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Research Article

P-Fuzzy Ideals and P-Fuzzy Filters in P-Algebras

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In this paper, we introduce the concept of p-fuzzy ideals and p-fuzzy filters in a p-algebra. We provide a set of equivalent conditions for a fuzzy ideal to be a p-fuzzy ideal and a p-algebra to be a Boolean algebra. It is proved that the class of p-fuzzy ideals forms a complete distributive lattice. Moreover, we show that there is an isomorphism between the class of p-fuzzy ideals and p-fuzzy filter.

1. Introduction

The concept of fuzzy sets was firstly introduced by Zadeh [1]. In 1971, Rosenfeld used the notion of a fuzzy subset of a set to introduce the concept of a fuzzy subgroup of a group [2]. Rosenfeld's paper inspired the development of fuzzy abstract algebra. Since then, several authors have developed interesting results on fuzzy theory (see [3–14]). In this paper, we introduce the concept of p-fuzzy ideals and p-fuzzy filters in p-algebra. We provide a set of equivalent conditions for a fuzzy ideal to be a p-fuzzy ideal and a p-algebra to be a Boolean algebra. Moreover, we prove that, for any fuzzy ideal of L, there is the smallest p-fuzzy ideal containing it. It is proved that the class of p-fuzzy ideals forms a complete distributive lattice. Moreover, we prove that the image and inverse image of a p-fuzzy ideal is a p-fuzzy ideal under a $*$-epimorphism mapping. Finally, we show that there is an isomorphism between the class of p-fuzzy ideals and p-fuzzy filters.

2. Preliminaries

In this section, we recall some definitions and basic results on p-algebra and fuzzy theory.

Definition 1 (see [15]). An algebra $L = (L; \wedge, \vee, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a p-algebra if the following conditions hold:

(1) $(L; \wedge, \vee, 0, 1)$ is a bounded lattice
(2) For all $a, b \in L$, $a \wedge b = 0 \Leftrightarrow a \wedge b^* = a$

Theorem 1 (see [15]). *For any two elements a, b of a p-algebra, we have the following:*

(1) $0^{**} = 0$
(2) $a \wedge a^* = 0$
(3) $a \le b \Rightarrow b^* \le a^*$
(4) $a \le a^{**}$
(5) $a^{***} = a^*$
(6) $(a \vee b)^* = a^* \wedge b^*$
(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

An element x of a p-algebra is called closed, if $x = x^{**}$.

Definition 2 (see [15]). A nonempty subset I of L is called an ideal of L if for any $x, y \in I$, $x \vee y \in I$ and $x \in I$, $y \in L$, $x \wedge y \in I$.

Definition 3 (see [15]). A nonempty subset F of L is called a filter of L if for any $x, y \in F$, $x \wedge y \in F$ and $x \in F$, $y \in L$, $x \vee y \in I$.

Definition 4 (see [16]). An ideal I of L is called a p-ideal if any $x \in I$, $x^{**} \in I$.

Definition 5 (see [16]). A filter F of L is called a p-filter if for any $x^{**} \in F$, $x \in F$.

Definition 6 (see [1]). Let X be any nonempty set. A mapping μ: $X \longrightarrow [0,1]$ is called a fuzzy subset of X.

The unit interval $[0,1]$ together with the operations min and max forms a complete lattice satisfying the infinite meet distributive law; i.e.,

$$\alpha\wedge\left(\bigvee_{\beta\in M}\beta\right) = \bigvee_{\beta\in M}(\alpha\wedge\beta), \tag{1}$$

for all $\alpha \in [0,1]$ and any $M\subseteq[0,1]$.

We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0,1]$, we have $\alpha\wedge\beta = \min\{\alpha,\beta\}$ and $\alpha\vee\beta = \max\{\alpha,\beta\}$.

The characteristics function of any set A is defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases} \tag{2}$$

Definition 7 (see [2]). Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu\cup\theta$ and $\mu\cap\theta$ of A as follows: for each $x \in A$, $(\mu\cup\theta)(x) = \mu(x)\vee\theta(x)$ and $(\mu\cap\theta)(x) = \mu(x)\wedge\theta(x)$.

Then, $\mu\cup\theta$ and $\mu\cap\theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i\colon i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\cup_{i\in I}\mu_i$ and the greatest lower bound $\cap_{i\in I}\mu_i$ of the μ_i's are given for each $x \in X$, $(\cup_{i\in I}\mu_i)(x) = \vee_{i\in I}\mu_i(x)$ and $(\cap_{i\in I}\mu_i)(x) = \wedge_{i\in I}\mu_i(x)$, respectively.

For each $t \in [0,1]$, the set

$$\mu_t = \{x \in A\colon \mu(x) \geq t\} \tag{3}$$

is called the level subset of μ at t [1].

Definition 8 (see [2]). Let f be a function from X into Y, μ be a fuzzy subset of X, and θ be a fuzzy subset of Y.

(1) The image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y defined by the following: for each $y \in Y$, $f(\mu)(y) = \begin{cases} \text{Sup}\{\mu(x)\colon x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise} \end{cases}$

(2) The preimage of θ under f, denoted by $f^{-1}(\theta)$, is a fuzzy subset of X defined by for each $x \in X$, $f^{-1}(\theta)(x) = \theta(f(x))$

Definition 9 (see [17]). A fuzzy subset μ of a bounded lattice L is called a fuzzy ideal of L, if for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(0) = 1$

(2) $\mu(x\vee y) \geq \mu(x)\wedge\mu(y)$

(3) $\mu(x\wedge y) \geq \mu(x)\vee\mu(y)$

Definition 10 (see [17]). A fuzzy subset μ of a bounded lattice L is called a fuzzy filter of L, if for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(1) = 1$

(2) $\mu(x\vee y) \geq \mu(x)\vee\mu(y)$

(3) $\mu(x\wedge y) \geq \mu(x)\wedge\mu(y)$

We define the binary operations "+" and "·" on the set of all fuzzy subsets of L as

$$\begin{aligned} (\mu+\theta)(x) &= \text{Sup}\{\mu(y)\wedge\theta(z)\colon y, z \in L,\ y\vee z = x\}, \\ (\mu\cdot\theta)(x) &= \text{Sup}\{\mu(y)\wedge\theta(z)\colon y, z \in L,\ y\wedge z = x\}. \end{aligned} \tag{4}$$

If μ and θ are fuzzy ideals of L, then $\mu\cdot\theta = \mu\wedge\theta = \mu\cap\theta$ and $\mu+\theta = \mu\vee\theta$ is a fuzzy ideal generated by $\mu\cup\theta$.

If μ and θ are fuzzy filters of L, then $\mu+\theta = \mu\wedge\theta$ (the pointwise infimum of μ and θ) and $\mu\cdot\theta = \mu\vee\theta$ (the supremum of μ and θ).

Theorem 2 (see [18]). *Let L be a lattice, $x \in L$ and $\alpha \in [0,1]$. Define a fuzzy subset α_x of L as*

$$\alpha_x(y) = \begin{cases} 1, & \text{if } y \leq x, \\ \alpha, & \text{if } y \nleq x, \end{cases} \tag{5}$$

which is a fuzzy ideal of L.

Remark 1 (see [18]). α_x is called the α-level principal fuzzy ideal corresponding to x.

Similarly, a fuzzy subset α^x of L defined as

$$\alpha^x(y) = \begin{cases} 1, & \text{if } x \leq y, \\ \alpha, & \text{if } x \nleq y, \end{cases} \tag{6}$$

is the α-level principal fuzzy filter corresponding to x.

Remark 2. Let L be a lattice with 0. Then, L is called 0-distributive if for any $a, b, c \in L$ with $a\wedge b = 0 = a\wedge c$ implying $a\wedge(b\vee c) = 0$.

Throughout the rest of this paper, L stands for the 0-distributive p-algebra unless otherwise mentioned.

3. P-Fuzzy Ideals

In this section, we study the concept of p-fuzzy ideals of a p-algebra. We provide a set of equivalent conditions for a fuzzy ideal to be a p-fuzzy ideal and a p-algebra to be a Boolean algebra. Moreover, we prove that, for any fuzzy ideal of L, there is the smallest p-fuzzy ideal containing it. It is proved that the class of p-fuzzy ideals forms a complete distributive lattice. Finally, we show that the image and inverse image of a p-fuzzy ideal is a p-fuzzy ideal under a $*$-epimorphism mapping.

Definition 11. A fuzzy ideal μ of L is called a p-fuzzy ideal if $\mu(x) = \mu(x^{**})$ for each $x \in L$.

Lemma 1. *A fuzzy ideal μ of L is a p-fuzzy ideal if and only if $\mu((a^*\wedge b^*)^*) \geq \mu(a)\wedge\mu(b)$ for all $a, b \in L$.*

Lemma 2. *For any $x \in L$, α_{x^*} is a p-fuzzy ideal.*

Proof. Let $a \in L$. If $\alpha_{x^*}(a) = 1$, then $a \leq x^*$ and $a^{**} \leq x^*$. Thus, $\alpha_{x^*}(a^{**}) = 1$. If $\alpha_{x^*}(a) = \alpha$, then $a \nleq x^*$ and $a^{**} \nleq x^*$. Thus, $\alpha_{x^*}(a^{**}) = \alpha$. Hence, α_{x^*} is a p-fuzzy ideal. □

Theorem 3. *α_x is a p-fuzzy ideal if and only if x is a closed element.*

Proof. Let α_x be a p-fuzzy ideal. Then, $\alpha_x(x^{**}) = 1$ and $x^{**} \leq x$. Since $x \leq x^{**}$, we have $x = x^{**}$. Thus, x is closed.

Conversely, suppose that x is a closed element. Let $a \in L$. If $\alpha_x(a) = 1$, then $a \leq x$ and $a^{**} \leq x$. Thus, $\alpha_x(a^{**}) = 1$. If $\alpha_x(a) = \alpha$, then $a \nleq x$ and $a^{**} \nleq x$. Thus, $\alpha_x(a^{**}) = \alpha$. Hence α_x is a p-fuzzy ideal. □

Corollary 1. *The following conditions on L are equivalent:*

(1) Every fuzzy ideal is a p-fuzzy ideal

(2) Every level principal fuzzy ideal is a p-fuzzy ideal

(3) L is Boolean algebra

Proof. The proofs of $1 \Rightarrow 2$ and $3 \Rightarrow 1$ are straightforward. To show that $2 \Rightarrow 3$, suppose that every level principal fuzzy ideal is a p-fuzzy ideal. Then, by the above theorem, every element of L is closed. For any $x, y \in L$, the supremum is given by $x \veebar y = (x^* \wedge y^*)^*$.

To show that L is distributive, it suffices to prove that

$$x \wedge (y \veebar z) \leq (x \wedge y) \veebar (x \wedge z), \quad \forall x, y, z \in L. \tag{7}$$

For this purpose, let $t = (x \wedge y) \vee (x \wedge z)$. Then, $x \wedge y \leq t = t^{**}$ gives $x \wedge y \wedge t^* = 0$ and $x \wedge t^* \leq y^*$. Similarly, $x \wedge t^* \leq z^*$ and therefore $x \wedge t^* \leq y^* \wedge z^* = (y^* \wedge z^*)^{**}$. It follows from this that $x \wedge t^* \wedge (y^* \wedge z^*)^* = 0$ and hence that $x \wedge (y \veebar z) = x \wedge (y^* \wedge z^*)^* \leq t^{**} = t = (x \wedge y) \veebar (x \wedge z)$.

To see that L is also complemented, observe that $1 = 0^*$ and $0 = 1^*$. Since every $x \in L$, we have $x \wedge x^* = 0$ and $x \vee x^* = (x^* \wedge x^{**})^* = 0^* = 1$. We see that the complement of x is x^*. Thus, L is complemented. Hence L is a Boolean algebra. □

Theorem 4. *A fuzzy subset μ of L is a p-fuzzy ideal if and only if every level subset of μ is a p-ideal of L.*

Corollary 2. *A nonempty subset I of L is a p-ideal if and only if χ_I is a p-fuzzy ideal.*

In the following result, we prove that, for any fuzzy ideal of L, there is the smallest p-fuzzy ideal containing it.

Theorem 5. *Let μ be a fuzzy ideal of L. Define*

$$p(\mu)(x) = \operatorname{Sup}\{\mu(a) \colon x \leq a^{**} \colon a \in L\}. \tag{8}$$

Then, $p(\mu)$ is the smallest p-fuzzy ideal containing μ and hence μ is a p-fuzzy ideal if and only if $\mu = p(\mu)$.

Proof. Let μ be a fuzzy ideal of L. Then clearly $p(\mu)$ is a fuzzy ideal of L. To prove $p(\mu)$ is a p-fuzzy ideal, let $x \in L$. Clearly $p(\mu)(x) \geq p(\mu)(x^{**})$. On the other hand, $p(\mu)(x) = \operatorname{Sup}\{\mu(a) \colon x \leq a^{**}\} \leq \operatorname{Sup}\{\mu(a) \colon x^{**} \leq a^{**}\} = p(\mu)(x^{**})$. Thus, $p(\mu)$ is a p-fuzzy ideal containing μ.

Now we proceed to show that $p(\mu)$ is the smallest p-fuzzy ideal containing μ. Let θ be a p-fuzzy ideal containing μ. Let $x, a \in L$ such that $x \leq a^{**}$. Then, $\theta(a^{**}) \leq \theta(x)$. Since θ is a p-fuzzy ideal and $\mu \subseteq \theta$, we get that $\theta(a) \leq \theta(x)$ and $\mu(a) \leq \theta(x)$. This shows that $\theta(x)$ is an upper bound of $\{\mu(a) \colon x \leq a^{**}, a \in L\}$. This implies $\operatorname{Sup}\{\mu(a) \colon x \leq a^{**}, a \in L\} \leq \theta(x)$. Thus, $p(\mu) \subseteq \theta$. So $p(\mu)$ is the smallest p-fuzzy ideal containing μ. □

Lemma 3. *If μ and θ are fuzzy ideals of L, then $\mu \subseteq \theta$ implies $p(\mu) \subseteq p(\theta)$.*

Lemma 4. *For any two fuzzy ideals μ and θ of L, $p(\mu \cap \theta) = p(\mu) \cap p(\theta)$.*

Proof. Let μ and θ be two fuzzy ideals of L. Clearly $p(\mu \cap \theta) \subseteq p(\mu) \cap p(\theta)$. To show the other inclusion, let $x \in L$. Then

$$\begin{aligned}
(p(\mu) \cap p(\theta))(x) &= \operatorname{Sup}\{\mu(a) \colon x \leq a^{**}, \quad a \in L\} \\
&\quad \wedge \operatorname{Sup}\{\theta(b) \colon x \leq b^{**}, b \in L\} \\
&\leq \operatorname{Sup}\{\mu(a \wedge b) \wedge \theta(a \wedge b) \colon x \leq a^{**} \wedge b^{**}\} \\
&\leq \operatorname{Sup}(\mu \cap \theta)(a \wedge b) \colon x \leq (a \wedge b)^{**} \\
&\leq \operatorname{Sup}\{(\mu \cap \theta)(c) \colon x \leq c^{**}, c \in L\} \\
&= p(\mu \cap \theta)(x).
\end{aligned} \tag{9}$$

Thus, $p(\mu \cap \theta) = p(\mu) \cap p(\theta)$. □

Lemma 5. *For any fuzzy ideal μ of L, the map $\mu \longrightarrow p(\mu)$ is a closure operator on $FI(L)$. That is,*

(1) $\mu \subseteq p(\mu)$

(2) $p(p(\mu)) = p(\mu)$

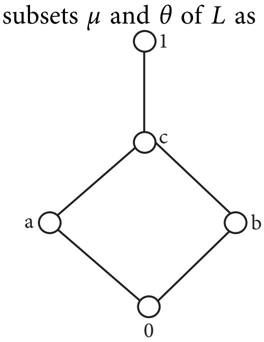
(3) $\mu \subseteq \theta \Rightarrow p(\mu) \subseteq p(\theta)$, for any two fuzzy ideals μ, θ of L

p-fuzzy ideals are simply the closed elements of $FI(L)$ with respect to the closure operator.

Every p-fuzzy ideal is a fuzzy ideal but the converse may not be true. For this, we have the following example.

Example 1. Consider the p-algebra $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.

Define fuzzy subsets μ and θ of L as follows:



$$\begin{aligned}&\mu(0)=1=\mu(a),\\&\mu(b)=\mu(c)=\mu(1)=0,\\&\theta(0)=\theta(b)=1,\\&\theta(a)=\theta(c)=\theta(1)=0.\end{aligned}\tag{10}$$

Then, it can be easily verified that μ and θ are p-fuzzy ideals of L. Moreover, we observe that the fuzzy ideal $\mu\vee\theta$ of L is not a p-fuzzy ideal.

The set of all p-fuzzy ideals of L is denoted by $\mathrm{FI}^p(L)$. We now prove that $\mathrm{FI}^p(L)$ is a lattice.

Theorem 6. *If $\lambda,\eta\in FI^p(L)$, then the supremum of λ and η is given by*

$$(\lambda\veebar\eta)(x)=\mathrm{Sup}\{\lambda(a)\wedge\eta(b)\colon x\leq(a^*\wedge b^*)^*,\ a,b\in L\}.\tag{11}$$

Proof. Put $\gamma=\lambda\veebar\eta$. Clearly $\gamma(0)=1$. For any $x,y\in L$,

$$\begin{aligned}\gamma(x)\wedge\gamma(y)&=\mathrm{Sup}\{\lambda(a_1)\wedge\eta(b_1)\colon x\leq(a_1^*\wedge b_1^*)^*,\ a_1,b_1\in L\}\\&\quad\wedge\mathrm{Sup}\{\lambda(a_2)\wedge\eta(b_2)\colon y\leq(a_2^*\wedge b_2^*)^*,\ a_2,b_2\in L\}\\&=\mathrm{Sup}\{\lambda(a_1)\wedge\lambda(a_2)\wedge\eta(b_1)\wedge\eta(b_2)\colon x\leq(a_1^*\wedge b_1^*)^*,\ y\leq(a_2^*\wedge b_2^*)^*\}\\&=\mathrm{Sup}\{\lambda(a_1\vee a_2)\wedge\eta(b_1\vee b_2)\colon x\leq(a_1^*\wedge b_1^*)^*,\ y\leq(a_2^*\wedge b_2^*)^*\}.\end{aligned}\tag{12}$$

If $x\leq(a_1^*\wedge b_1^*)^*$ and $y\leq(a_2^*\wedge b_2^*)^*$, then $x\vee y\leq((a_1\vee a_2)^*\wedge(b_1\vee b_2)^*)^*$. Thus

$$\begin{aligned}\gamma(x)\wedge\gamma(y)&\leq\mathrm{Sup}\left\{\lambda(a_1\vee a_2)\wedge\eta(b_1\vee b_2)\colon x\vee y\leq\left((a_1\vee a_2)^*\wedge(b_1\vee b_2)^*\right)^*\right\}\\&\leq\mathrm{Sup}\{\lambda(c_1)\wedge\eta(c_2)\colon x\vee y\leq(c_1^*\wedge c_2^*)^*\}\\&=\gamma(x\vee y).\end{aligned}\tag{13}$$

Thus, $\gamma(x\vee y)\geq\gamma(x)\wedge\gamma(y)$.

On the other hand,

$$\begin{aligned}\gamma(x)&\leq\mathrm{Sup}\{\lambda(a)\wedge\eta(b)\colon x\leq(a^*\wedge b^*)^*\}\\&\leq\mathrm{Sup}\{\lambda(a)\wedge\eta(b)\colon x\wedge y\leq(a^*\wedge b^*)^*\}\\&=\gamma(x\wedge y).\end{aligned}\tag{14}$$

Thus, $\gamma(x\wedge y)\geq\gamma(x)\vee\gamma(y)$.

To show γ is a p-fuzzy ideal, let $x\in L$. Then clearly $\gamma(x^{**})\leq\gamma(x)$.

$$\begin{aligned}\gamma(x)&=\mathrm{Sup}\{\lambda(a)\wedge\eta(b)\colon x\leq(a^*\wedge b^*)^*\}\\&\leq\mathrm{Sup}\{\lambda(a)\wedge\eta(b)\colon x^{**}\leq(a^*\wedge b^*)^*\}\\&=\gamma(x^{**}).\end{aligned}\tag{15}$$

Thus, γ is a p-fuzzy ideal of L.

Now we proceed to show that γ is the smallest p-fuzzy ideal containing λ and η. Clearly $\lambda\subseteq\gamma$ and $\eta\subseteq\gamma$. Let θ be any p-fuzzy ideal containing λ and η. For any $x\in L$,

$$\begin{aligned}\gamma(x)&=\mathrm{Sup}\{\lambda(a)\wedge\eta(b)\colon x\leq(a^*\wedge b^*)^*\}\\&\leq\mathrm{Sup}\{\theta(a)\wedge\theta(b)\colon x\leq(a^*\wedge b^*)^*\}\\&=\mathrm{Sup}\{\theta(a\vee b)\colon x\leq(a\vee b)^{**}\}\\&\leq\mathrm{Sup}\{\theta(c)\colon x\leq c^{**}\}\\&=p(\theta)(x).\end{aligned}\tag{16}$$

This shows that $p(\theta)=\theta$. Thus, $\gamma\subseteq\theta$. So γ is the smallest p-fuzzy ideal of L containing λ and η. □

Theorem 7. *The set $FI^p(L)$ of p-fuzzy ideal of L forms a complete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly $(\mathrm{FI}^p(L),\subseteq)$ is a partially ordered set. For $\lambda,\gamma\in\mathrm{FI}^p(L)$, the infimum and the supremum of λ and γ are $\lambda\wedge\gamma=\lambda\cap\gamma$ and $\lambda\veebar\gamma$, respectively. Thus, $(\mathrm{FI}^p(L),\wedge,\veebar)$ is a lattice.

To show the distributivity, it suffices to show $\lambda\cap(\gamma\veebar\eta)\subseteq(\lambda\cap\gamma)\veebar(\lambda\cap\eta)$ for all $\lambda,\gamma,\eta\mathrm{FI}^p(L)$. For any $x\in L$,

$$\begin{aligned}(\lambda \cap (\gamma \veebar \eta))(x) &= \mathrm{Sup}\{\lambda(x)\wedge(\gamma(a)\wedge\eta(b)) \colon x \le (a^{*}\wedge b^{*})^{*}, a, b \in L\}\\ &= \mathrm{Sup}\{(\lambda(x)\wedge\gamma(a))\wedge(\lambda(x)\wedge\eta(b)) \colon x \le (a^{*}\wedge b^{*})^{*}\}\\ &= \mathrm{Sup}\{(\lambda(x)\wedge\gamma(a^{**}))\wedge(\lambda(x)\wedge\eta(b^{**})) \colon x \le (a^{*}\wedge b^{*})^{*}\}\\ &\le \mathrm{Sup}\{(\lambda(x\wedge a^{**})\wedge\gamma(x\wedge a^{**}))\wedge(\lambda(x\wedge b^{**})\wedge\eta(x\wedge b^{**})) \colon x \le (a^{*}\wedge b^{*})^{*}\}\\ &= \mathrm{Sup}\{(\lambda\cap\gamma)(x\wedge a^{**})\wedge(\lambda\cap\eta)(x\wedge b^{**}) \colon x \le (a^{*}\wedge b^{*})^{*}\}.\end{aligned} \tag{17}$$

If $x \le (a^{*}\wedge b^{*})^{*}$, then $x\wedge a^{*}\wedge b^{*} = 0$. Since L is a p-algebra, $a\wedge b^{*} = a\wedge(a\wedge b)^{*}$ for all $a, b \in L$, and we get that $x \le ((x\wedge a^{**})^{*}\wedge(x\wedge b^{**})^{*})^{*}$. Thus

$$\begin{aligned}(\lambda \cap (\gamma \veebar \eta))(x) &\le \mathrm{Sup}\{(\lambda\cap\gamma)(x\wedge a^{**})\wedge(\lambda\cap\eta)(x\wedge b^{**}) \colon x \le ((x\wedge a^{**})^{*}\wedge(x\wedge b^{**})^{*})^{*}\}\\ &\le \mathrm{Sup}\{(\lambda\cap\gamma)(c)\wedge(\lambda\cap\eta)(d) \colon x \le (c^{*}\wedge d^{*})^{*},\ c, d \in L\}\\ &\le ((\lambda\cap\gamma)\veebar(\lambda\cap\eta))(x).\end{aligned} \tag{18}$$

Hence $\lambda \cap (\gamma\veebar\eta) = (\lambda\cap\gamma)\veebar(\lambda\cap\eta)$. So $\mathrm{FI}^{p}(L)$ is distributive.

Now we show the completeness. Since $(\mathrm{FI}^{p}, \subseteq)$ is a poset and χ_L and $\chi_{\{0\}}$ are the greatest and least elements of FI^{p}, respectively, then $\mathrm{FI}^{p}(L)$ is a complete distributive lattice. □

Theorem 8. *Let γ be a p-fuzzy ideal of L and η be a fuzzy filter of L such that $\gamma\cap\eta \le \beta$, $\alpha \in [0, 1)$. Then, there exists a prime p-fuzzy ideal θ of L such that $\gamma\subseteq\theta$ and $\theta\cap\eta \le \alpha$.*

Proof. Put $\Gamma = \{\lambda \in \mathrm{FI}^{p}(L) \colon \gamma\subseteq\lambda \text{ and } \lambda\cap\eta \le \beta\}$. Since $\gamma \in \Gamma$, Γ is nonempty and it forms a poset together with the inclusion ordering of fuzzy sets. Let $\mathscr{A} = \{\mu_i\}_{i\in I}$ be any chain in Γ. Then clearly $\cup_{i\in I}\mu_i$ is a p-fuzzy ideal. Since $\mu_i\cap\eta \le \alpha$ for each $i \in I$, then $(\cup_{i\in I}\mu_i)\cap\eta \le \alpha$. Thus, $\cup_{i\in I}\mu_i \in \Gamma$. By applying Zorn's lemma, we get a maximal element. Let us say $\theta \in \Gamma$; that is, θ is a p-fuzzy ideal of L such that $\gamma\subseteq\theta$ and $\theta\cap\eta \le \alpha$.

Now we proceed to show θ is a prime fuzzy ideal. Assume that θ is not prime fuzzy ideal. Let $\mu_1, \mu_2 \in \mathrm{FI}(L)$ and $\mu_1\cap\mu_2\subseteq\theta$ such that $\mu_1\nsubseteq\theta$ and $\mu_2\nsubseteq\theta$. If we put $\theta_1 = p(\mu_1\vee\theta)$ and $\theta_2 = p(\mu_2\vee\theta)$, then both θ_1 and θ_2 are p-fuzzy ideals of L properly containing θ. Since θ is maximal in Γ, we get $\theta_1 \notin \Gamma$ and $\theta_2 \notin \Gamma$. Thus, $\theta_1\cap\eta\nleq\alpha$ and $\theta_2\cap\eta\nleq\alpha$. This implies there exist $x, y \in L$ such that $(\theta_1\cap\eta)(x) > \alpha$ and $(\theta_2\cap\eta)(y) > \alpha$. So $((\theta_1\cap\theta_2)\cap\eta)(x\wedge y) > \alpha$.

$$\begin{aligned}&\Rightarrow p(\theta\vee(\mu_1\wedge\mu_2))(x\wedge y)\wedge\eta(x\wedge y) > \alpha\\ &\Rightarrow p(\theta)(x\wedge y)\wedge\eta(x\wedge y) > \alpha\\ &\Rightarrow \theta(x\wedge y)\wedge\eta(x\wedge y) > \alpha\\ &\Rightarrow (\theta\cap\eta)(x\wedge y) > \alpha.\end{aligned} \tag{19}$$

This is a contradiction. Hence θ is prime p-fuzzy ideal of L.

Let L and M be p-algebras. Then, a lattice morphism $f \colon L \longrightarrow M$ is said to be a $*$-morphism if $f(x^{*}) = f(x)^{*}$ for all $x, y \in L$. □

Theorem 9. *Let $f \colon L \longrightarrow M$ be a $*$-epimorphism. If μ is a p-fuzzy ideal of L, then $f(\mu)$ is a p-fuzzy ideal of M.*

Proof. Let μ be a p-fuzzy ideal of L. Then, $f(\mu)$ is a fuzzy ideal of M. To show $f(\mu)$ is a p-fuzzy ideal, let $y \in M$. Since $y \le y^{**}$, we have $f(\mu)(y^{**}) \le f(\mu)(y)$. On the other hand,

$$f(\mu)(y) = \mathrm{Sup}\{\mu(x) \colon x \in f^{-1}(y)\}. \tag{20}$$

Since $x \in f^{-1}(y)$ and f is a $*$-morphism, we have $x^{**} \in f^{-1}(y^{**})$. Thus

$$\begin{aligned}f(\mu)(y) &\le \mathrm{Sup}\{\mu(x^{**}) \colon x^{**} \in f^{-1}(y^{**})\}\\ &\le \mathrm{Sup}\{\mu(a) \colon a \in f^{-1}(y^{**})\}\\ &= f(\mu)(y^{**}).\end{aligned} \tag{21}$$

Thus, $f(\mu)(y) \le f(\mu)(y^{**})$. So $f(\mu)$ is a p-fuzzy ideal of M. □

Theorem 10. *Let $f \colon L \longrightarrow M$ be a $*$-epimorphism. If θ is a p-fuzzy ideal of M, then $f^{-1}(\theta)$ is a p-fuzzy ideal of L.*

Theorem 11. *Let $f \colon L \longrightarrow M$ be a $*$-epimorphism. Then, the map $g \colon FI^{p} \longrightarrow FM^{p}$ defined by $\mu\mapsto f(\mu)$ is a lattice epimorphism.*

Proof. Let $\mu, \theta \in \mathrm{FI}^{p}(L)$. Then, $\mu\cap\theta$ and $\mu\veebar\theta$ are p-fuzzy ideals of L. Thus, by Theorem 9, $f(\mu\cap\theta)$ and $f(\mu\veebar\theta)$ are p-fuzzy ideals of M. Clearly $f(\mu\cap\theta)\subseteq f(\theta)\cap f(\mu)$. For any $y \in M$,

$$(f(\mu)\cap f(\theta))(y) = \operatorname{Sup}\left\{\mu(a)\colon a \in f^{-1}(y),\ a \in L\right\}\wedge\operatorname{Sup}\left\{\theta(b)\colon b \in f^{-1}(y),\ b \in L\right\}. \tag{22}$$

If $a \in f^{-1}(y)$ and $b \in f^{-1}(y)$, then $a\wedge b \in f^{-1}(y)$. Thus,

$$\begin{aligned}(f(\mu)\cap f(\theta))(y) &\leq \operatorname{Sup}\left\{\mu(a\wedge b)\wedge\theta(a\wedge b)\colon a\wedge b \in f^{-1}(y)\right\}\\ &= \operatorname{Sup}\left\{(\mu\cap\theta)(a\wedge b)\colon a\wedge b \in f^{-1}(y)\right\}\\ &\leq \operatorname{Sup}\left\{(\mu\cap\theta)(c)\colon c \in f^{-1}(y)\right\}\\ &= f(\mu\cap\theta)(y).\end{aligned} \tag{23}$$

So $f(\mu)\cap f(\theta) = f(\mu\cap\theta)$.

Again clearly, $f(\mu)\veebar f(\theta)\subseteq f(\mu\veebar\theta)$. For any $x \in M$,

$$\begin{aligned}(f(\mu)\veebar f(\theta))(x) &= \operatorname{Sup}\{f(\mu)(x_1)\wedge f(\theta)(x_2)\colon x \leq (x_1^*\wedge x_2^*)^*\}\\ &= \operatorname{Sup}\left\{\operatorname{Sup}\left\{\mu(b_1)\colon b_1 \in f^{-1}(x_1)\right\}\wedge\operatorname{Sup}\left\{\mu(b_2)\colon b_2 \in f^{-1}(x_2)\right\}x \leq (x_1^*\wedge x_2^*)^*\right\},\\ f(\mu\veebar\theta)(x) &= \operatorname{Sup}\left\{(\mu\veebar\theta)(a)\colon a \in f^{-1}(x),\ a \in L\right\}\\ &= \operatorname{Sup}\left\{\operatorname{Sup}\{\mu(a_1)\wedge\theta(a_2)\colon a \leq (a_1^*\wedge a_2^*)^*\}\colon a \in f^{-1}(x)\right\}.\end{aligned} \tag{24}$$

If $f(a) = x$ and $a \leq (a_1^*\wedge a_2^*)^*$, then $x = f(a) \leq (f(a_1)^*\wedge f(a_2)^*)^*$. Put $f(a_1) = y_1$ and $f(a_2) = y_2$. Then, $a_1 \in f^{-1}(y_1)$, $a_2 \in f^{-1}(y_2)$ and $x \leq (y_1^*\wedge y_2^*)^*$. Based on this fact, we have

$$\begin{aligned}f(\mu\veebar\theta)(x) &\leq \operatorname{Sup}\left\{\operatorname{Sup}\left\{\mu(a_1)\colon a_1 \in f^{-1}(y_1)\right\}\wedge\operatorname{Sup}\left\{\theta(a_2)\colon a_2 \in f^{-1}(y_2)\right\}\colon \leq (y_1^*\wedge y_2^*)^*\right\}\\ &= \operatorname{Sup}\{f(\mu)(y_1)\wedge f(\theta)(y_2)\colon x \leq (y_1^*\wedge y_2^*)^*\}\\ &= (f(\mu)\veebar f(\theta))(x).\end{aligned} \tag{25}$$

Thus, $f(\mu\veebar\theta) = f(\mu)\veebar f(\theta)$. So g is a homomorphism. Now we proceed to show g is an epimorphism. Let $\mu \in \mathrm{FI}^p(M)$. Then, by Theorem 10, $f^{-1}(\mu) \in \mathrm{FI}^p(L)$. Since f is onto, we have $f(f^{-1}(\mu)) = \mu$. Thus, g is onto. So g is a lattice epimorphism. ☐

4. P-Fuzzy Filters

In this section, we study the concept of p-fuzzy filter of a p-algebra. We prove that the class of p-fuzzy filters forms a complete distributive lattice.

Definition 12. A fuzzy filter μ of L is called a p-fuzzy filter of L if $\mu(x) = \mu(x^{**})$ for any $x \in L$.

Theorem 12. *A fuzzy subset μ of L is a p-fuzzy filter if and only if every level subset of μ is a p-filter of L.*

Corollary 3. *A nonempty subset I of L is a p-filter if and only if χ_I is a p-fuzzy filter.*

The set of all dense elements of L is denoted by D.

Corollary 4. *χ_D is the smallest p-fuzzy filter of L.*

Proof. Since the set of all dense elements of L is a p-filter of L, by the above corollary χ_D is a p-fuzzy filter of L. To show χ_D is the smallest p-fuzzy filter, let μ be a p-fuzzy filter of L. Then clearly $\chi_D(x) \leq \mu(x)$, $\forall x \notin D$. If $x \in D$, then $x^{**} \in D$. Since μ is a p-fuzzy filter, we get that $\mu(x) = \mu(x^{**}) = 1$. This implies $\chi_D\subseteq\mu$. Thus, χ_D is the smallest p-fuzzy filter. ☐

Corollary 5. *$\chi_{[1)}$ is p-fuzzy filter of L if and only if $\chi_D = \chi_{[1)}$.*

Proof. Suppose $\chi_{[1)}$ is p-fuzzy filter of L. Then, by the above corollary $\chi_{[1)} = \chi_D$. Conversely, let $\chi_D = \chi_{[1)}$. Since χ_D is a p-fuzzy filer, we get that $\chi_{[1)}$ is a p-fuzzy filter of L. ☐

Theorem 13. *A proper fuzzy filter μ of L is a p-fuzzy filter if and only if for any $x, y \in L$ such that $x^* = y^*$ and $\mu(x) = \mu(y)$.*

The set of all p-fuzzy filters of a p-algebra L is denoted by $\mathrm{FF}^p(L)$. The following result shows that $\mathrm{FF}^p(L)$ forms a lattice.

Theorem 14. *If $\lambda, \eta \in FF^p(L)$, then the supremum of λ and η is given by*

$$(\lambda \veebar \eta)(x) = \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : x \leq (a \wedge b)^*,\ a, b \in L\}. \quad (26)$$

Proof. Put $\gamma = \lambda \veebar \eta$. Clearly $\gamma(1) = 1$. For any $x, y \in L$,

$$\begin{aligned}\gamma(x) \wedge \gamma(y) &= \operatorname{Sup}\{\lambda(a_1) \wedge \eta(b_1) : x^* \leq (a_1 \wedge b_1)^*,\ a_1, b_1 \in L\} \wedge \operatorname{Sup}\{\lambda(a_2) \wedge \eta(b_2) : y^* \leq (a_2 \wedge b_2)^*,\ a_2, b_2 \in L\} \\ &= \operatorname{Sup}\{\lambda(a_1) \wedge \lambda(a_2) \wedge \eta(b_1) \wedge \eta(b_2) : x^* \leq (a_1 \wedge b_1)^*, y^* \leq (a_2 \wedge b_2)^*\} \\ &= \operatorname{Sup}\{\lambda(a_1 \wedge a_2) \wedge \eta(b_1 \wedge b_2) : x^* \leq (a_1 \wedge b_1)^*, y^* \leq (a_2 \wedge b_2)^*\}.\end{aligned} \quad (27)$$

If $x^* \leq (a_1 \wedge b_1)^*$ and $y^* \leq (a_2 \wedge b_2)^*$, then $(x \wedge y)^* \leq ((a_1 \wedge a_2) \wedge (b_1 \wedge b_2))^*$. Thus

$$\gamma(x) \wedge \gamma(y) \leq \operatorname{Sup}\{\lambda(a_1 \wedge a_2) \wedge \eta(b_1 \wedge b_2) : (x \wedge y)^* \leq ((a_1 \wedge a_2) \wedge (b_1 \wedge b_2))^*\} \leq \operatorname{Sup}\{\lambda(c_1) \wedge \eta(c_2) : (x \wedge y)^* \leq (c_1 \wedge c_2)^*\} = \gamma(x \wedge y). \quad (28)$$

Thus, $\gamma(x \wedge y) \geq \gamma(x) \wedge \eta(y)$.

On the other hand,

$$\begin{aligned}\gamma(x) &\leq \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : x^* \leq (a \wedge b)^*\} \\ &\leq \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : x^* \wedge y^* \leq (a \wedge b)^*\} \\ &= \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : (x \vee y)^* \leq (a \wedge b)^*\} \\ &= \gamma(x \vee y).\end{aligned} \quad (29)$$

Thus, $\gamma(x \vee y) \geq \gamma(x) \vee \gamma(y)$.

To show γ is a p-fuzzy filter of L, let $x \in L$.

$$\begin{aligned}\gamma(x^{**}) &= \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : x^{***} \leq (a \wedge b)^*\} \\ &= \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : x^* \leq (a \wedge b)^*\} \\ &= \gamma(x).\end{aligned} \quad (30)$$

Thus, γ is a p-fuzzy filter of L. We now show that γ is the smallest p-fuzzy filter containing λ and η. Let θ be a p-fuzzy filter containing λ and η. For any $x \in L$,

$$\begin{aligned}\gamma(x) &\leq \operatorname{Sup}\{\lambda(a) \wedge \eta(b) : x^* \leq (a \wedge b)^*\} \\ &\leq \operatorname{Sup}\{\theta(a) \wedge \theta(b) : x^* \leq (a \wedge b)^*\} \\ &= (\theta \veebar \theta)(x) \\ &= \theta(x).\end{aligned} \quad (31)$$

Thus, γ is the smallest p-fuzzy filter containing λ and η. □

Theorem 15. *The set $FF^p(L)$ of p-fuzzy filter of L forms a complete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly $(\mathrm{FF}^p(L), \subseteq)$ is a partially ordered set. For $\lambda, \gamma \in \mathrm{FF}^p(L)$, the infimum and the supremum of λ and γ are $\lambda \wedge \gamma = \lambda \cap \gamma$ and $\lambda \veebar \gamma$, respectively. Thus, $(\mathrm{FF}^p(L), \wedge, \veebar)$ is a lattice.

To show the distributivity, it suffices to show $\lambda \cap (\gamma \veebar \eta) \subseteq (\lambda \cap \gamma) \veebar (\lambda \cap \eta)$ for all $\lambda, \gamma, \eta \mathrm{FI}^p(L)$. For any $x \in L$,

$$\begin{aligned}(\lambda \cap (\gamma \veebar \eta))(x) &= \operatorname{Sup}\{\lambda(x) \wedge (\gamma(a) \wedge \eta(b)) : x^* \leq (a \wedge b)^*, a, b \in L\} \\ &= \operatorname{Sup}\{(\lambda(x) \wedge \gamma(a)) \wedge (\lambda(x) \wedge \eta(b)) : x^* \leq (a \wedge b)^*\} \\ &\leq \operatorname{Sup}\{(\lambda(x \vee a) \wedge \gamma(x \vee a)) \wedge (\lambda(x \vee b) \wedge \eta(x \vee b)) : x^* \leq (a \wedge b)^*\} \\ &= \operatorname{Sup}\{(\lambda((x \vee a)^{**}) \wedge \gamma((x \vee a)^{**})) \wedge (\lambda((x \vee b)^{**}) \wedge \eta((x \vee b)^{**})) : x^* \leq (a \wedge b)^*\} \\ &= \operatorname{Sup}\{(\lambda \cap \gamma)((x \vee a)^{**}) \wedge (\lambda \cap \eta)((x \vee b)^{**}) : x^* \leq (a \wedge b)^*\}.\end{aligned} \quad (32)$$

If $x^* \leq (a \wedge b)^*$, then $x^* \wedge (a \wedge b)^{**} = 0$. We can easily verify that $x^* \leq ((x \vee a)^{**} \wedge (x \vee b)^{**})^*$. Thus,

$$\begin{aligned}(\lambda\cap(\gamma\veebar\eta))(x) &\leq \mathrm{Sup}\{(\lambda\cap\gamma)((x\vee a)^{**})\wedge(\lambda\cap\eta)((x\vee b)^{**})\colon x^{*}\leq((x\vee a)^{**}\wedge(x\vee b)^{**})^{*}\}\\ &\leq \mathrm{Sup}\{(\lambda\cap\gamma)(c)\wedge(\lambda\cap\eta)(d)\colon x^{*}\leq(c\wedge d)^{*},\ c,d\in L\}\\ &\leq((\lambda\cap\gamma)\veebar(\lambda\cap\eta))(x).\end{aligned} \tag{33}$$

Thus, $\lambda\cap(\gamma\veebar\eta)=(\lambda\cap\gamma)\veebar(\lambda\cap\eta)$. So $\mathrm{FF}^{p}(L)$ is distributive. □

5. Relation between $\mathbf{FI}^{p}(L)$ and $\mathbf{FF}^{p}(L)$

In this section, we show that there is an isomorphism between the class of p-fuzzy ideals and p-fuzzy filters.

Lemma 6. *Let μ be a p-fuzzy ideal. Define*

$$\mu^{\dagger}(x)=\mu(x^{*}). \tag{34}$$

Then, $\mu^{\dagger}$ is a p-fuzzy filter.

Proof. Let μ be a p-fuzzy ideal of L. Since $1^{*}=0$, we get that $\mu^{\dagger}(1)=1$. For any $x,y\in L$, $\mu^{\dagger}(x)\wedge\mu^{\dagger}(y)=\mu(x^{*})\wedge\mu(y^{*})=\mu(x^{*}\vee y^{*})=\mu((x^{*}\vee y^{*})^{**})=\mu((x\wedge y)^{*})=\mu^{\dagger}(x\wedge y)$. Thus, $\mu^{\dagger}$ is a fuzzy filter. To show $\mu^{\dagger}$ is a p-fuzzy ideal, let $x\in L$. Then, $\mu^{\dagger}(x^{**})=\mu(x^{*})=\mu^{\dagger}(x)$. Hence $\mu^{\dagger}$ is a p-fuzzy filter of L. □

Lemma 7. *Let θ be a fuzzy filter. Define*

$$\theta^{\ddagger}(x)=\theta(x^{*}). \tag{35}$$

Then, $\theta^{\ddagger}$ is a p-fuzzy ideal.

Proof. Let θ be a fuzzy filter. Since $1=0^{*}$, we get $\theta^{\ddagger}(0)=1$. For any $x,y\in L$,

$$\theta^{\ddagger}(x\vee y)=\theta(x^{*}\wedge y^{*})=\theta(x^{*})\wedge\theta(y^{*})=\theta^{\ddagger}(x)\wedge\theta^{\ddagger}(y). \tag{36}$$

Thus, $\theta^{\ddagger}$ is a fuzzy ideal of L. To show $\theta^{\ddagger}$ is a p-fuzzy ideal, let $x\in L$. Then, $\theta^{\ddagger}(x^{**})=\theta(x^{*})=\theta^{\ddagger}(x)$. Thus, $\theta^{\ddagger}$ is a p-fuzzy ideal of L. □

Theorem 16. *Let μ be a fuzzy filter of L. Then, $(\mu^{\ddagger})^{\dagger}=\mu$ if and only if μ is a p-fuzzy filter.*

Proof. Let μ be a fuzzy filter of L and $(\mu^{\ddagger})^{\dagger}=\mu$. Then, by Lemmas 6 and 7, μ is a p-fuzzy filter. Conversely, let μ be a p-fuzzy filter and $x\in L$. Then, $(\mu^{\ddagger})^{\dagger}(x)=\mu^{\ddagger}(x^{*})=\mu(x^{**})=\mu(x)$. Thus, $(\mu^{\ddagger})^{\dagger}=\mu$. □

Theorem 17. *In L, $FI^{p}(L)\cong FF^{p}(L)$.*

Proof. Define $f\colon \mathrm{FI}^{p}(L)\longrightarrow \mathrm{FF}^{p}(L)$ by $f(\mu)=\mu^{\dagger}$. Since $\mu\in\mathrm{FI}^{p}(L)$, by Lemma 6, $f(\mu)\in\mathrm{FF}^{p}(L)$. Let $\mu,\theta\in\mathrm{FI}^{p}(L)$ such that $f(\mu)=f(\theta)$. Then, $\mu^{\dagger}=\theta^{\dagger}$. Since $\mu^{\dagger}$ and $\theta^{\dagger}$ are p-fuzzy filters of L, then $\mu^{\dagger\ddagger}$ and $\theta^{\dagger\ddagger}$ are p-fuzzy ideals of L and $\mu^{\dagger\ddagger}=\theta^{\dagger\ddagger}$. To show $\mu=\theta$, let $x\in L$. Then, $\mu^{\dagger\ddagger}(x)=\theta^{\dagger\ddagger}(x)$. This implies $\mu(x^{**})=\theta(x^{**})$. Since μ and θ are p-fuzzy ideal, we get that $\mu(x)=\theta(x)$. Thus, f is one to one.

Let $\theta\in\mathrm{FF}^{p}(L)$. Then, by Theorem 16, $\theta=\theta^{\ddagger\dagger}$. Since θ is a fuzzy filter, then $\theta^{\ddagger}$ is a p-fuzzy ideal of L and $f(\theta^{\ddagger})=\theta$. So f is onto.

Let $\mu,\theta\in\mathrm{FI}^{p}(L)$ and $x\in L$. Then

$$\begin{aligned}f(\mu\veebar\theta)(x) &= \mathrm{Sup}\{\mu(a)\wedge\theta(b)\colon x^{*}\leq(a^{*}\wedge b^{*})^{*},\ a,b\in L\}\\ &= \mathrm{Sup}\{\mu(a^{**})\wedge\theta(b^{**})\colon x^{*}\leq(a^{*}\wedge b^{*})^{*}\}\\ &= \mathrm{Sup}\{\mu^{\dagger}(a^{*})\wedge\theta^{\dagger}(b^{*})\colon x^{*}\leq(a^{*}\wedge b^{*})^{*}\}\\ &\leq \mathrm{Sup}\{\mu^{\dagger}(c)\wedge\theta^{\dagger}(d)\colon x^{*}\leq(c\wedge d)^{*},\ c,d\in L\}\\ &=(\mu^{\dagger}\veebar\theta^{\dagger})(x)\\ &=(f(\mu)\veebar f(\theta))(x).\end{aligned} \tag{37}$$

Thus, $f(\mu\veebar\theta)\subseteq(f(\mu)\veebar f(\theta))$. On the other hand,

$$\begin{aligned}(f(\mu)\veebar f(\theta))(x) &= \mathrm{Sup}\{\mu^{\dagger}(a)\wedge\theta^{\dagger}(b)\colon x^{*}\leq(a\wedge b)^{*},\ a,b\in L\}\\ &= \mathrm{Sup}\{\mu(a^{*})\wedge\theta(b^{*})\colon x^{*}\leq(a\wedge b)^{*}\}\\ &= \mathrm{Sup}\{\mu(a^{*})\wedge\theta(b^{*})\colon x^{*}\leq(a^{**}\wedge b^{**})^{*}\}\\ &\leq \mathrm{Sup}\{\mu(c)\wedge\theta(d)\colon x^{*}\leq(c^{*}\wedge d^{*})^{*},\ c,d\in L\}\\ &=(\mu\veebar\theta)(x^{*})\\ &=(\mu\veebar\theta)^{\dagger}(x)\\ &=f(\mu\veebar\theta)(x).\end{aligned} \tag{38}$$

Thus, $f(\mu\veebar\theta)=f(\mu)\veebar f(\theta)$. Similarly, $f(\mu\cap\theta)=f(\mu)\cap f(\theta)$. Hence $\mathrm{FI}^{p}(L)\cong\mathrm{FF}^{p}(L)$. □

6. Conclusion

In this paper, we introduced the concept of p-fuzzy ideals and p-fuzzy filters in a p-algebra. We provide a set of equivalent conditions for a fuzzy ideal to be a p-fuzzy ideal and a p-algebra to be a Boolean algebra. It is proved that the class of p-fuzzy ideals forms a complete distributive lattice. We also studied the image and inverse image of p-fuzzy ideals under a $*$-epimorphism mapping. Moreover, we prove that there is an isomorphism between the class of p-fuzzy ideals and p-fuzzy filter. Our future work will focus on σ-fuzzy ideals in a (0, 1) distributive lattice.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this study. The author is one of the academic staff of University of Gondar, College of

Natural and Computational Sciences, Department of Mathematics, Gondar. Ethiopia.

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