))\widetilde{\sqsubseteq}f((\widetilde{F},A))\widetilde{\sqcap}f((\widetilde{G},B))$.

(5) If $(\widetilde{F},A)\,\widetilde{\sqsubseteq}\,(\widetilde{G},B)$ for nontrivial case in Definition 19, $e'\in p(E)\subseteq E'$, $y\in Y^*$, then

$$\begin{aligned} h_{f((\widetilde{F},A))(e_\prime)}(y) &= \bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e')\cap A}\widetilde{F}_A(n)\right)(h(x))\\ &=\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e')\cap A}\widetilde{F}_A(n)\right)(h(x))\\ &\,\widetilde{\leq}\bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e')\cap B}\widetilde{G}_B(n)\right)(h(x))\\ &= h_{f((\widetilde{G},B))(e')}(y), \quad \text{which completes the proof.} \end{aligned} \tag{14}$$

Remark 3.1. The inclusion in Theorem 3, parts 2 and 4, cannot be replaced by equality relation. Moreover, the converse of part 5 is not necessarily true as shown in the following example.

Example 6. Let $X = \{(2/a), (3/b), (4/c)\}$ and $Y = \{(3/x), (4/y), (2/z)\}$ be two msets, $E = \{e_1, e_2, e_3, e_4\}$ and $E\prime = \{e_1', e_2', e_3'\}$. Also, let $f = (u, p)\colon HF^2SM(\widetilde{X}_E) \longrightarrow HF^2SM(\widetilde{Y}_{E'})$ be a mapping such that $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E\prime$ be two mappings defined as $u(a) = x$, $u(b) = z$, and $u(c) = x$ and $p(e_1) = e_2'$, $p(e_2) = e_1'$, $p(e_3) = e_1'$, and $p(e_4) = e_1'$. Then,

(1) $f(\widetilde{1}_{\widetilde{X}_E}) = \{(e_1', \{< (3/x), \{1,1\} >, < (4/y), \{0,0\} >, < (2/z), \{1,1\} >\}), (e_1', \{< (3/x), \{1,1\} >, < (4/y), \{0,0\} >, < (2/z), \{1,1\} >\})\} \neq \widetilde{1}_{\widetilde{Y}_{E'}}$.

(2) Let $(\widetilde{F}, A) = \{(e_3, \{< (2/a), \{0.6, 0.7\} >, < (3/a), \{0.2, 0.4\} >, < (4/c), \{0.5, 0.8\} >\})\}$ and $(\widetilde{G}, B) = \{(e_3, \{\{< (2/a), \{0.2, 0.3\} >, < (3/b), \{0.1, 0.6\} >, < (4/c), \{0.6, 0.7\} >\})\}$ be two hesitant fuzzy soft msets in $HF^2SM(\widetilde{X}_E)$. Assume that $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = (\widetilde{H}, C)$, where $C = A \cap B$, then $(\widetilde{H}, C) = \{(e_3, \{< (2/a), \{0.2, 0.3\} >, < (3/b), \{0.1, 0.4\} >, < (4/c), \{0.5, 0.7\} >\})\}$. Therefore,

$$
\begin{aligned}
h_{f((\widetilde{H},C))(e_1')}(x) &= \bigvee_{q \in u^{-1}(x)} \left(\widetilde{\cup}_{n \in p^{-1}(e_1') \cap C} \widetilde{H}_C(n) \right)(h(q)) \\
&= \bigvee_{q \in \{a,c\}} \left(\widetilde{\cup}_{n \in \{e_2, e_3, e_4\} \cap C} \widetilde{H}_C(n) \right)(h(q)) \\
&= \{0.2, 0.3\} \vee \{0.5, 0.7\} \\
&= \{0.5, 0.7\}.
\end{aligned}
\tag{15}
$$

In similar way, $h_{f((\widetilde{H},C))(e_1')}(y) = \{0,0\}$ and $h_{f((\widetilde{H},C)))(e_1')}(z) = \{0.1, 0.4\}$. Also, $h_{f((\widetilde{H},C))(e_2')}(x) = h_{f((\widetilde{H},C))(e_2')}(y) = h_{f((\widetilde{H},C))(e_2')}(z) = \{0,0\}$ as $p^{-1}(e_2') \cap C = \phi$, but $f((\widetilde{F}, A)) = \{(e_1', \{< (3/x), \{0.6, 0.8\} >, < (4/y), \{0,0\} >, < (2/z), \{0.2, 0.4 >\}), (e_2', \{< (3/x), \{0,0\} >, < (4/y), \{0,0\} >, < (2/z), \{0,0\} >\})\}$, $f((\widetilde{G}, B)) = \{, (\{< (3/x), \{0.6, 0.7\} >, < (4/y), \{0,0\} >, < (2/z), \{0.1, 0.6\} >\}), (e_2', \{< (3/x), \{0,0\} >, < (4/y), \{0,0\} >, < (2/z), \{0,0\} >\})\}$. Hence, $f((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B)) \neq f((\widetilde{F}, A))\widetilde{\sqcap}\, f((\widetilde{G}, B))$.

(3) Let $(\widetilde{F}, A) = \{(e_3, \{< (2/a), \{0.6, 0.7\} >, < (3/b), \{0.2, 0.4\} >, < (4/c), \{0.5, 0.8\} >\})\}$ and $(\widetilde{G}, B) = \{(e_3, \{< (2/a), \{0.5, 0.6\} >, < (3/b), \{0.3, 0.5\} >, < (4/c), \{0.7, 0.9\} >\})\}$ be two hesitant fuzzy soft msets in $HF^2SM(\widetilde{X}_E)$. Then, $f((\widetilde{F}, A)) = \{(e_1', \{< (3/x), \{0.6, 0.8\} >, < (4/y), \{0,0\} >, < (2/z), \{0.2, 0.4\} >\}), (e\prime, \{< (3/x), \{0,0\} >, < (4/y), \{0,0\} >, < (2/z), \{0,0\} >\})\}$ and $f((\widetilde{G}, B)) = \{(e_1', \{< (3/x), \{0.7, 0.9\} >, < (4/y), \{0,0\} >, < (2/z), \{0.3, 0.5\} >\}), (e_1', \{< (3/x), \{0,0\} >, < (4/y), \{0,0\} >, < (2/z), \{0,0\} >\})\}$. Hence, $f((\widetilde{F}, A))\widetilde{\sqsubseteq}\, f((\widetilde{G}, B))$ but $(\widetilde{F}, A)\ \widetilde{\sqsubseteq}(\widetilde{G}, B)$.

Theorem 4. *Let $f = (u, p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E\prime})$ be a mapping such that $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E\prime$ be two mappings. If $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are two hesitant fuzzy soft msets in $HF^kSM(\widetilde{Y}_{E\prime})$ and $(\widetilde{F}_i, A_i)$ is a family of hesitant fuzzy soft msets in $HF^kSM(\widetilde{Y}_{E\prime})$, then*

(1) $f^{-1}(\widetilde{0}_{\widetilde{Y}_E}) = \widetilde{0}_{\widetilde{X}_E}$.

(2) $f^{-1}(\widetilde{1}_{\widetilde{Y}_{E'}}) = \widetilde{1}_{\widetilde{X}_E}$.

(3) $f^{-1}((\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B)) = f^{-1}((\widetilde{F}, A))\widetilde{\sqcup} f^{-1}((\widetilde{G}, B))$. In general, $f^{-1}(\widetilde{\sqcup}_i(\widetilde{F}_i, A_i)) = \widetilde{\sqcup}_i f^{-1}((\widetilde{F}_i, A_i))$.

(4) $f^{-1}((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B)) = f^{-1}((\widetilde{F}, A))\widetilde{\sqcap} f^{-1}((\widetilde{G}, B))$. In general, $f^{-1}(\widetilde{\sqcap}_i(\widetilde{F}_i, A_i)) = \widetilde{\sqcap}_i f^{-1}((\widetilde{F}_i, A))_i$.

(5) If $(\widetilde{F}, A)\widetilde{\sqsubseteq}(\widetilde{G}, B)$, then $f^{-1}((\widetilde{F}, A))\widetilde{\sqsubseteq} f^{-1}((\widetilde{G}, B))$.

Proof. The proof of parts 1and 2 is obvious.

(3) Let $(\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B) = (\widetilde{H}, C)$, where $C = A \cup B$. For $e \in p^{-1}(C) \subseteq E$, $x \in X^*$, and for nontrivial case, we have

$$
\begin{aligned}
h_{f^{-1}((\widetilde{H},C))(e)}(x) &= \widetilde{H}_C(p(e))(h(u(x))) \\
&= \begin{cases} \widetilde{F}_A(p(e))(h(u(x))), & \text{if } p(e) \in A - B, \\ \widetilde{G}_B(p(e))(h(u(x))), & \text{if } p(e) \in B - A, \\ \left(\widetilde{F}_A(p(e))\widetilde{\cup}\widetilde{G}_B(p(e))\right)(h(u(x))), & \text{if } p(e) \in A \cap B. \end{cases}
\end{aligned}
\tag{16}
$$

By using Definition 22, we have

$$\begin{aligned} h_{\left[f^{-1}((\widetilde{F},A))\widetilde{\sqcup} f^{-1}((\widetilde{G},B))\right](e)}(x) &= h_{f^{-1}((\widetilde{F},A))(e)}(x) \vee h_{f^{-1}((\widetilde{G},B))(e)}(x) \\ &= \widetilde{F}_A(p(e))(h(u(x)))) \vee \widetilde{G}_B(p(e))(h(u(x))) \\ &= \begin{cases} \widetilde{F}_A(p(e))(h(u(x))), & \text{if } p(e) \in A - B, \\ \widetilde{G}_B(p(e))(h(u(x))), & \text{if } p(e) \in B - A, \\ \left(\widetilde{F}_A(p(e)) \widetilde{\cup} \widetilde{G}_B(p(e)))(h(u(x)))\right), & \text{if } p(e) \in A \cap B. \end{cases} \end{aligned} \tag{17}$$

Hence, $f^{-1}((\widetilde{F},A)\widetilde{\sqcup}(\widetilde{G},B)) = f^{-1}((\widetilde{F},A))\widetilde{\sqcup} f^{-1}((\widetilde{G},B))$.

(4) Let $(\widetilde{F},A)\widetilde{\sqcap}(\widetilde{G},B) = (\widetilde{H},C)$, where $C = A\cap B$. For $e \in p^{-1}(C) \subseteq E$, $x \in X^*$, and for nontrivial case, we have

$$\begin{aligned} h_{f^{-1}((\widetilde{H},C))(e)}(x) &= \widetilde{H}_C(p(e))(h(u(x))) \\ &= \left(\widetilde{F}_A(p(e)) \widetilde{\cap} \widetilde{G}_B(p(e))\right)(h(u(x))) \\ &= (\widetilde{F}_A(p(e))) \widetilde{\cap} \widetilde{G}_B(p(e))(h(u(x))) \\ &= [(\widetilde{F}_A(p(e)))(h(u(x)))] \wedge \left[\widetilde{G}_B(p(e))(h(u(x)))\right] \\ &= h_{f^{-1}((\widetilde{F},A))(e)}(x) \wedge h_{f^{-1}((\widetilde{G},B))(e)}(x). \end{aligned} \tag{18}$$

Hence,
$f^{-1}((\widetilde{F},A)\widetilde{\sqcap}(\widetilde{G},B)) = f^{-1}((\widetilde{F},A))\widetilde{\sqcap} f^{-1}((\widetilde{G},B))$.

(5) If $(\widetilde{F},A)\widetilde{\sqsubseteq}(\widetilde{G},B)$, then for $p(e) \in A$, we have

$$\begin{aligned} h_{f^{-1}((\widetilde{F},A))(e)}(x) &= \widetilde{F}_A(p(e))(h(u(x))) \\ &\widetilde{\leq} \left(\widetilde{G}_A(p(e))\right)(h(u(x))) \\ &= h_{f^{-1}((\widetilde{G},A))(e)}(x), \quad p(e) \in A \\ &\widetilde{\leq} h_{f^{-1}((\widetilde{G},B))(e)}(x), \quad p(e) \in B. \end{aligned} \tag{19}$$

Hence, $f^{-1}((\widetilde{F},A))\widetilde{\sqsubseteq} f^{-1}((\widetilde{G},B))$.

Remark 3.2. The converse in Theorem 4 part 5 is not necessarily true as shown in the following example.

Example 7. Let $X = \{(1/a), (2/b), (3/c)\}$ and $Y = \{(4/x), (3/y), (2/z)\}$ be two msets, $E = \{e_1, e_2, e_3, e_4\}$ and $E\prime = \{e_1', e_2', e_3'\}$. Also, let $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E\prime$ be two mappings defined as $u(a) = y$, $u(b) = z$, and $u(c) = z$ and $p(e_1) = e_1'$, $p(e_2) = e_1'$, $p(e_3) = e_2'$, and $p(e_4) = e_2'$. Choose two hesitant fuzzy soft msets $(\widetilde{F},A)$ and $(\widetilde{G},B)$ in $HF^2SM(\widetilde{Y}_{E'})$ such as $((\widetilde{F},A) = \{(e_3', \{<(4/x), \{0.3, 0.5\}>, <(3/y), \{0.5, 1\}>, <(2/z), \{0.8, 0.9\}>\})\})$ and $(\widetilde{G},B) = \{(e_3', \{<(4/x), \{0.4, 0.7\}>, <(3/y), \{0, 0.2\}>, <(2/z), \{0.3, 1\}>\})\}$. Then, the inverse image of $(\widetilde{F},A)$ under $f = (u,p)\colon HF^2SM(\widetilde{X}_E) \longrightarrow HF^2SM(\widetilde{Y}_{E'})$ is obtained as $f^{-1}((\widetilde{F},A)) = \widetilde{0}_{\widetilde{X}_E}$. Also, $f^{-1}((\widetilde{G},B)) = \widetilde{0}_{\widetilde{X}_E}$. Hence, $f^{-1}((\widetilde{F},A)) = f^{-1}((\widetilde{G},B))$, but $(\widetilde{F},A)\widetilde{\sqsubseteq}(\widetilde{G},B)$.

Definition 23. Let $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E\prime$ be two mappings. An *HFSM* mapping $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ is called

(1) One-one (or injection) if p and u are one-one (or injection)
(2) Onto (or surjection) if p and u are onto (or surjection)
(3) Bijection if p and u are bijection

Theorem 5. *Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ and $g = (r,t)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be two HFSMmappings of dimension k. Then, f and g are equal if and only if $u = r$ and $p = t$.*

Proof. Immediate.

Definition 24. Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ and $g = (r,t)\colon HF^kSM(\widetilde{Y}_{E'}) \longrightarrow HF^kSM(\widetilde{Z}_{E''})$ be two *HFSM*mappings of dimension k. Their composition $g\circ f$is also a hesitant fuzzy soft multimapping with dimension k from $HF^kSM(\widetilde{X}_E)$ into $HF^kSM(\widetilde{Z}_{E''})$ such that, for every $(\widetilde{F},A)$ in $HF^kSM(\widetilde{X}_E)$,

$$(g\circ f)(\widetilde{F},A) = g(f((\widetilde{F},A))). \tag{20}$$

This composition is defined as, for $e'' \in t(E\prime) \subseteq E''$ and $z \in Z^*$,

$$\begin{aligned} h_{g(f((\widetilde{F},A)))(e'')}(z) &= \vee_{x\in u^{-1}(r^{-1}(z))} \widetilde{\cup}_{n\in t^{-1}(e'')\cap p(A)} \\ &\quad \cdot (f((\widetilde{F},A)))(n)(h(x)). \end{aligned} \tag{21}$$

Theorem 6. *Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ and $g = (r,t)\colon HF^kSM(\widetilde{Y}_{E'}) \longrightarrow HF^kSM(\widetilde{Z}_{E''})$ be two HFSMmappings of dimension k. Then,*

(1) $g\circ f$ is injection if f and g are injection or, equivalently, if $r\circ u$ and $t\circ p$ are injection
(2) $g\circ f$ is surjection if f and gare surjection or, equivalently, if $r\circ u$ and $t\circ p$ are surjection
(3) $g\circ f$ is bijection if f and g are bijection or, equivalently, if $r\circ u$ and $t\circ p$ are bijection

Proof

(1) Let $(\widetilde{F},A), (\widetilde{G},B) \widetilde{\in} HF^kSM(\widetilde{X}_E)$ such that $g\circ f((\widetilde{F},A)) = g\circ f((\widetilde{G},B))$. Therefore, $g(f$

$((\widetilde{F},A))) = g(f((\widetilde{G},B)))$. Since g is injection, then $f((\widetilde{F},A)) = f((\widetilde{G},B))$. Also, f is injection, so $(\widetilde{F},A) = (\widetilde{G},B)$. Hence, $g\circ f$ is injection.

(2) Let $(\widetilde{H},C)\widetilde{\in} HF^kSM(\widetilde{Z}_{E''})$, then there exists $(\widetilde{G},B)\widetilde{\in} HF^kSM(\widetilde{Y}_{E'})$ such that $g((\widetilde{G},B)) = (\widetilde{H},C)$ as g is surjection. Since f is also surjection, then there exists $(\widetilde{F},A)\widetilde{\in} HF^kSM(\widetilde{X}_E)$ such that $f((\widetilde{F},A)) = (\widetilde{G},B)$. Thus, $g(f((\widetilde{F},A)))) = g((\widetilde{G},B)) = (\widetilde{H},C)$ which completes the proof.

(3) Immediately by part 1 and 2.

Definition 25. A bijection hesitant fuzzy soft multimapping $f = (u,p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ is called invertable. Also, the inverse of f, denoted by f^{-1}, is defined as $f^{-1} = (u^{-1},p^{-1})$: $HF^kSM(\widetilde{Y}_{E\prime}) \longrightarrow HF^kSM(\widetilde{X}_E)$, for each $(\widetilde{F},A)$ in $HF^kSM(\widetilde{Y}_{E\prime})$, $f^{-1}((\widetilde{F},A))$ in $HF^kSM(\widetilde{X}_E)$.

Theorem 7. *Let $f = (u,p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ and $g = (r,t)$: $HF^kSM(\widetilde{Y}_{E'}) \longrightarrow HF^kSM(\widetilde{Z}_{E''})$ be two bijection HFSMmappings of dimension k. Then, $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.*

Proof. If f and $g$are bijection $HFSM$mappings with dimension k, then there exist f^{-1}: $HF^kSM(\widetilde{Y}_{E'}) \longrightarrow HF^kSM(\widetilde{X}_E)$ and g^{-1}: $HF^kSM(\widetilde{Z}_{E''}) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ defined as $f^{-1}((\widetilde{G},B)) = (\widetilde{F},A)$, whenever $f((\widetilde{F},A)) = (\widetilde{G},B)$, $(\widetilde{F},A)\widetilde{\in} HF^kSM(\widetilde{X}_E)$, $(\widetilde{G},B)\widetilde{\in} HF^kSM(\widetilde{Y}_{E\prime})$, and $g^{-1}((\widetilde{H},C)) = (\widetilde{G},B)$, whenever $g((\widetilde{G},B)) = (\widetilde{H},C)$ and $(\widetilde{H},C)\widetilde{\in} HF^kSM(\widetilde{Z}_{E''})$. Hence, $(g\circ f)((\widetilde{F},A)) = g(f((\widetilde{F},A))) = g((\widetilde{G},B)) = (\widetilde{H},C)$. Since f,g are bijection, then $g\circ f$ is also bijection. Therefore, $(g\circ f)^{-1}$exists such that $(g\circ f)^{-1}((\widetilde{H},C)) = (\widetilde{F},A)$. Also, $(f^{-1}\circ g^{-1})((\widetilde{H},C)) = f^{-1}(g^{-1}((\widetilde{H},C))) = f^{-1}((\widetilde{G},B)) = (\widetilde{F},A)$. Then, $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

Theorem 8. *Let $f = (u,p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ and $g = (r,t)$: $HF^kSM(\widetilde{Y}_{E\prime}) \longrightarrow HF^kSM(\widetilde{Z}_{E'})$ be two HFSMmappings of dimension k. Then,*

(1) $f(f^{-1}((\widetilde{F},A)))\widetilde{\sqsubseteq}(\widetilde{F},A)$, where $(\widetilde{F},A)\widetilde{\in} HF^kSM(\widetilde{Y}_{E\prime})$

(2) $(\widetilde{F},A)\widetilde{\sqsubseteq} f^{-1}(f((\widetilde{F},A)))$, where $(\widetilde{F},A)\widetilde{\in} HF^kSM(\widetilde{X}_E)$

Proof

(1) For $y\in Y^*$, $e\prime\in p(E)\subset E\prime$ and $(\widetilde{F},A)\widetilde{\in} HF^kSM(\widetilde{Y}_{E\prime})$, we have

$$
\begin{aligned}
h_{f\left(f^{-1}((\widetilde{F},A))(e\prime)\right)}(y) &= \bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e\prime)\cap p^{-1}(A)}\left(f^{-1}((\widetilde{F},A))(n)\right)(h(x))\right)\\
&= \bigvee_{x\in u^{-1}(y)}\left(\widetilde{\cup}_{n\in p^{-1}(e\prime)\cap p^{-1}(A)}\left(\left(f^{-1}((\widetilde{F},A))\right)(n)\right)\right)(h(x))\\
&= \bigvee_{x\in u^{-1}(y)}\ \bigvee_{n\in p^{-1}(e\prime)\cap p^{-1}(A)} f^{-1}((\widetilde{F},A))(n)(h(x))\\
&= \bigvee_{x\in u^{-1}(y)}\ \bigvee_{n\in p^{-1}(e\prime)\cap p^{-1}(A)} \widetilde{F}_A(p(n)(h(u(x))))\\
&= \bigvee_{n\in p^{-1}(e\prime)\cap p^{-1}(A)} \widetilde{F}_A((p(n))(h(y)))\\
&\widetilde{\leq} \bigvee_{n\in p^{-1}(e\prime)} \widetilde{F}_A(p(n))(h(y))\\
&= h_{(\widetilde{F},A)(e\prime)}(y).
\end{aligned}
\tag{22}
$$

Hence, $f(f^{-1}((\widetilde{F},A)))\widetilde{\sqsubseteq}(\widetilde{F},A)$.

(2) The proof is similar to that of part 1.

Remark 3.3. The inclusion in Theorem 9 parts 1 and 2 cannot be replaced by equality relation as shown in the following example.

Example 8

(1) From Example 7, $f(f^{-1}(\widetilde{F},A)) = \widetilde{0}_{\widetilde{Y}_{p(E)}} \neq (\widetilde{F},A)$.

(2) From Example 6 part 2, $f^{-1}(f(\widetilde{H},C)) = \{(e_1, \{<(2/a),\{0,0\}>, <(3/b), \{0,0\}>, <(4/c),\{0,0\}>\}), (e_{20},\{<(2/a),\{0.5,0.7\}>, <(3/b),\{0.1,0.4\}>, <(4/c),\{0.5,0.7\}>\}), (e_3, \{<(2/a),\{0.5,0.7\}>, <(3/b),\{0.1,0.4\}>, <(4/c), \{0.5,0.7\}>\}), (e_4, \{<(2/a), \{0.5,0.7\}>, <(3/b),\{0.1,0.4\}>, <(4/c),\{0.5,0.7\}>\})\}$ but $(\widetilde{H},C) = \{(e_3,\{<(2/a),\{0.2,0.3\}>, <(3/b), \{0.1,0.4\}>, <4/c, \{0.5,0.7\}>\})\}$. Hence, $f^{-1}(f(\widetilde{H},C)) \neq (\widetilde{H},C)$.

Corollary 1. *Let $f = (u,p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be an HFSM mapping of dimension k. Then,*

(1) $f(f^{-1}((\widetilde{F},A))) = (\widetilde{F},A)$, where $(\widetilde{F},A)\widetilde{\in} HF^kSM(\widetilde{Y}_{E'})$ if f is surjection

(2) $(\widetilde{F}, A) = f^{-1}(f((\widetilde{F}, A)))$, *where* $(\widetilde{F}, A) \widetilde{\in} HF^kSM(\widetilde{X}_E)$ *if* f *is injection*

Proof. Immediate by using Theorem 8.

Definition 26. A hesitant fuzzy soft multimapping f with dimension k, where $f = (u_o, p_o)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{X}_E)$ is said to be identity if u_o, p_o are identity mappings.

Theorem 9. *Let* $f = (u, p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E\prime})$ *and* $g = (r, t)$: $HF^kSM(\widetilde{Z}_{E''}) \longrightarrow HF^kSM(\widetilde{X}_E)$ *be two HFSMmappings of dimension k. Then, for the identity mapping* $i = (u_o, p_o)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{X}_E)$, *we have*

(1) $f \circ i = f$

(2) $i \circ g = g$

Proof. Immediate.

4. Continuous Mappings on Hesitant Fuzzy Soft Multispaces

The aim of this section is to introduce the concept of hesitant fuzzy soft multitopology. Therefore, some types of hesitant fuzzy soft multimapping are presented in detail such as continuity, open, closed, and homeomorphism. Also, their properties and results are obtained.

Definition 27. The subcollection $\widetilde{\tau}_E$ of members of $HF^kSM(X)_E$ is called a hesitant fuzzy soft multitopology of dimension $k$on $(\widetilde{X}, E)$, if the following conditions are satisfied:

(1) $\widetilde{0}_{\widetilde{X}_E}, \widetilde{1}_{\widetilde{X}_E} \widetilde{\in} \widetilde{\tau}_E$

(2) If $(\widetilde{A}, E), (\widetilde{B}, E) \widetilde{\in} \widetilde{\tau}_E$, then $(\widetilde{A}, E) \widetilde{\sqcap} (\widetilde{B}, E) \widetilde{\in} \widetilde{\tau}_E$

(3) If $(\widetilde{A}_i, E) \widetilde{\in} \widetilde{\tau}_E$, $i \in I$, then $\widetilde{\sqcup}_{i\in I} (\widetilde{A}_i, E) \widetilde{\in} \widetilde{\tau}_E$

The pair $(\widetilde{X}_E, \widetilde{\tau}_E)$ is called a hesitant fuzzy soft multitopological space. Each member of $\widetilde{\tau}_E$ is called an open hesitant fuzzy soft mset. Also, the complement of an open hesitant fuzzy soft mset is called closed. The family of all closed hesitant fuzzy soft msets is denoted by $\widetilde{\tau}_E^c$.

Definition 28. Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space. A subfamily $\widetilde{\beta}_E$ is called a hesitant fuzzy soft multibasis for $\widetilde{\tau}_E$ if every member of $\widetilde{\tau}_E$ can be written as arbitrary hesitant fuzzy soft multiunion of some elements of $\widetilde{\beta}_E$.

Definition 29. Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space and $(\widetilde{F}, A)$ be an HF^kSMset over $(\widetilde{X}, E)$. The closure of $(\widetilde{F}, A)$ is denoted by $\overline{(\widetilde{F}, A)}$ and defined as

$$\overline{(\widetilde{F}, A)} = \widetilde{\sqcap}\left\{(\widetilde{G}, B) : (\widetilde{G}, B) \widetilde{\in} \widetilde{\tau}_E^c, (\widetilde{F}, A) \widetilde{\sqsubseteq} (\widetilde{G}, B)\right\}. \quad (23)$$

Theorem 10. *Let* $(\widetilde{X}_E, \widetilde{\tau}_E)$ *be a hesitant fuzzy soft multitopological space and* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *be two* HF^kSM*sets over* $(\widetilde{X}, E)$, *then*

(1) $\overline{(\widetilde{F}, A)}$ *is the smallest closed hesitant fuzzy soft mset containing* $(\widetilde{F}, A)$

(2) $(\widetilde{F}, A)$ *is a closed hesitant fuzzy soft mset if and only if* $\overline{(\widetilde{F}, A)} = (\widetilde{F}, A)$

(3) $\overline{\overline{(\widetilde{F}, A)}} = \overline{(\widetilde{F}, A)}$

(4) If $(\widetilde{F}, A) \widetilde{\sqsubseteq} (\widetilde{G}, B)$, *then* $\overline{(\widetilde{F}, A)} \widetilde{\sqsubseteq} \overline{(\widetilde{G}, B)}$

Proof. The proof is omitted.

Definition 30. Let $(f = (u, p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'}))$ be a mapping such that $u: X^* \longrightarrow Y^*$ and $p: E \longrightarrow E\prime$ be two mappings. Let $\widetilde{\tau}_E$ and $\widetilde{\eta}_{E\prime}$ be two hesitant fuzzy soft multitopologies of dimension k over $\widetilde{X}_E$ and $\widetilde{Y}_{E\prime}$, respectively. A function f is said to be

(1) Continuous if $f^{-1}((\widetilde{G}, B)) \widetilde{\in} \widetilde{\tau}_E$ for all $(\widetilde{G}, B) \widetilde{\in} \widetilde{\eta}_{E'}$

(2) Open if $f((\widetilde{F}, A)) \widetilde{\in} \widetilde{\eta}_{E'}$ for all $(\widetilde{F}, A) \widetilde{\in} \widetilde{\tau}_E$

(3) Closed if $f((\widetilde{F}, A)) \widetilde{\in} \widetilde{\eta}_{E'}^c$ for all $(\widetilde{F}, A) \widetilde{\in} \widetilde{\tau}_E^c$

(4) Homeomorphism if it is bijection, continuous, and its inverse f^{-1} is also continuous

Theorem 11. *Let* $f = (u, p)$: $HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ *be a mapping such that* $u: X^* \longrightarrow Y^*$ *and* $p: E \longrightarrow E\prime$ *be two mappings. Let* $\widetilde{\tau}_E$ *and* $\widetilde{\eta}_{E\prime}$ *are two hesitant fuzzy soft multitopologies of dimension k over* $\widetilde{X}_E$ *and* $\widetilde{Y}_{E\prime}$ *respectively. Then, the following conditions are equivalent:*

(1) f *is continuous*

(2) $f^{-1}((\widetilde{G}, B)) \widetilde{\in} \widetilde{\tau}_E$ *for all* $(\widetilde{G}, B) \widetilde{\in} \widetilde{\beta}_{E'}$, *where* $\widetilde{\beta}_{E'}$ *is a base for* $\widetilde{\eta}_{E'}$

(3) $f^{-1}((\widetilde{G}, B))$ *is* $\widetilde{\tau}_E$*–closed for all* $\widetilde{\eta}_{E'}$*–closed* $(\widetilde{G}, B)$

(4) $f^{-1}((\widetilde{G}, B)^\circ) \widetilde{\sqsubseteq} (f^{-1}((\widetilde{G}, B)))^\circ$ *for all* $(\widetilde{G}, B) \widetilde{\in} HF^kSM(\widetilde{Y}_{E'})$

(5) $f(\overline{(\widetilde{F}, A)}) \widetilde{\sqsubseteq} \overline{f((\widetilde{F}, A))}$, *where* $(\widetilde{F}, A) \widetilde{\in} HF^kSM(\widetilde{X}_E)$

(6) $\overline{f^{-1}((\widetilde{G}, B))} \widetilde{\sqsubseteq} f^{-1}(\overline{(\widetilde{G}, B)})$, *where* $(\widetilde{G}, B) \widetilde{\in} HF^kSM(\widetilde{Y}_{E'})$.

Proof. The proof is omitted for parts 1, 2, and 3, and these statements are equivalent.

(1) Since $(\widetilde{G}, B)^\circ \widetilde{\sqsubseteq} (\widetilde{G}, B)$, by using Theorem 4, we get $f^{-1}((\widetilde{G}, B)^\circ) \widetilde{\sqsubseteq} f^{-1}((\widetilde{G}, B))$. Therefore, by using Theorem 1, $(f^{-1}((\widetilde{G}, B)^\circ))^\circ \widetilde{\sqsubseteq} (f^{-1}((\widetilde{G}, B)))^\circ$, but f is continuous, so $f^{-1}((\widetilde{G}, B)^\circ)$ is $\widetilde{\tau}_E$–open. Hence, $f^{-1}((\widetilde{G}, B)^\circ) \widetilde{\sqsubseteq} (f^{-1}((\widetilde{G}, B)))^\circ$.

(2) By using part 4, $f^{-1}(\overline{f((\widetilde{F}, A))})$ is $\widetilde{\tau}_E$–closed, but $(\widetilde{F}, A) \widetilde{\sqsubseteq} f^{-1}(\overline{f((\widetilde{F}, A))})$. Then, $\overline{(\widetilde{F}, A)} \widetilde{\sqsubseteq} f^{-1}(\overline{f((\widetilde{F}, A))})$. Now, by using Theorems 3 and 8, we have $f(\overline{(\widetilde{F}, A)})) \widetilde{\sqsubseteq} f((f^{-1}((\overline{f((\widetilde{F}, A))}))) \widetilde{\sqsubseteq} \overline{f((\widetilde{F}, A))}))$.

(3) Since $(\widetilde{G},B)\sqsubseteq\overline{(\widetilde{G},B)}$, by using Theorem 4, we get $f^{-1}((\widetilde{G},B))\sqsubseteq f^{-1}(\overline{(\widetilde{G},B)}) = \overline{f^{-1}((\widetilde{G},B))}$ as f is continuous. Therefore, $\overline{f^{-1}((\widetilde{G},B))}\sqsubseteq f^{-1}(\overline{(\widetilde{G},B)})$.

Remark 4. The inclusion in Theorem 11 parts 4, 5, and 6 cannot be replaced by equality relation as shown in the following example.

Example 9. From Example 4, let $(\widetilde{X}_E,\widetilde{\tau}_E) = \{\widetilde{0}_{\widetilde{X}_E},\widetilde{1}_{\widetilde{X}_E},(\widetilde{F},E)\}$ be a hesitant fuzzy soft multitopological space with dimension 3, where $(\widetilde{F},E) = \{(e_1,\widetilde{0}_{\widetilde{X}}),(e_2,\widetilde{0}_{\widetilde{X}}),(e_3,\{<(2/a),\{0,0.2,0.4\}>,<(3/b),\{0.2,0.3,0.4\}>,<(1/c),\{0.1,0.5,0.7\}>,<(4/d),\{0,0.2,0.4\}>\}),(e_4,\{<(2/a),\{0,0.2,0.4\}>,<(3/b),\{0.2,0.3,0.4\}>,<(1/c),\{0.1,0.5,0.7\}>,<(4/d),\{0,0.2,0.4\}>\})\}$, (for short, $\widetilde{0}_{\widetilde{X}} = \{<(2/a),\{0,0,0\}>,<(3/b),\{0,0,0\}>,<1/c,\{0,0,0\}>,<(4/d),\{0,0,0\}>\}$) and $(\widetilde{Y}_{E'},\widetilde{\eta}_{E'}) = \{\widetilde{0}_{\widetilde{Y}_{E'}},\widetilde{1}_{\widetilde{Y}_{E'}},(\widetilde{H},E')\}$ be a hesitant fuzzy soft multitopological space with dimension 3, where $(\widetilde{H},E') = \{(e_1',\{<(3/x),\{0.1,0.5,0.7\}>,<(1/y),\{0,0.2,0.4\}>,<(2/z),\{0.2,0.3,0.4\}>\}),(e_2',\{<(3/x),\{0.2,0.4,0.6\}>,<(1/y),\{0.1,0.2,0.3\}>,<(2/z),\{0,0,0.2\}>\}),(e_3',\widetilde{0}_{\widetilde{Y}})\}$, then

(1) Choose $(\widetilde{G},E') = \{(e_1',\widetilde{1}_{\widetilde{Y}}),(e_2',\{<(3/x),\{0.3,0.4,0.7\}>,<(1/y),\{0.1,0.3,0.5\}>,<(2/z),\{0.2,0.3,0.5\}>\}),((e_3',\widetilde{1}_{\widetilde{Y}})\}$ is an element in $HF^kS(\widetilde{Y}_{E'})$, therefore $(\widetilde{G},E')^{\circ} = (\widetilde{H},E')$. Then, $f^{-1}((\widetilde{G},E')^{\circ}) = (\widetilde{F},E)$. Now, we need to estimate $f^{-1}((\widetilde{G},E'))$; so,

$$\begin{aligned} h_{f^{-1}((\widetilde{G},E'))(e_1)}(a) &= \widetilde{G}_{E'}(p(e_1))(h(u(a))) \\ &= \widetilde{G}_{E'}(e_3')(h(y)) \\ &= \{1,1,1\}. \end{aligned} \tag{24}$$

By the similar way, we get $f^{-1}((\widetilde{G},E')) = \widetilde{1}_{\widetilde{X}_E}$. Hence, $(f^{-1}((\widetilde{G},E')))^{\circ} = \widetilde{1}_{\widetilde{X}_E}$. Then, $(f^{-1}((\widetilde{G},E')))^{\circ} \neq f^{-1}((\widetilde{G},E')^{\circ})$.

(2) Choose $(\widetilde{G},E) = \{(e_1,\{<(2/a),\{0.1,0.3,0.5\}>,<(3/b),\{0,0,0\}>,<1/c,\{0,0,0\}>,<4/d,\{0,0,0\}>\}),(e_2,\{<(2/a),\{0,0,0\}>,<(3/b),\{0,0,0\}>,<(1/c),\{0.2,0.3,0.4\}>,<(4/d),\{0.3,0.7,0.9\}>\}),(e_3,\widetilde{0}_{\widetilde{X}}),(e_4,\widetilde{0}_{\widetilde{X}})\}$ is an element in $HF^kSM(\widetilde{X}_E)$. Then, $\overline{(\widetilde{G},E)} = \{(e_1,\widetilde{1}_{\widetilde{X}}),(e_2,\widetilde{1}_{\widetilde{X}}),(e_3,\{<(2/a),\{0.6,0.8,1\}>,<(3/b),\{0.6,0.7,0.8\}>,<(1/c),\{0.3,0.5,0.9\}>,<(4/d),\{0.6,0.8,1\}>\}),(e_4,\{<(2/a),\{0.6,0.8,1\}>,<(3/b),\{0.6,0.7,0.8\}>,<(1/c),\{0.3,0.5,0.9\}>,<(4/d),\{0.6,0.8,1\}>\})\}$.

$$\begin{aligned} h_{f(\overline{(\widetilde{G},E)})}(e_1')(x) &= \bigvee_{q\in u^{-1}(x)}\left(\widetilde{\sqcup}_{n\in p^{-1}(e_1')\cap E}\overline{\widetilde{G}}_E(n)(h(x))\right) \\ &= \bigvee_{q\in\{c\}}\left(\widetilde{\sqcup}_{n\in\{e_3,e_4\}}\overline{\widetilde{G}}_E(n)\right)(h(x)) \\ &= \{0.3,0.5,0.9\}. \end{aligned} \tag{25}$$

By the similar way, we get $\overline{f((\widetilde{G},E))} = \{(e_1',\{<(3/x),\{0.3,0.5,0.9\}>,<(1/y),\{0.6,0.8,1\}>,<(2/z),\{0.6,0.7,0.8\}>\}),(e_3',\widetilde{1}_{\widetilde{Y}})\}$, but $(f(\overline{(\widetilde{G},E)})) = \{(e_3',\{<(3/x),\{0.3,0.5,0.9\}>,<(1/y),\{0.6,0.8,1\}>,<(2/z),\{0.6,0.7,0.8\}>\}),(e_2',\{<(3/x),\{0.4,0.6,0.8\}>,<(1/y),\{0.7,0.8,0.9\}>,<(2/z),\{0.8,1,1\}>\}),(e_3',\widetilde{1}_{\widetilde{Y}})\}$. Hence, $(f(\overline{(\widetilde{G},E)})) \neq \overline{f((\widetilde{G},E))}$. Also, one may extend an example for part 6in Theorem 11 by the same technique.

Theorem 12. *Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ and $g = (r,t)\colon HF^kSM(\widetilde{Y}_{E'}) \longrightarrow HF^kSM(\widetilde{Z}_{E''})$ be two HFSMmappings of dimension k and $\widetilde{\tau}_E,\widetilde{\eta}_{E'}$, and $\widetilde{\zeta}_{E'}$ be three topologies over $\widetilde{X}_E,\widetilde{Y}_{E'}$, and $\widetilde{Z}_{E'}$, respectively. If f,g are continuous, then $g\circ f$ is also continuous.*

Proof. Immediate.

Theorem 13. *Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be an HFSMmapping of dimension k and $\widetilde{\tau}_E$ and $\widetilde{\eta}_{E'}$ be two topologies over $\widetilde{X}_E$ and $\widetilde{Y}_{E'}$, respectively. If $f((\widetilde{F},A))\widetilde{\in}\widetilde{\eta}_{E'}$ for every $(\widetilde{F},A)\widetilde{\in}\widetilde{\beta}_E$, where $\widetilde{\beta}_E$ is a base for $\widetilde{\tau}_E$, then f is an open hesitant fuzzy soft multimapping.*

Proof. Let $(\widetilde{G},B)\widetilde{\in}\widetilde{\tau}_E$. Then, $\{(\widetilde{G},B) = \widetilde{\sqcup}\{(\widetilde{H},C)\colon (\widetilde{H},C)\widetilde{\in}\widetilde{\beta}_E\}\}$. Therefore, $f((\widetilde{G},B)) = f(\widetilde{\sqcup}\{(\widetilde{H},C)\colon (\widetilde{H},C)\widetilde{\in}\widetilde{\beta}_E\}) = \widetilde{\sqcup}\{f((\widetilde{H},C))\colon (\widetilde{H},C)\widetilde{\in}\widetilde{\beta}_E\}$. According to the given hypothesis, $f((\widetilde{H},C))\widetilde{\in}\widetilde{\eta}_{E'}$; hence, $f((\widetilde{G},B))\widetilde{\in}\widetilde{\eta}_{E'}$ which completes the proof.

Theorem 14. *Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be a mapping such that $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E'$ be two mappings. Let $\widetilde{\tau}_E$ and $\widetilde{\eta}_{E'}$ are two hesitant fuzzy soft multitopologies of dimension k over $\widetilde{X}_E$ and $\widetilde{Y}_{E'}$, respectively. If $(\widetilde{F},A)\widetilde{\in}HF^kSM(\widetilde{X}_E)$, then*

(1) f is open if and only if $f((\widetilde{F},A)^{\circ})\sqsubseteq(f((\widetilde{F},A)))^{\circ}$

(2) f is closed if and only if $\overline{f((\widetilde{F},A))}\sqsubseteq f(\overline{(\widetilde{F},A)})$

Proof

(1) Let f be an open hesitant fuzzy soft multimapping and since $(\widetilde{F},A)^{\circ}\sqsubseteq(\widetilde{F},A)$. Then, by using Theorem 3, $f((\widetilde{F},A)^{\circ})\sqsubseteq f((\widetilde{F},A))$. By taking the interior for both sides, $(f((\widetilde{F},A)^{\circ}))^{\circ}\sqsubseteq(f((\widetilde{F},A)))^{\circ}$ but f is open, then $(f((\widetilde{F},A)^{\circ}))^{\circ} = f((\widetilde{F},A)^{\circ})$. Hence, $f((\widetilde{F},A)^{\circ})\sqsubseteq(f((\widetilde{F},A)))^{\circ}$. Conversely, let $(\widetilde{G},B)\widetilde{\in}\widetilde{\tau}_E$ and by using the given hypothesis, we have $f((\widetilde{G},B)^{\circ})\sqsubseteq(f((\widetilde{G},B)))^{\circ}$. Then, $f((\widetilde{G},B))\sqsubseteq(f((\widetilde{G},B)))^{\circ}$, but we know that $(f((\widetilde{G},B)))^{\circ}\sqsubseteq f((\widetilde{G},B))$. Hence, f is an open hesitant fuzzy soft multimapping.

(2) By the similar way of part 1.

Theorem 15. *Let $f = (u,p)\colon HF^kSM(\widetilde{X}_E) \longrightarrow HF^kSM(\widetilde{Y}_{E'})$ be a mapping such that $u\colon X^* \longrightarrow Y^*$ and $p\colon E \longrightarrow E'$ be two mappings. Let $\widetilde{\tau}_E$ and $\widetilde{\eta}_{E'}$ are two hesitant fuzzy soft multitopologies of dimension k over $\widetilde{X}_E$ and $\{\widetilde{Y}_{E'}\}$, respectively. Then, the following conditions are equivalent:*

(1) *f is a homeomorphism hesitant fuzzy soft multimapping*

(2) *f is a bijection, open, and continuous hesitant fuzzy soft multimapping*

(3) *f is a bijection, closed, and continuous hesitant fuzzy soft multi mapping*

Proof. Straightforward.

5. Connectedness on Hesitant Fuzzy Soft Multitopological Spaces

The aim of this section is to introduce the concept of hesitant fuzzy soft multiconnected space and present their results and properties in detail. Moreover, the concept of hesitant fuzzy soft multi-subspace is introduced.

Definition 31. Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space with dimension k. A hesitant fuzzy soft multiseparation of $\widetilde{X}_E$ is a pair $(\widetilde{F}, E)$, $(\widetilde{G}, E)$of no-null open hesitant fuzzy soft msets over $\widetilde{X}_E$ such that

$$\begin{aligned}(\widetilde{F}, E) \widetilde{\sqcup} (\widetilde{G}, E) &= \widetilde{1}_{\widetilde{X}_E},\\ (\widetilde{F}, E) \widetilde{\sqcap} (\widetilde{G}, E) &= \widetilde{0}_{\widetilde{X}_E}.\end{aligned} \tag{26}$$

Definition 32. Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space with dimension k. It is said to be hesitant fuzzy soft multiconnected if there does not exist a hesitant fuzzy soft multiseparation of $\widetilde{X}_E$. Otherwise, $(\widetilde{X}_E, \widetilde{\tau}_E)$ is said to be a hesitant fuzzy soft multi-disconnected.

Example 10. Let $X = \{(2/a), (3/b)\}$ be an mset, $E = \{e_1, e_2\}$ be a set of parameters, and $\widetilde{\tau}_E = \{\widetilde{0}_{\widetilde{X}_E}, \widetilde{1}_{\widetilde{X}_E}, (\widetilde{F}, E), (\widetilde{G}, E)\}$ be a hesitant fuzzy soft multitopology with dimension 2 over $\widetilde{X}_E$, where $(\widetilde{F}, E) = \{(e_1, \{<(2/a), \{0.1, 0.3\}>, <(3/b), \{0.2, 0.5\}>\}), (e_2, \{<(2/a), \{0.2, 0.4\}>, <(3/b), \{0.3, 0.6\}>\})\}$, and $(\widetilde{G}, E) = \{(e_1, \{<(2/a), \{0.2, 0.4\}>, <(3/b), \{0.3, 0.7\}>\}), (e_2, \{<(2/a), \{0.4, 0.6\}>, <(3/b), \{0.4, 0.8\}>\})\}$. Since, $(\widetilde{F}, E) \widetilde{\sqcup} (\widetilde{G}, E) = (\widetilde{G}, E) \neq \widetilde{1}_{\widetilde{X}_E}$, and $(\widetilde{F}, E) \widetilde{\sqcap} (\widetilde{G}, E) = (\widetilde{F}, E) \neq \widetilde{0}_{\widetilde{X}_E}$, then a hesitant fuzzy soft multitopological space $(\widetilde{X}_E, \widetilde{\tau}_E)$ is connected.

Theorem 16. *Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space with dimension k. If the only hesitant fuzzy soft msets over $\widetilde{X}_E$ that are both open and closed in $(\widetilde{X}_E, \widetilde{\tau}_E)$ are $\widetilde{0}_{\widetilde{X}_E}$ and $\widetilde{1}_{\widetilde{X}_E}$, then a hesitant fuzzy soft multitopological space $(\widetilde{X}_E, \widetilde{\tau}_E)$is connected.*

Proof. Let $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$be a hesitant fuzzy soft multiseparation of $\widetilde{X}_E$. If $(\widetilde{F}, E) = \widetilde{1}_{\widetilde{X}_E}$, then $(\widetilde{G}, E) = \widetilde{0}_{\widetilde{X}_E}$ which is a contradiction. Hence, $(\widetilde{F}, E) \neq \widetilde{1}_{\widetilde{X}_E}$. Since $(\widetilde{F}, E) \widetilde{\sqcup} (\widetilde{G}, E) = \widetilde{1}_{\widetilde{X}_E}$ and $(\widetilde{F}, E) \widetilde{\sqcap} (\widetilde{G}, E) = \widetilde{0}_{\widetilde{X}_E}$, then $(\widetilde{F}, E) = (\widetilde{G}, E)^c$. Therefore, $(\widetilde{F}, E)$ is both open and closed hesitant fuzzy soft mset different from $\widetilde{0}_{\widetilde{X}_E}$ and $\widetilde{1}_{\widetilde{X}_E}$ which is a contradiction. Hence, a hesitant fuzzy soft multitopological space $(\widetilde{X}_E, \widetilde{\tau}_E)$is connected.

Example 11. By using Theorem 16, the hesitant fuzzy soft multi-indiscrete topological space $(\widetilde{X}_E, \widetilde{\tau}_E)$with dimension $k$is connected.

Remark 5. The converse of Theorem 16 is not necessarily true as shown in the following example.

Example 12. Let $X = \{(2/a), (3/b)\}$ be an mset, $E = \{e_1, e_2\}$ be a set of parameters, and $\widetilde{\tau}_E = \{\widetilde{0}_{\widetilde{X}_E}, \widetilde{1}_{\widetilde{X}_E}, (\widetilde{F}, E)\}$ be a hesitant fuzzy soft multitopology with dimension 2 over $\widetilde{X}_E$, where $(\widetilde{F}, E) = \{(e_1, \{<(2/a), \{0, 1\}>, <(3/b), \{0, 1\}>\}), (e_2, \{<(2/a), \{0, 1\}>, <3/b, \{0, 1\}>\})\}$. Then, a hesitant fuzzy soft multitopological space $(\widetilde{X}_E, \widetilde{\tau}_E)$ is connected, but there exists an open and closed hesitant fuzzy soft mset $(\widetilde{F}, E)$ different from $\widetilde{0}_{\widetilde{X}_E}$ and $\widetilde{1}_{\widetilde{X}_E}$.

Example 13. Let $X = \{(2/a), (3/b)\}$ be an mset, $E = \{e_1, e_2\}$ be a set of parameters, and $\widetilde{\tau}_E = \{\widetilde{0}_{\widetilde{X}_E}, \widetilde{1}_{\widetilde{X}_E}, (\widetilde{F}, E), (\widetilde{G}, E)\}$ be a hesitant fuzzy soft multitopology with dimension 2 over $\widetilde{X}_E$, where $(\widetilde{F}, E) = \{(e_1, \{<(2/a), \{1, 1\}>, <(3/b), \{0, 0\}>\}), (e_2, \{<(2/a), \{0, 0\}>, <(3/b), \{1, 1\}>\})\}$ and $(\widetilde{G}, E) = \{(e_1, \{<(2/a), \{0, 0\}>, <(3/b), \{1, 1\}>\}), (e_2, \{<(2/a), \{1, 1\}>, <(3/b), \{0, 0\}>\})\}$. Since, $(\widetilde{F}, E) \neq \widetilde{0}_{\widetilde{X}_E}$, $(\widetilde{G}, E) \neq \widetilde{0}_{\widetilde{X}_E}$ and $(\widetilde{F}, E) \widetilde{\sqcup} (\widetilde{G}, E) = \widetilde{1}_{\widetilde{X}_E}$, $(\widetilde{F}, E) \widetilde{\sqcap} (\widetilde{G}, E) = \widetilde{0}_{\widetilde{X}_E}$, a hesitant fuzzy soft multitopological space $(\widetilde{X}_E, \widetilde{\tau}_E)$ is disconnected.

Definition 33. Let $(\widetilde{X}_E, \widetilde{\tau}_E)$ be a hesitant fuzzy soft multitopological space with dimension k and $\widetilde{Y}_E$ be a nonempty hesitant fuzzy soft multi-subset of $\widetilde{X}_E$. The family $\widetilde{\tau}_{\widetilde{Y}_E} = \{(\widetilde{Y}, E) \widetilde{\sqcap} (\widetilde{F}, E)$: $(\widetilde{F}, E) \widetilde{\in} \widetilde{\tau}_E\}$ is said to be a hesitant fuzzy soft multitopology with dimension k on $\widetilde{Y}_E$, and $(\widetilde{Y}_E, \widetilde{\tau}_{\widetilde{Y}_E})$is called a hesitant fuzzy soft multi-subspace of $(\widetilde{X}_E, \widetilde{\tau}_E)$.

Theorem 17. *If the hesitant fuzzy soft msets with dimension k, $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$, form a hesitant fuzzy soft multiseparation of $\widetilde{X}_E$ and $(\widetilde{Y}_E, \widetilde{\tau}_{\widetilde{Y}_E})$ is a hesitant fuzzy soft multiconnected subspace of $(\widetilde{X}_E, \widetilde{\tau}_E)$, then $\widetilde{Y}_E$ lies entirely within either $(\widetilde{F}, E)$ or $(\widetilde{G}, E)$.*

Proof. Since $\widetilde{Y}_E \widetilde{\sqsubseteq} (\widetilde{F}, E) \widetilde{\sqcup} (\widetilde{G}, E) = \widetilde{1}_{\widetilde{X}_E}$, then $(\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{F}, E)) \widetilde{\sqcup} (\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{G}, E)) = \widetilde{Y}_E$, and $\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{F}, E)$, $\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{G}, E)$ are $\widetilde{\tau}_{\widetilde{Y}_E}$ −open. Suppose that $\widetilde{Y}_E$ does not lie entirely within neither $(\widetilde{F}, E)$ nor $(\widetilde{G}, E)$. Then, $\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{F}, E) \neq \widetilde{0}_{\widetilde{Y}_E}$ and $\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{G}, E) \neq \widetilde{0}_{\widetilde{Y}_E}$. Also, $(\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{F}, E)) \widetilde{\sqcap} (\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{G}, E)) = \widetilde{Y}_E \widetilde{\sqcap} ((\widetilde{F}, E) \widetilde{\sqcap} (\widetilde{G}, E)) = \widetilde{Y}_E \widetilde{\sqcap} \widetilde{0}_{\widetilde{X}_E} = \widetilde{0}_{\widetilde{Y}_E}$. Hence, $\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{F}, E)$ and $\widetilde{Y}_E \widetilde{\sqcap} (\widetilde{G}, E)$ are two hesitant fuzzy soft multiseparation of $\widetilde{Y}_E$, i.e., $\widetilde{Y}_E$ is disconnected which is a contradiction. Then, $\widetilde{Y}_E$ lies entirely within either $(\widetilde{F}, E)$ or $(\widetilde{G}, E)$.

6. Conclusions

The fuzzy set theory, which was originally introduced by Zadeh [1] in 1965, is one of the most efficient decision aid techniques providing the ability to deal with imprecise and vague information. Nonetheless, to cope with imperfect or imprecise information that two or more sources of vagueness appear simultaneously, the traditional fuzzy set shows some limitations. Hence, it has been extended into several different forms, such as the type 2fuzzy set, the type *n*fuzzy set, the interval-valued fuzzy set, and the fuzzy multisets. All these extensions are based on the same rationale that it is not clear to assign the membership degree of an element to a fixed set. Recently, the concept of hesitant fuzzy sets is introduced firstly in 2010 by Torra [23] which permits the membership to have a set of possible values and presents some of its basic operations in expressing uncertainty and vagueness. Torra and Narukawa [24] established the similarities and differences with the hesitant fuzzy sets and the previous generalization of fuzzy sets such as intuitionistic fuzzy sets, type 2 fuzzy sets, and type *n* fuzzy sets. Therefore, other authors [25, 26] introduced the concept of hesitant fuzzy soft sets, and they presented some of the applications in decision-making problems. In 2015, Dey and Pal [27] presented the concept of a hesitant multifuzzy soft topological space. In 2019, Kandil et al. [28] introduced some important and basic issues of hesitant fuzzy soft multisets and studied some of its structural properties such as the neighborhood hesitant fuzzy soft multisets, interior hesitant fuzzy soft multisets, hesitant fuzzy soft multitopological spaces, and hesitant fuzzy soft multi-basis. Finally, they showed how to apply the concept of hesitant fuzzy soft multisets in decision-making problems.

In this paper, we introduced some important and basic issues of hesitant fuzzy soft multisets. The main properties of the current branch are studied, and some operations of this type of sets are established. Also, the concept of hesitant fuzzy soft multitopological spaces is defined. It should be mentioned that the concept of hesitant fuzzy soft multisets is a generalization of the previous concepts such as hesitant fuzzy soft sets, hesitant fuzzy multisets, hesitant fuzzy sets, and fuzzy sets. The concept of mapping on hesitant fuzzy soft multisets is introduced, and some results for this type of mappings are presented. The notions of inverse image and identity mapping are introduced, and their basic properties are investigated in detail. Also, the types of mappings on hesitant fuzzy soft multisets are given, and their properties are established. Therefore, the composition of two hesitant fuzzy soft multi mapping with the same dimension is presented. Moreover, we introduce the concepts of hesitant fuzzy soft multitopologies and hesitant fuzzy soft multi-subspaces. Some types of hesitant fuzzy soft multimapping such as continuity, open, closed, and homeomorphism are presented in detail. Also, their properties and results are investigated. Finally, the concept of hesitant fuzzy soft multiconnected space is introduced. The future work in this approach is introducing the near continuous hesitant fuzzy soft multimappings. Also, we will investigate the concepts of locally connected, hyperconnected in hesitant fuzzy soft multispaces and their applications in real-life problems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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