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Research Article

L-Fuzzy Prime Spectrums of ADLs

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The notion of an Almost Distributive Lattice (ADL) is a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebra and Boolean rings. In this paper, the set of all L-fuzzy prime ideals of an ADL with truth values in a complete lattice L satisfying the infinite meet distributive law is topologized and the resulting space is discussed.

1. Introduction

The concept of prime ideal is vital in the study of structure theory of distributive lattices in general and of Boolean algebras in particular [2]. In this context, we recall the work of Stone [1] on the representation of distributive lattices by algebra of sets. In fact, he proved that a lattice L is distributive if and only if any ideal of L is the intersection of all prime ideals containing it. Also, he introduced a topology on the set of all prime ideals of a given Boolean algebra B in such a way that B is isomorphic with the Boolean algebra of clopen subsets of resulting space.

Swamy and Rao [2] have introduced the notion of an Almost Distributive Lattice (ADL) which is algebra $(A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ satisfying all the axioms of a distributive lattice with zero except $\wedge$ commutative, $\vee$ commutative, and right distributivity of $\vee$ over $\wedge$. In fact, in any ADL, three conditions are equivalent.

Next, Rosenfold [3] introduced the notion of fuzzy groups; many researchers are turned into fuzzifying various algebra. Santhi Sundar Raj et al. [4–6] have introduced the concepts of fuzzy prime ideals of an ADL and studied them deeply.

In this paper, we introduce a topology on the set of all L-fuzzy prime ideals of an ADL A and the resulting space is called the L-fuzzy prime spectrums of A, denoted by $\mathbf{F_L}\text{spec}(A)$ or X. For an L-fuzzy ideal λ of A, open subset of X is of the form $X(\lambda) = \{\mu \in X\colon \lambda \nleq \mu\}$ and $V(\lambda) = \{\mu \in X\colon \lambda \leq \mu\}$ is a closed set. In particular, we prove that $\{X(a)\colon a \in S\}$ is a base for a topology on X. Furthermore, it is proved that the space $\mathbf{F_L}\text{spec}(A)$ is compact and it contains a subspace homeomorphic with the spectrum of A which is dense in it. Also, it is proved that the space X is a Hausdorff space if and only if the space is a T_1-space and, further, it is noted that the space X is a T_1-space if and only if every L-fuzzy prime ideal is an L-fuzzy maximal ideal and L-fuzzy minimal prime ideal of A. Finally, it is proven that if A and B are isomorphic ADLs, then the space $\mathbf{F_L}\text{spec}(A)$ is homeomorphic with the space $\mathbf{F_L}\text{spec}(B)$.

Throughout this paper, A stands for an ADL $(A, \wedge, \vee, 0)$ with a maximal element and L stands for a complete lattice $(L, \wedge, \vee, 0, 1)$ satisfying the infinite meet distributive law; that is, $(x \wedge (\vee_{y \in S} y)) = \vee_{y \in S}(x \wedge y)$ for any $S \subseteq L$ and $x \in L$.

2. Preliminaries

In this section, we recall some definitions and basic results mostly taken from [2, 4].

Definition 1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$:

(1) $0 \wedge a = 0$

(2) $a \vee 0 = a$

(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(6) $(a \vee b) \wedge b = b$

Any bounded below distributive lattice is an ADL. Any nonempty set X can be made into an ADL which is not a lattice by fixing an arbitrarily chosen element 0 in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0, \\ b, & \text{if } a \neq 0, \end{cases} \qquad a \vee b = \begin{cases} b, & \text{if } a = 0, \\ a, & \text{if } a \neq 0. \end{cases} \tag{1}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL.

Definition 2. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define $a \leq b$ if $a = a \wedge b (\Longleftrightarrow a \vee b = b)$. Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 1. *The following hold for any a, b and c in an ADL A:*

(1) $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$

(2) $a \wedge a = a = a \vee a$

(3) $a \wedge b \leq b \leq b \vee a$

(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$

(5) $a \wedge b = b \Longleftrightarrow a \vee b = a$

(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative)

(7) $a \vee (b \vee a) = a \vee b$

(8) $a \leq b \Longrightarrow a \wedge b = a = b \wedge a (\Longleftrightarrow a \vee b = b = b \vee a)$

(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(11) $a \wedge b = b \wedge a \Longleftrightarrow a \vee b = b \vee a$

(12) $a \wedge b = \inf\{a, b\} \Longleftrightarrow a \wedge b = b \wedge a \Longleftrightarrow a \vee b = sup\{a, b\}$

Definition 3. Let I be a nonempty subset of an ADL A. Then I is called an ideal of A if $a, b \in I \Rightarrow a \vee b \in I$ and $a \wedge x \in I$ for all $x \in A$.

As a consequence, for any ideal I of $A, x \wedge a \in I$ for all $a \in I$ and $x \in A$. An element $m \in A$ is said to be maximal if, for any $x \in A, m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 4. Let $L = (L, \wedge, \vee)$ and $M = (M, \wedge, \vee)$ be lattices and let $f: L \longrightarrow M$ be a mapping. Then f is called (1) an order homomorphism (or isotone) if $a \leq b$ in $L \Rightarrow f(a) \leq f(b)$ in M and (2) a lattice homomorphism if, for any $a, b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$.

Theorem 2. *Let L and M be lattices and let $f: L \longrightarrow M$ be a bijection. Then f is a lattice isomorphism if and only if both f and f^{-1} are order homomorphisms.*

Definition 5. Let $(A, \wedge, \vee, 0)$ and $(A', \wedge', \vee', 0')$ be ADLs. A mapping $f: A \longrightarrow A'$ is called a homomorphism if the following are satisfied for any x and $y \in A$: (1) $f(x \vee y) = f(x) \vee' f(y)$. (2) $f(x \wedge y) = f(x) \wedge' f(y)$. (3) $f(0) = 0'$.

Definition 6. Let X and Y be topological spaces and let $f: X \longrightarrow Y$ be a mapping; then f is said to be continuous if and only if inverse image of every open set in Y is open in X.

Definition 7. Let X and Y be topological spaces and let $f: X \longrightarrow Y$ be a mapping; then f is said to be open if and only if image of every open set in X is open in Y.

Definition 8. Let X and Y be topological spaces; then a bijection $\phi: X \longrightarrow Y$ is said to be a homeomorphism if it is a continuous open mapping.

Definition 9. An L-fuzzy subset λ of A is said to be an L-fuzzy ideal of A, if $\lambda(0) = 1$ and $\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$.

Theorem 3. *Let λ be an L-fuzzy subset of A. Then λ is an L-fuzzy ideal if and only if (1) $\lambda(0) = 1$, (2) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y)$, and (3) $\lambda(x \wedge y) \geq \lambda(x) \vee \lambda(y)$, for all $x, y \in A$.*

Definition 10. Let χ_S denote the characteristic function of any subset S of an ADL A; that is,

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases} \tag{2}$$

Definition 11. A proper L-fuzzy ideal λ of A is called an L-fuzzy prime ideal of A if, for any $x, y \in A, \lambda(x \wedge y) = \lambda(x)$ or $\lambda(y)$.

Theorem 4. *Let λ be a proper L-fuzzy ideal of A. Then the following are equivalent to each other: (1) For each $\alpha \in L$, $\lambda_\alpha = A$ or λ_α is a prime ideal of A. (2) λ is an L-fuzzy prime ideal of A. (3) For any $x, y \in A$, $\lambda(x \wedge y) \leq \lambda(x) \vee \lambda(y)$ and either $\lambda(x) \leq \lambda(y)$ or $\lambda(y) \leq \lambda(x)$.*

Theorem 5. *Let λ be an L-fuzzy prime ideal of A and let 0 be a prime element in A. Then λ is an L-fuzzy minimal prime ideal of A if and only if λ_α is a minimal prime ideal of A, for all $\alpha \in L$.*

Definition 12. A proper L-fuzzy ideal λ of A is called an L-fuzzy maximal ideal of A if, for each $\alpha \in L$, either $\lambda_\alpha = A$ or λ_α is a maximal ideal of A.

3. Topological Space on L-Fuzzy Prime Ideals

In this section, we introduce the Zariski topology on the set of L-fuzzy prime ideals of an Almost Distributive Lattice A. Our definition of L-fuzzy prime ideal offers us an appropriate setting to introduce a topology on the set of L-fuzzy prime ideals of A. A topology is introduced on the set of all L-fuzzy prime ideals of A to obtain the space called the hull-kernel topology on the set of all L-fuzzy prime ideals and denoted by $\mathbf{F_L}\mathrm{spec}(A)$ or X. First, we have the following.

Theorem 6. *Let A and B be ADLs. Let $p: A \longrightarrow B$ be a lattice homomorphism and $p(0) = 0$. If $\lambda: A \longrightarrow L$ and $\mu: B \longrightarrow L$ are L-fuzzy ideals of A and B, respectively, then (1) $p^{-1}(\mu)$ is an L-fuzzy ideal of A, (2) $p(\lambda)$ is an L-fuzzy ideal of B if p is an epimorphism, and (3) $p(p^{-1}(\mu)) = \mu$.*

Proof. Define $p^{-1}(\mu): A \longrightarrow L$ and $p(\lambda): B \longrightarrow L$ as $p^{-1}(\mu)(x) = \mu(p(x))$ for each $x \in A$ and $p(\lambda)(y) = \mathrm{Sup}\{\lambda(x):\ p(x) = y, x \in A\}$ for each $y \in B$. Then,

(1) $p^{-1}(\mu)(0) = \mu(p(0)) = \mu(0) = 1$, as μ is an L-fuzzy ideal of B. Let $x, y \in A$. Then,

$$\begin{aligned} p^{-1}(\mu)(x \vee y) &= \mu(p(x \vee y)) \\ &= \mu(p(x) \vee p(y)) \\ &\quad \cdot (\text{since } p \text{ is lattice homomorphism}) \\ &= \mu(p(x)) \wedge \mu(p(y)) \\ &\quad \cdot (\text{since } \mu \text{ is an } L-\text{fuzzy ideal}) \\ &= p^{-1}(\mu)(x) \wedge p^{-1}(\mu)(y). \end{aligned} \tag{3}$$

Also, let $x \le y$ in A. As p is homomorphism, we get $p(x) \le p(y)$. But then $\mu(p(x)) \ge \mu(p(y))$ (since μ is an L-fuzzy prime ideal). That is, $p^{-1}(\mu)(x) \ge p^{-1}(\mu)(y)$. Therefore, $p^{-1}(\mu)$ is antitone. Thus, $p^{-1}(\mu)$ is an L-fuzzy ideal of A.

(2) Clearly, $p(\lambda)(0) = 1$. Let $p: A \longrightarrow B$ be a lattice epimorphism. Let $a, b \in B$. Then there exists $x, y \in A$ such that $p(x) = a$ and $p(y) = b$. Thus, $p(x \vee y) = p(x) \vee p(y) = a \vee b$. Now,

$$\begin{aligned} p(\lambda)(a \vee b) &= \mathrm{Sup}\{\lambda(z):\ p(z) = a \vee b, z \in A\} \\ &= \bigvee_{z \in p^{-1}(a \vee b)} \lambda(z) \\ &\ge \bigvee_{x \in p^{-1}(a), y \in p^{-1}(b)} \lambda(x \vee y) \\ &\ge \bigvee_{x \in p^{-1}(a), y \in p^{-1}(b)} \lambda(x) \wedge \lambda(y) \\ &\quad \cdot (\text{since } \lambda \text{ is an } L-\text{fuzzy ideal}) \\ &= \left(\bigvee_{x \in p^{-1}(a)} \lambda(x)\right) \wedge \left(\bigvee_{y \in p^{-1}(b)} \lambda(y)\right) \\ &= p(\lambda)(a) \wedge p(\lambda)(b), \end{aligned} \tag{4}$$

Thus, $p(\lambda)(a \vee b) \ge p(\lambda)(a) \wedge p(\lambda)(b)$. Similarly, $p(\lambda)(a \wedge b) \ge p(\lambda)(a) \vee p(\lambda)(b)$. Let $a \le b$ in B. Then, $a \vee b = b$. Therefore, whenever $p(x) = a$ and $p(b) = y$, we have $p(x \vee y) = p(x) \vee p(y) = a \vee b$. Now,

$$\begin{aligned} p(\lambda)(a) \wedge p(\lambda)(b) &= \left(\bigvee_{c \in p^{-1}(a)} \lambda(c)\right) \wedge \left(\bigvee_{d \in p^{-1}(b)} \lambda(d)\right) \\ &= \bigvee_{c \in p^{-1}(a), d \in p^{-1}(b)} \lambda(c) \wedge \lambda(d) \\ &= \bigvee_{c \vee d \in p^{-1}(b)} \lambda(c \vee d) \\ &\quad \cdot (\text{since } \lambda \text{ is an } L-\text{fuzzy ideal}) \\ &= p(\lambda)(b), \end{aligned} \tag{5}$$

This shows that $p(\lambda)(b) \le p(\lambda)(a)$. Thus, $p(\lambda)$ is an antitone map. Therefore, $p(\lambda)$ is an L-fuzzy ideal of A.

(3) Let $b \in B$. Then,

$$\begin{aligned} p\left(p^{-1}(\mu)\right)(b) &= \bigvee_{y \in p^{-1}(b)} p^{-1}(\mu)(y) \\ &= \bigvee_{y \in p^{-1}(b)} \mu(p(y)) \\ &= \mu(b). \end{aligned} \tag{6}$$

Thus, $p(p^{-1}(\mu)) = \mu$. □

Theorem 7. *If an L-fuzzy subset λ of A is an L-fuzzy prime ideal of A, then λ is a homomorphism from $(A, \wedge, \vee, 0)$ to $(L, \vee, \wedge)$.*

Proof. As λ is an L-fuzzy ideal, then

$$\begin{aligned} &\lambda(0) = 1, \\ &\lambda(x \vee y) = \lambda(x) \wedge \lambda(y), \quad \text{for all}, x, y \in A. \end{aligned} \tag{7}$$

Since λ is an L-fuzzy prime ideal of A, we have $\lambda(x \wedge y) = \lambda(x)$ or $\lambda(x \wedge y) = \lambda(y)$. In either case, we get

$$\lambda(x \wedge y) \le \lambda(x) \vee \lambda(y). \tag{8}$$

Also, $x \wedge y \le y$ and $y \wedge x \le x$ and λ is antitone (being an L-fuzzy ideal) implying that $\lambda(y) \le \lambda(x \wedge y)$ and $\lambda(x) \le \lambda(y \wedge x) = \lambda(x \wedge y)$. Thus,

$$\lambda(x) \vee \lambda(y) \le \lambda(x \wedge y). \tag{9}$$

From (8) and (9), we have

$$\lambda(x) \vee \lambda(y) = \lambda(x \wedge y). \tag{10}$$

Therefore, from (7) and (10), λ is a homomorphism from $(A, \wedge, \vee, 0)$ to $(L, \vee, \wedge)$.

In Theorem 6, we have proved that inverse image of an L-fuzzy ideal of an ADL A is an L-fuzzy ideal again. In the case of L-fuzzy prime ideals, we have the following. □

Theorem 8. *Let A and B be ADLs. Let $p: A \longrightarrow B$ be a lattice homomorphism. If λ is an L-fuzzy prime ideal of B, then $p^{-1}(\lambda)$ is an L-fuzzy prime ideal of A.*

Proof. By Theorem 6, $p^{-1}(\lambda)$ is an L-fuzzy ideal of A. Let $x, y \in A$. Then,

$$\begin{aligned} p^{-1}(\lambda)(x\wedge y) &= \lambda(p(x\wedge y)) \\ &= \lambda(p(x)\wedge p(y)) \\ &= \lambda(p(x)),\ \text{or}\ \lambda(p(y)) \\ &= p^{-1}(\lambda)(x),\ \text{or}\ p^{-1}(\lambda)(y). \end{aligned} \tag{11}$$

Therefore, $p^{-1}(\lambda)$ is an L-fuzzy prime ideal of A. □

Theorem 9. *Let A and B be ADLs. Let $p\colon A \longrightarrow B$ be a lattice isomorphism. If λ is an L-fuzzy prime ideal of A, then $p(\lambda)$ is an L-fuzzy prime ideal of B and $p^{-1}(p(\lambda)) = \lambda$.*

Proof. By Theorem 6 (2), $p(\lambda)$ is an L-fuzzy ideal of B. Let $a, b \in B$. Then, $a = p(x)$ and $b = p(y)$, for some $x, y \in A$. Therefore, $p(x\wedge y) = a\wedge b$. Now, if $t \in p^{-1}(a)$, then $p(t) = a = p(x)$ implies that $t = x$ (since p is injective).

Therefore,

$$\begin{aligned} p(\lambda)(a) &= \operatorname{Sup}\{\lambda(t)\colon p(t) = a, t \in A\} \\ &= \bigvee_{t\in p^{-1}(a)} \lambda(t) \\ &= \lambda(x), \end{aligned} \tag{12}$$

Similarly, $p(\lambda)(b) = \lambda(y)$ and if $z \in p^{-1}(a\wedge b)$, then $z = x\wedge y$. Thus,

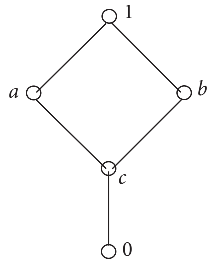
$$\begin{aligned} p(\lambda)(a\wedge b) &= \operatorname{Sup}\{\lambda(z)\colon p(z) = a\wedge b, z \in A\} \\ &= \bigvee_{z\in p^{-1}(a\wedge b)} \lambda(z) \\ &= \lambda(x\wedge y) \\ &= \lambda(x),\ \text{or}\ \lambda(y). \end{aligned} \tag{13}$$

Thus, $p(\lambda)$ is an L-fuzzy prime ideal of B. Also, let $x \in A$. Then,

$$\begin{aligned} p^{-1}(p(\lambda))(x) &= p(\lambda)(p(x)) \\ &= \operatorname{Sup}\{\lambda(y)\colon p(y) = p(x), y \in A\} \\ &= \bigvee_{x\in p^{-1}(p(y))} \lambda(y), \quad (\text{since } p \text{ is injective}) \\ &= \lambda(x). \end{aligned} \tag{14}$$

Therefore, $p^{-1}(p(\lambda)) = \lambda$. □

Example 1. Consider the lattice $A = \{0, a, b, c, 1\}$ whose Hasse diagram is given below:



and let $B = \{0, a, b, c\}$ and let $\vee$ and $\wedge$ be binary operations on B defined by

∨	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

∧	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then, $(B, \wedge, \vee, 0)$ is an ADL, which is not a lattice $(a\wedge b \neq b\wedge a)$. Let $L = [0, 1]$ be the closed unit interval of real numbers. Then L is a frame with respect to the usual ordering. Define L-fuzzy subsets μ and λ of A and B, respectively, by $\mu(0) = 1$ and $\mu(x) = 0.5$, for all $x \neq 0$; $\lambda(0) = 1$, $\lambda(a) = \lambda(b) = 0$, and $\lambda(c) = 0.5$; and define a function $p\colon A \longrightarrow B$ by $p(0) = 0, p(b) = b, p(c) = c, p(a) = p(1) = a$. Then, we observe that $\mu_\alpha = A$ if $0 \leq \alpha \leq 0.5$ and $\mu_\alpha = \{0\}$ if $0.5 < \alpha \leq 1$. Thus, by Theorem 4, μ is an L-fuzzy prime ideal of A. Also, $\lambda_{0.5} = \{0, c\}$ and $\lambda_1 = \{0\}$ are prime ideals of B. Therefore, λ is an L-fuzzy prime ideal of B (by4). For any x and $y \in A$,

$$\begin{aligned} p(0) &= 0, \\ p(x\vee y) &= p(x)\vee p(y), \\ p(x\wedge y) &= p(x)\wedge p(y). \end{aligned} \tag{15}$$

Thus, p is a lattice homomorphism. Also, $x \leq y \Rightarrow p(x) \leq p(y)$. Thus, p is isotone and, for each $y \in B$, there exists $x \in A$ such that $p(x) = y$. Hence, p is a bijection map. Therefore, p is a lattice isomorphism. By the above Theorems 6 and 8, $p(\lambda)$ and $p^{-1}(\mu)$ are L-fuzzy prime ideals of A and B, respectively, since λ and μ are L-fuzzy prime ideals.

In the following, we obtain a topological space by introducing Zariski topology on the set of L-fuzzy prime ideals of ADLs.

Definition 13. Let $A = (A, \wedge, \vee, 0)$ be a nontrivial ADL and let X be the set of all L-fuzzy prime ideals of A. For any L-fuzzy subset Θ of A, we define

$$\begin{aligned} V(\Theta) &= \{\lambda \in X\colon \Theta \leq \lambda\}, \\ X(\Theta) &= \{\lambda \in X\colon \Theta \nleq \lambda\} = X - V(\Theta). \end{aligned} \tag{16}$$

The complement of $V(\Theta)$ in $X = X(\Theta)$.

Now, we prove some properties of V and X.

Theorem 11. *Let Θ and σ be L-fuzzy subsets of A. Then, we have the following: (1) if $\Theta \leq \sigma$, then $V(\sigma) \subseteq V(\Theta)$ and $X(\Theta) \subseteq X(\sigma)$; (2) $V(\sigma) \cup V(\Theta) \subseteq V(\sigma\wedge\Theta)$ and $X(\sigma) \cup X(\Theta) \subseteq X(\sigma\wedge\Theta)$; (3) $V(\Theta) = V(\langle\Theta\rangle)$ and $X(\Theta) = X(\langle\Theta\rangle)$, where $\langle\Theta\rangle$ is the smallest L-fuzzy ideal containing Θ (4) $V(\chi_{\{0\}}) = X$ and $V(\chi_{\{1\}}) = \varnothing$.*

Proof

(1)

$$\begin{aligned}\mu \in V(\sigma) &\Rightarrow \sigma \leq \mu \\ &\Rightarrow \Theta \leq \mu \,(\text{since } \Theta \leq \sigma) \\ &\Rightarrow \mu \in V(\Theta).\end{aligned} \tag{17}$$

Therefore, $V(\sigma) \subseteq V(\Theta)$. Also,

$$\begin{aligned}\nu \notin X(\sigma) &\Rightarrow \sigma \leq \nu \\ &\Rightarrow \Theta \leq \nu \,(\text{since } \Theta \leq \sigma) \\ &\Rightarrow \nu \notin X(\Theta),\end{aligned} \tag{18}$$

Therefore, $X(\Theta) \subseteq X(\sigma)$.

(2)

$$\begin{aligned}\mu \in V(\sigma) \cup V(\Theta) &\Rightarrow \mu \in V(\sigma), \quad \text{or } \mu \in V(\Theta), \\ &\Rightarrow \sigma \leq \mu, \quad \text{or } \Theta \leq \mu \\ &\Rightarrow \sigma \wedge \Theta \leq \mu, \\ &\quad \cdot (\text{since } \sigma \leq \mu \Rightarrow \sigma \wedge \Theta \leq \mu \wedge \Theta \leq \mu),\end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned}\Theta \leq \mu &\Rightarrow \sigma \wedge \Theta \leq \sigma \wedge \mu \leq \mu, \\ &\Rightarrow \mu \in V(\sigma \wedge \Theta),\end{aligned} \tag{20}$$

Therefore, $V(\sigma) \cup V(\Theta) \subseteq V(\sigma \wedge \Theta)$.

(3) Clearly $V(\langle\Theta\rangle) \subseteq V(\Theta)$, since $\Theta \leq \langle\Theta\rangle$ and by (1). On the other hand, let $\mu \in V(\Theta)$. Then $\Theta \leq \mu$; it follows that $\langle\Theta\rangle \leq \mu$. Hence, $\mu \in V(\langle\Theta\rangle)$. Therefore, $V(\Theta) = V(\langle\Theta\rangle)$. Also, clearly $X(\Theta) \subseteq X(\langle\Theta\rangle)$, since $\Theta \leq \langle\Theta\rangle$ and by (1). On the other hand,

$$\begin{aligned}\mu \in X(\langle\Theta\rangle) &\Rightarrow \langle\Theta\rangle \nleq \mu \\ &\Rightarrow \Theta \nleq \mu \\ &\Rightarrow \mu \in X(\Theta),\end{aligned} \tag{21}$$

It follows that $X(\langle\Theta\rangle) \subseteq X(\Theta)$. Therefore, $X(\Theta) = X(\langle\Theta\rangle)$.

(4) Let $\lambda \in X$. Then $\lambda(0) = 1$. Therefore, $V(\chi_{\{0\}}) = X$ and $V(\chi_{\{1\}}) = \varnothing$ (as λ is an L-fuzzy prime ideal). □

Remark 1. In general, equality does not hold in Theorem 7. Equality holds if Θ and σ are crisp ideals of A. The following example shows that, in a case of L-fuzzy subsets of A, equality does not hold even if Θ and σ are L-fuzzy ideals.

Example 2. Let $A = \{0, a, b, c\}$ and $L = \{0, s, t, 1\}$ with $0 < s < t < 1$ and let $\vee$ and $\wedge$ be binary operations on A defined by

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then, $(A, \wedge, \vee, 0)$ is an ADL. Now define λ: $A \longrightarrow L$ by $\lambda(0) = 1, \lambda(a) = \lambda(b) = 0$ and $\lambda(c) = s$. Then $\lambda_0 = A, \lambda_s = \{0, c\}$, and $\lambda_1 = \{0\}$ are prime ideals of A. Therefore, λ is an L-fuzzy prime ideal of A. Define Θ: $A \longrightarrow L$ and σ: $A \longrightarrow L$ as $\Theta(0) = 1, \Theta(a) = 0, \Theta(b) = \Theta(c) = s$, and $\sigma(0) = 1, \sigma(a) = \sigma(b) = s, \sigma(c) = t$. Clearly, Θ and σ are L-fuzzy ideals of A and $\Theta \leq \sigma$. Then, $V(\sigma) = \{(0, 1)\}$ and $V(\Theta) = \{(0, 1), (a, 0), (c, s)\}$. From this, $V(\sigma) \subseteq V(\Theta)$. Also, $\sigma \wedge \Theta = \{(0, 1), (a, 0), (b, s), (c, s)\}$ and hence $V(\sigma \wedge \Theta) = V(\Theta)$. Now, $V(\sigma) \cup V(\Theta) = \{(0, 1), (a, 0), (c, s)\} \subseteq V(\sigma \wedge \Theta)$ but $\{(c, s)\} \notin V(\sigma) \cup V(\Theta)$. Hence, $V(\sigma) \cup V(\Theta) \subset V(\sigma \wedge \Theta)$.

Recall that if I is an ideal of A, then the characteristic function χ_I of I is an L-fuzzy ideal of A. For such L-fuzzy ideals, we have the following.

Theorem 12. *Let I and J be ideals of A. Then, (1) $V(\chi_I) \cup V(\chi_J) = V(\chi_{I\cap J})$ and (2) $X(\chi_I) \cup X(\chi_J) = X(\chi_{I\cap J})$.*

Proof

(1) Since $I \cap J \subseteq I, J$, $\chi_{I\cap J} \leq \chi_I, \chi_J$. Then, by Theorem 7 (2), we have $V(\chi_I) \cup V(\chi_J) \subseteq V(\chi_{I\cap J})$. On the other hand, let $\mu \in V(\chi_{I\cap J})$. Then, $\chi_{I\cap J} \leq \mu$, and it follows that $\mu(x) = 1$, for each $x \in I \cap J$. If $\chi_I \nleq \mu$ and $\chi_J \nleq \mu$, then there exists $x \in I, y \in J$ such that $\mu(x) \neq 1$ and $\mu(y) \neq 1$. But, as $x \wedge y \in I \cap J$, then $\mu(x \wedge y) = 1$. As μ is an L-fuzzy prime ideal of A, then $1 = \mu(x \wedge y) \leq \mu(x) \vee \mu(y)$. It follows that $\mu(x) \vee \mu(y) = 1$. This implies that either $\mu(x) = 1$ or $\mu(y) = 1$, which gives a contradiction with the choice of x and y. So, $\chi_I \leq \mu$ or $\chi_J \leq \mu$. Therefore, $\mu \in V(\chi_I)$ or $\mu \in V(\chi_J)$ and hence $\mu \in V(\chi_I) \cup V(\chi_J)$.

(2) Clearly, $X(\chi_{I\cap J}) \subseteq X(\chi_I) \cup X(\chi_J)$. On the other hand,

$$\begin{aligned}\mu \in X(\chi_I) \cup X(\chi_J) &\Rightarrow \mu \in X(\chi_I) \quad \text{or } \mu \in X(\chi_J) \\ &\Rightarrow \chi_I \nleq \mu, \quad \text{or } \chi_J \nleq \mu \\ &\Rightarrow \chi_I \cap \chi_J \nleq \mu \\ &\Rightarrow \chi_{I\cap J} \nleq \mu \\ &\Rightarrow \mu \in X(\chi_{I\cap J}).\end{aligned} \tag{22}$$

Therefore, $X(\chi_I)\cup X(\chi_J)\subseteq X(\chi_{I\cap J})$. □

Corollary 1. *For any subsets S of an ADL A and letting $a,b\in A$,*

(1) $V(\chi_S)=\cap\{V(\chi_{\{a\}}): a\in S\}$
(2) $V(\chi_{\{a\}})\bigcup V(\chi_{\{b\}})=V(\chi_{\{a\wedge b\}})$

Theorem 13. *For any subsets S of an ADL A, we have the following:*

(1) (1) $\chi_S=\cup_{a\in S}\chi_{\{a\}}$
(2) (2) $\langle\chi_S\rangle=\chi_{(S]}$
(3) (3) $\langle\chi_{\{a\}}\rangle=\chi_{(a]}$ *for every* $a\in A$
(4) (4) $X(\chi_S)=X(\chi_{(S]})$
(5) (5) $V(\chi_S)=V(\chi_{(S]})$

Proof. Clearly $\chi_S\le\chi_{(S]}$. Let λ be an L-fuzzy ideal of A such that $\chi_S\le\lambda$. Then, $\lambda(a)=1$, for all $a\in S$. Now, we shall prove that $\chi_{(S]}\le\lambda$. For any $t\in A$, we have

$$t\in(S]\Rightarrow t=\left(\bigvee_{i=1}^{n}a_i\right)\wedge x,\quad \text{for some } x\in A \text{ and } a_1,a_2,\dots,a_n\in S \text{ and } \chi_S(t)=1. \tag{23}$$

Now,

$$\begin{aligned}\lambda(t)&=\lambda\left(\left(\bigvee_{i=1}^{n}a_i\right)\wedge x\right)\\&=\lambda\left(\bigvee_{i=1}^{n}(a_i\wedge x)\right)\\&=\bigwedge_{i=1}^{n}\lambda(a_i\wedge x)\\&\ge\bigwedge_{i=1}^{n}(\lambda(a_i)\vee\lambda(x)),\quad (by\dots)\\&=(\lambda(a_1)\vee\lambda(x))\wedge(\lambda(a_2)\vee\lambda(x))\wedge\cdots\wedge(\lambda(a_n)\vee\lambda(x))\\&=(1\vee\lambda(x))\wedge(1\vee\lambda(x))\wedge\cdots\wedge(1\vee\lambda(x))\\&=1\wedge1\wedge\cdots\wedge1=1.\end{aligned} \tag{24}$$

Therefore, $\lambda(t)\ge1$ and hence $\lambda(t)=1$. Therefore, $\chi_{(S]}\le\lambda$. This shows that $\chi_{(S]}$ is the smallest L-fuzzy ideal of A containing χ_S. Thus, $\langle\chi_S\rangle=\chi_{(S]}$. □

Theorem 14. *If $\{\lambda_i\}_{i\in\Delta}$ is a family of L-fuzzy subsets of A, then*

$$V\left(\bigcup_{i\in\Delta}\lambda_i\right)=\bigcap_{i\in\Delta}V(\lambda_i). \tag{25}$$

Proof

$$\begin{aligned}\mu\in V\left(\bigcup_{i\in\Delta}\lambda_i\right)&\Longleftrightarrow\bigcup_{i\in\Delta}\lambda_i\le\mu,\\&\Longleftrightarrow\lambda_i\le\mu,\quad\text{for each } i\in\Delta\\&\Longleftrightarrow\mu\in V(\lambda_i),\quad\text{for each } i\in\Delta\\&\Longleftrightarrow\mu\in\bigcap_{i\in\Delta}V(\lambda_i).\end{aligned} \tag{26}$$

This shows that $V(\cup_{i\in\Delta}\lambda_i)=\cap_{i\in\Delta}V(\lambda_i)$. □

Theorem 15. *Let $\tau=\{X(\Theta): \Theta$ is an $L-$fuzzy subset of $A\}$. Then the pair (X,τ) is a topological space.*

Proof. Consider L-fuzzy subsets of A defined by $\sigma(x)=0$ and $\Theta(x)=1$, for all $x\in A$. Then, $V(\sigma)=\{\lambda\in X: \sigma\le\lambda\}=X$. Therefore, $X(\sigma)=X-V(\sigma)=\varnothing$. Also, $V(\Theta)=\{\lambda\in X: \Theta\le\lambda\}=\varnothing$. Therefore, $X(\Theta)=X-V(\Theta)=X$. Let μ and ν be L-fuzzy subsets of A. Then, $V(\mu)\cup V(\nu)=V(\langle\mu\rangle)\cup V(\langle\nu\rangle)=V(\langle\mu\rangle\wedge\langle\mu\rangle)$. Now,

$$\begin{aligned}X(\mu)\cap X(\nu)&=(X-V(\mu))\cap(X-V(\nu))\\&=X-(V(\mu)\cup V(\nu))\\&=X-(V(\langle\mu\rangle)\cup V(\langle\nu\rangle))\\&=X-(V(\langle\mu\rangle\wedge\langle\nu\rangle))\\&=X(\langle\mu\rangle\wedge\langle\nu\rangle)\in\tau.\end{aligned} \tag{27}$$

Also, let $\{\lambda_i: i\in I\}$ be nonempty collection of L-fuzzy ideals of A. Then, we have $V(\bigcup\{\lambda_i: i\in I\})=\cap V(\{\lambda_i: i\in I\})$ (by the above theorem). Now,

$$\begin{aligned}\bigcup_{i\in I}X(\lambda_i)&=\bigcup_{i\in I}(X-V(\lambda_i))=X-\bigcap_{i\in I}V(\lambda_i)\\&=X-V\left(\bigcup_{i\in I}\lambda_i\right)=X\left(\bigcup_{i\in I}\lambda_i\right).\end{aligned} \tag{28}$$

Therefore, (X,τ) is a topological space. □

Definition 14. The topological space (X,τ), as in Theorem 15, is called L-fuzzy prime spectrum of an ADL A and is denoted by $\mathbf{F_L}\mathrm{spec}(A)$ or X.

In the following section, we introduce base of $\mathbf{F_L}\mathrm{spec}(A)$.

Definition 15. Let $A=(A,\wedge,\vee,0)$ be a nontrivial ADL and let X be the set of all L-fuzzy prime ideals of A. For any $a\in A$, we define $X(a)=X(\chi_{\{a\}})$. That is,

$$X(a)=\{\lambda\in X: \chi_{\{a\}}\nleq\lambda\}, \tag{29}$$

and hence

$$V(a)=\{\lambda\in X: \chi_{\{a\}}\le\lambda\}. \tag{30}$$

Theorem 16. *Let X be the set of L-fuzzy prime ideals of an ADL A. Then the following hold for any a and $b\in A$:*

(1) $X(a) \subseteq X(b) \Longleftrightarrow (a] \subseteq (b] \Longleftrightarrow b \wedge a = a$
(2) $X(a) = X(b) \Longleftrightarrow a \sim b \Longleftrightarrow (a] = (b]$
(3) $X(a) = \varnothing \Longleftrightarrow a = 0$
(4) $X(a) = X \Longleftrightarrow a$ *is maximal element in* A
(5) $X(a) \cap X(b) = X(a \wedge b) = X(b \wedge a)$
(6) $X(a) \cup X(b) = X(a \vee b) = X(b \vee a)$

Proof

$$\begin{aligned} X(a) = \varnothing &\Rightarrow X - X(a) = X \\ &\Rightarrow \chi_{(a]} \leq \lambda, \quad \text{for every } \lambda \in X \\ &\Rightarrow 1 \leq \lambda(a) \\ &\Rightarrow \lambda(a) = 1 \\ &\Rightarrow a \in \lambda_1, \quad \text{for every } \lambda \in X. \end{aligned} \tag{31}$$

Let P be a prime ideal of A. Then, χ_P is an L-fuzzy prime ideal of A. Put $\mu = \chi_P$. Then, $\mu_1 = P$. Therefore, $a \in P$, for every prime ideal P of A. Hence, $a \in \bigcup P = \{0\}$. Therefore, $a = 0$. Conversely, if $a = 0$, then there is no ideal not containing 0. It follows that there is L-fuzzy prime ideal λ not containing $\chi_{(0]}$ and hence $X(\chi_{(0]}) = \varnothing$. Therefore, $X(0) = \varnothing$.

(4) Suppose that a is maximal in A. Then, $\chi_{(a]} = \chi_A$. Then, no proper L-fuzzy ideal contains χ_A and hence $X(\chi_{(a]}) = X(\chi_A) = X$. Conversely,

$$\begin{aligned} X(a) = X &\Rightarrow \chi_{(a]} \nleq \lambda, \quad \text{for every } \lambda \in X \\ &\Rightarrow 1 \nleq \lambda(a) \\ &\Rightarrow a \notin \lambda_1. \end{aligned} \tag{32}$$

Let P be a prime ideal of A. Then, χ_P is an L-fuzzy prime ideal of A and let $\lambda = \chi_P$. Then $\lambda_1 = P$ and hence $a \notin \bigcup\{P\colon P \text{ is a prime ideal of } A\}$. Therefore, a is maximal.

(5)

$$\begin{aligned} \lambda \in X(a) \cap X(b) &\Longleftrightarrow \chi_{(a]} \nleq \lambda, \quad \text{and } \chi_{(b]} \nleq \lambda \\ &\Longleftrightarrow 1 \nleq \lambda(a), \quad \text{and } 1 \nleq \lambda(b) \\ &\Longleftrightarrow a, b \notin \lambda_1 \\ &\Longleftrightarrow a \wedge b \notin \lambda_1, \quad (\text{since } \lambda_1 \text{ is prime ideal of } A) \\ &\Longleftrightarrow 1 \nleq \lambda(a \wedge b) \\ &\Longleftrightarrow \chi_{(a \wedge b]} \nleq \lambda \\ &\Longleftrightarrow \lambda \in X(a \wedge b). \end{aligned} \tag{33}$$

Thus, $X(a) \cap X(b) = X(a \wedge b) = X(b \wedge a)$ (since $a \wedge b \sim b \wedge a$).

(6)

$$\begin{aligned} \lambda \in X(a) \cup X(b) &\Longleftrightarrow \chi_{(a]} \nleq \lambda, \quad \text{or } \chi_{(b]} \nleq \lambda, \\ &\Longleftrightarrow 1 \nleq \lambda(a), \quad \text{or } 1 \nleq \lambda(b) \\ &\Longleftrightarrow a \notin \lambda_1, \quad \text{or } b \notin \lambda_1 \\ &\Longleftrightarrow a \vee b \notin \lambda_1 \\ &\Longleftrightarrow \chi_{(a \vee b]} \nleq \lambda \\ &\Longleftrightarrow \lambda \in X(a \vee b). \end{aligned} \tag{34}$$

Thus, $X(a) \cup X(b) = X(a \vee b) = X(b \vee a)$ (since $a \vee b \sim b \vee a$). □

Theorem 17. *For any subset* S *of an ADL* A *and* $X = \mathbf{F_L}spec(A)$, *we have the following:*

(1) (1) $X(\chi_S) = \bigcup\{X(a)\colon a \in S\}$
(2) (2) $V(\chi_S) = \cap\{V(a)\colon a \in S\}$

Proof

(1) $\lambda \in X(\chi_S)$. Then, $\lambda \in X$ and $\chi_{(S]} \nleq \lambda$ and, hence, $1 \nleq \lambda(t)$, for some $t \in (S]$. Since $t \in (S], t = (\vee_{i=1}^{n}) \wedge x$, for some $x \in A$ and $a_1, a_2, \ldots, a_n \in S$. Let $a = \vee_{i=1}^{n} a_i$. Then, $a \in (S]$. Now, we shall prove that $\chi_{(a]} \nleq \lambda$. Clearly, $t \in (a]$ and hence $\chi_{(a]}(t) = 1 \nleq \lambda(t)$. Therefore, $\chi_{(a]} \nleq \lambda$ and hence $\lambda \in X(a)$. On the other hand, let $\lambda \in X$ such that $\chi_{(a]} \nleq \lambda$, for some $a \in S$. Then, $1 \nleq \lambda(t)$ for some $t \in (a]$. This implies that $\chi_{(S]} \nleq \lambda$ and hence $\lambda \in X(\chi_S)$. Thus, $X(\chi_S) = \cup_{a \in S} X(a)$.

(2)

$$\begin{aligned} V(\chi_S) &= X - X(\chi_S) \\ &= X - \bigcup\{X(a)\colon a \in S\}, \; (\text{since by } (1)) \\ &= \cap\{X - X(a)\colon a \in S\} \\ &= \cap\{V(a)\colon a \in S\}. \end{aligned} \tag{35}$$

□

Theorem 18. $\{X(a)\colon a \in S\}$ *is a base for a topology on* X.

Proof. By Theorem 16 (4), $X(m) = X$, for any maximal element m of A. Therefore, $X = \bigcup\{X(a)\colon a \in A\}$. Let $a, b \in A$. Then, for any $\lambda \in X$, we have

$$\begin{aligned} \lambda \in X(a) \cap X(b) &\Longleftrightarrow \chi_{(a]} \nleq \lambda, \quad \text{and } \chi_{(b]} \nleq \lambda, \\ &\Longleftrightarrow 1 \nleq \lambda(a), \quad \text{and } 1 \nleq \lambda(b) \\ &\Longleftrightarrow a, b \notin \lambda_1 \\ &\Longleftrightarrow a \wedge b \notin \lambda_1, \quad (\text{since } \lambda_1 \text{ is prime ideal of } A) \\ &\Longleftrightarrow 1 \nleq \lambda(a \wedge b) \\ &\Longleftrightarrow \chi_{(a \wedge b]} \nleq \lambda \\ &\Longleftrightarrow \lambda \in X(a \wedge b). \end{aligned} \tag{36}$$

Thus, $X(a) \cap X(b) = X(a \wedge b) = X(b \wedge a)$ (since $a \wedge b \sim b \wedge a$). Therefore, $\{X(a)\colon a \in A\}$ form a base for a topology on X. □

Theorem 19. *Let A be an ADL and $X = \mathbf{F_L}spec(A)$. Then, any closed subset of X is of the form $h(\chi_I)$ for some ideal I of A and any open subset of X is of the form $X(\chi_I)$ for some ideal I of A.*

Proof. For each $a \in A$, $X(a)$ is open in X and $X(\chi_I) = \{\lambda \in X\colon \chi_I \nleq \lambda\} = \cup_{a\in I} X(a)$ and hence $X(\chi_I)$ is open in X. Since $h(\chi_I) = X - X(\chi_I)$, it follows that $h(\chi_I)$ is closed in X. On the other hand, let Y be a closed subset of X. Then, $X - Y$ is open in X and hence $X - Y = \cup_{a\in S} X(a)$ for some subset S of A, since $\{X(a)\colon a \in A\}$ is a base for a topology on X. If $I = (S]$, the ideal generated by S in A, then $\chi_S \le \lambda \Longleftrightarrow \chi_I \le \lambda$ for every $\lambda \in$ X. Now, we shall prove that $X - Y = X(\chi_I)$. Let $\lambda \in X$. Then,

$$\begin{aligned} \chi_I \nleq \lambda &\Rightarrow \chi_S \nleq \lambda \\ &\Rightarrow 1 \nleq \lambda(a), \quad \text{for some } a \in A \\ &\Rightarrow \chi_{\{a\}} \nleq \lambda \\ &\Rightarrow \lambda \in X(\chi_{\{a\}}) = X(a). \end{aligned} \tag{37}$$

On the other hand,

$$\begin{aligned} \lambda \in X - Y &\Rightarrow \lambda \in X(a), \quad \text{for some } a \in S, \\ &\Rightarrow \chi_{\{a\}} \nleq \lambda, \\ &\Rightarrow \chi_S \nleq \lambda, \quad \left(\text{since } \chi_S = \cup_{a\in S} \chi_{\{a\}}\right) \\ &\Rightarrow \chi_{(S]} \nleq \lambda \\ &\Rightarrow \chi_I \nleq \lambda \\ &\Rightarrow \lambda \in X(\chi_I). \end{aligned} \tag{38}$$

Therefore, $X - Y = X(\chi_I)$ and hence $Y = h(\chi_I)$. □

Theorem 20. *Let A be an ADL and $X = \mathbf{F_L}spec(A)$. Then, for any $Y \subseteq X$, Y is a compact open subset of X if and only if $Y = X(a)$ for some $a \in A$.*

Proof. Suppose that Y is compact open. Since Y is open, $Y = \cup_{a\in S} X(a)$ for some $S \subseteq A$. Also, Y is compact and $\{X(a)\colon a \in S\}$ is a cover of Y. Then, there exists $a_1, a_2, \ldots, a_n \in S$ such that $Y = X(a_1) \cup X(a_2) \cup \cdots \cup X(a_n) = X(a)$, where $a = \vee_{i=1}^{n} a_i$.

Conversely, suppose that $Y = X(a)$ for some $a \in A$. Therefore, Y is open. Now, we prove that $X(a)$ is compact. Suppose that $X(a) \subseteq \cup_{b\in S} X(b)$, for some $S \subseteq A$. Then $a \in (S]$; otherwise, if $a \notin (S]$, then there exists a prime ideal P of A such that $(S] \subseteq P$ and $a \notin P$. Let $\lambda =$ χ_{_P}. Then, λ is an L-fuzzy prime ideal of A, since P is prime and also $\lambda_1 = P$. Since $a \notin \lambda_1, 1 \nleq \lambda(a)$, it follows that χ_{_(a]}$\nleq \lambda$ and hence $\lambda \in X(a)$ but $\lambda \notin X(b)$, for every $b \in S$, which is a contradiction to our assumption. Therefore, $a \in (S]$ and hence $a = (\vee_{i=1}^{n} b_i) \wedge x$, for some $x \in A$ and $b_1, b_2, \ldots, b_n \in S$. This implies that $X(a) \subseteq \cup_{i=1}^{n} X(b_i)$. Thus, $X(a)$ is compact. □

Corollary 2. *Every basic open set $X(a)$ in $\mathbf{F_L}spec(A)$ is compact.*

Corollary 3. *Let A be an ADL. Then, $\mathbf{F_L}spec(A)$ is compact if and only if A has a maximal element.*

Proof. Let $X = \mathbf{F_L}\mathrm{spec}(A)$. Suppose that X is compact. Then, $X = X(a)$ for some $a \in A$ (by Corollary 2) and hence a is maximal element in A. On the other hand, if A has a maximal element, say m, then $X = X(m)$, which is compact (by 2). □

Theorem 21. *For any subset S of A, S is closed in $\mathbf{F_L}spec(A)$ if and only if there exists $T \subseteq A$ such that $S = V(\chi_T)$.*

Proof. Suppose that S is closed in X. Then, $X - S$ is open in X and

$$\begin{aligned} X - S &= \cup \left\{X\left(\chi_{\{t\}}\right) | t \in T\right\}, \quad \text{for some } T \subseteq A, \\ &= \cup \left\{X - V\left(\chi_{\{t\}}\right) | t \in T\right\} \\ &= X - \cap \left\{V\left(\chi_{\{t\}}\right) | t \in T\right\} \\ &= X - V(\chi_T). \end{aligned} \tag{39}$$

Therefore, $S = V(\chi_T)$, for some $T \subseteq A$. Conversely, suppose that $S = V(\chi_T)$, for some subset T of A. Then, $V(\chi_T) = V(\cup_{t\in T} \chi_{\{t\}}) = \cup_{t\in T} V(\chi_{\{t\}})$ and hence $V(\chi_T)$ is closed in $\mathbf{F_L}\mathrm{spec}(A)$, as $V(\chi_{\{t\}})$ is closed in S. Therefore, the subset S of A is closed in $\mathbf{F_L}\mathrm{spec}(A)$. □

Theorem 22. *Let $\mathscr{P}$ denote the set of all prime ideals of A. Then the set $\mathscr{D} = \{\chi_p\colon p \in \mathscr{P}\}$ is dense in $\mathbf{F_L}spec(A)$.*

Proof. Let P be a prime ideal of A.Then χ_p is an L-fuzzy prime ideal of A. Then $D \subseteq X$. Let $\lambda \in X - D$. Let $X(\chi_{\{a\}})$ be a basic open subset of X containing λ. Now,

$$\begin{aligned} \lambda \in X\left(\chi_{\{a\}}\right) &\Rightarrow \lambda(a) \neq 1 \\ &\Rightarrow a \notin \lambda_1 \\ &\Rightarrow \chi_{\lambda_1}(a) = 0 \\ &\Rightarrow \chi_{\lambda_1} \in X\left(\chi_{\{a\}}\right). \end{aligned} \tag{40}$$

Thus, $\chi_{\lambda_1} \in D$, since λ_1 is a prime ideal of A. Therefore, $X(\chi_{\{a\}})$ contains a point of D. Thus, every member of $X - D$ is a limit point of D. Thus, $\overline{D} = X$. □

Theorem 23. *For any subset $\mathscr{S}$ of X, $\overline{\mathscr{S}} = V(\chi_{\mathscr{T}})$, where $\mathscr{T} = \cap\{\lambda_1\colon \lambda_1 \in \mathscr{S}\}$.*

Proof. Let $\mu \in S$. Then,

$$\begin{aligned} \chi_T(a) = 1 &\Rightarrow a \in T \\ &\Rightarrow a \in \mu_1 \\ &\Rightarrow \mu(a) = 1 \\ &\Rightarrow \chi_T \le \mu. \end{aligned} \tag{41}$$

Therefore, $\mu \in V(\chi_T)$. Thus, $S \subseteq V(\chi_T)$. But then $V(\chi_T)$ is closed set containing S. Hence, $\overline{S} \subseteq V(\chi_T)$. On the other hand, let $\lambda \in V(\chi_T)$. Let $\lambda \notin S$. Let $X(\chi_{\{a\}})$ be a basic open set containing λ. Then,

$$\begin{aligned}\lambda \in X\left(\chi_{\{a\}}\right) &\Rightarrow \lambda \notin V\left(\chi_{\{a\}} s\right)\\ &\Rightarrow \lambda(a) \neq 1\\ &\Rightarrow \chi_T(a) \neq 1, \quad (\text{since } \lambda \in V(\chi_T))\\ &\Rightarrow a \notin T = \cap\{\sigma_1 : \sigma_1 \in \mathcal{S}\}\\ &\Rightarrow \nu(a) \neq 1, \quad \text{for some } \nu \in S \Rightarrow \chi_{\{a\}} \nleq \nu\\ &\Rightarrow \nu \notin V\left(\chi_{\{a\}}\right)\\ &\Rightarrow \nu \in X\left(\chi_{\{a\}}\right).\end{aligned} \tag{42}$$

Thus, $\nu \in (X(\chi_{\{a\}} \cap S)) - \{\lambda\}$. This shows that λ is a limit point of S. Thus, $V(\chi_T) \subseteq \overline{S}$. Therefore, $\overline{\mathcal{S}} = V(\chi_{\mathcal{T}})$. ☐

Corollary 4. *For any subset $S \in X$, $\overline{\{\lambda\}} = V(\chi_{\lambda_1})$ for any $\mu, \nu \in X$, $\{\overline{\mu}\} = \{\overline{\nu}\}$ if and only if $\mu_1 = \nu_1$.*

First, let us recall that a topological space X is said to be T_0−space if, for any $x \neq y \in X$, there exists an open set containing x and not containing y or vice versa, that X is called a T_1−space if, for any $x \neq y \in X$, there exists open sets G and H in X such that $x \in G - H$ and $y \in H - G$, and that X is called a Hausdorff space (T_2−space) if, for any $x \neq y \in X$, there exist disjoint open sets G and H in X containing x and y, respectively.

Theorem 24. *For any ADL A, $\mathbf{F_L}$spec(A) is a T_0−space.*

Proof. Let $\mu \neq \lambda \in \mathbf{F_L}\mathrm{spec}(A)$. That is, μ and λ are distinct L-fuzzy prime ideals of A. Then, either $\mu \nleq \lambda$ or $\lambda \nleq \mu$. Suppose that $\mu \nleq \lambda$. Then, there exists $\chi_{\{a\}} \in \mu$ such that $\chi_{\{a\}} \nleq \lambda$. Then $\lambda \in X(a)$ and $\mu \notin X(a)$. Also, each $X(a)$ is open set containing λ but not μ. On the other hand, if $\lambda \nleq \mu$, then there exists an open set containing μ but not λ. Therefore, $\mathbf{F_L}\mathrm{spec}(A)$ is a T_0−space. ☐

Theorem 25. *Let A be an ADL with maximal elements. Then the following are equivalent to each other:*

(1) $\mathbf{F_L}$spec(A) is a T_2− space (or a Hausdorff space)
(2) $\mathbf{F_L}$spec(A) is a T_1− space
(3) Every L-fuzzy prime ideal of A is an L-fuzzy maximal ideal of A
(4) Every L-fuzzy prime ideal of A is an L-fuzzy minimal prime ideal of A

Proof

(1) (1)⇒(2): clear.

(2) (2)⇒(3): suppose that $\mathbf{F_L}\mathrm{spec}(A)$ is a T_1−space and μ is an L-fuzzy prime ideal of A. Since $\{\mu\}$ is closed in $\mathbf{F_L}\mathrm{spec}(A)$, $\{\overline{\mu}\} = \{\mu\}$. By Theorem 19, we have $\{\overline{\mu}\} = h(\chi_{\mu_1}) = \{\mu\}$, which implies that there is no L-fuzzy prime ideal of A containing μ other than μ itself. That is, μ is an L-fuzzy maximal ideal of A.

(3) (3)⇒(4): suppose that every L-fuzzy prime ideal of A is an L-fuzzy maximal ideal of A. Suppose that μ is an L-fuzzy prime ideal of A. Let λ be an L-fuzzy prime ideal of A such that $\lambda \leq \mu$. By assumption, $\lambda = \mu$. Therefore, μ is an L-fuzzy minimal prime ideal of A.

(4) (4)⇒(1): suppose that every L-fuzzy prime ideal of A is an L-fuzzy minimal prime ideal of A. Let $\mu \neq \lambda \in \mathbf{F_L}\mathrm{spec}(A)$. That is, μ and λ are distinct L-fuzzy prime ideals of A. Then, either $\mu \nleq \lambda$ or $\lambda \nleq \mu$. Suppose that $\mu \nleq \lambda$. Then, there exists $\chi_{\{a\}} \in \mu$ such that $\chi_{\{a\}} \nleq \lambda$. Then, $\lambda \in X(a)$ and $\mu \notin X(a)$. Also, each $X(a)$ is open set containing λ but not μ. On the other hand, if $\lambda \nleq \mu$, then there exists an open set containing μ but not λ. Hence, $a \in \mu_1$ and $a \notin \lambda_1$. By the minimality of μ_1, there exists $b \notin \mu_1$ such that $a \wedge b = 0$. Therefore, $X(a) \cap X(b) = X(a \wedge b) = \varnothing$. Hence, $\mathbf{F_L}\mathrm{spec}(A)$ is a Hausdorff space. ☐

Theorem 26. *Let A and B be ADLs and let $p\colon A \longrightarrow B$ be a lattice homomorphism. Define $g\colon \mathbf{F_L}spec(B) \longrightarrow \mathbf{F_L}spec(A)$ by $g(\lambda) = p^{-1}(\lambda)$, for each $\lambda \in \mathbf{F_L}spec(B)$. Then, (1) g is continuous mapping. (2) If p is onto, then g is one-to-one.*

Proof. Clearly g is a well-defined map.

(1) Let $h(\chi_{\{t\}})$ be any basic closed set in $\mathbf{F_L}\mathrm{spec}(A)$ and $t \in A$. Then,

$$\begin{aligned}g^{-1}\left(h\left(\chi_{\{t\}}\right)\right) &= \left\{\lambda \in \mathbf{F_L}\mathrm{spec}(B) | g(\lambda) \in h\left(\chi_{\{t\}}\right)\right\}\\ &= \left\{\lambda \in \mathbf{F_L}\mathrm{spec}(B) | \chi_{\{t\}} \leq g(\lambda)\right\}\\ &= \left\{\lambda \in \mathbf{F_L}\mathrm{spec}(B) | g(\lambda)(t) = 1\right\}\\ &= \left\{\lambda \in \mathbf{F_L}\mathrm{spec}(B) | p^{-1}(\lambda)(t) = 1\right\}\\ &= \left\{\lambda \in \mathbf{F_L}\mathrm{spec}(B) | \lambda(p(t)) = 1\right\}\\ &= \left\{\lambda \in \mathbf{F_L}\mathrm{spec}(B) | \chi_{\{p(t)\}} \leq \lambda\right\}\\ &= h\left(\chi_{\{p(t)\}}\right).\end{aligned} \tag{43}$$

Therefore, inverse image under g of any basic open set in $\mathbf{F_L}\mathrm{spec}(A)$ is a closed set in $\mathbf{F_L}\mathrm{spec}(B)$ and hence g is continuous.

(2) Let p be onto and $\lambda, \mu \in \mathbf{F_L}\mathrm{spec}(B)$ such that $g(\lambda) = g(\mu)$. Then, for each $a \in A$,

$$\begin{aligned}g(\lambda)(a) &= g(\mu)(a)\\ &\Rightarrow p^{-1}(\lambda)(a) = p^{-1}(\mu)(a)\\ &\Rightarrow \lambda(p(a)) = \mu(p(a))\\ &\Rightarrow \lambda = \mu.\end{aligned} \tag{44}$$

Therefore, g is a one-to-one map. ☐

Theorem 27. *Let $p\colon A \longrightarrow B$ be an isomorphism. Then, the space $\mathbf{F_L}spec(A)$ is homeomorphic with the space $\mathbf{F_L}spec(B)$.*

Proof. Let $p: A \longrightarrow B$ be an isomorphism. Define the function $g: \mathbf{F_L}\mathrm{spec}(B) \longrightarrow \mathbf{F_L}\mathrm{spec}(A)$ by $g(\mu) = p^{-1}(\mu)$, for each $\mu \in \mathbf{F_L}\mathrm{spec}(B)$, and $f: \mathbf{F_L}\mathrm{spec}(A) \longrightarrow \mathbf{F_L}\mathrm{spec}(B)$ by $f(\lambda) = p(\lambda)$, for each $\lambda \in \mathbf{F_L}\mathrm{spec}(A)$. Then, g and f are well defined and inverse of each other (by Theorems 8 and 9). Thus, by the above theorem, f and g are continuous. Therefore, $\mathbf{F_L}\mathrm{spec}(A)$ and $\mathbf{F_L}\mathrm{spec}(B)$ are homeomorphisms (by Definition 12). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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