Research Article

Novel Generalized Divergence Measure on Intuitionistic Fuzzy Sets and Its Application

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In the present paper, we introduce a new parametric fuzzy divergence measure on intuitionistic fuzzy sets. Some properties of the proposed measure are also being studied. In addition, the application of the intuitionistic fuzzy divergence measure in decision making and consequently choosing the best medicines and treatment for the patients has also been discussed. There are some diseases for which vaccine is not available. In that case, we have devised a method to choose the best treatment for the patients based on the results of clinical trials.

1. Introduction

Measuring the information has been one of the important topics of keen interest in the field of information theory because of its numerous applications in pattern recognition, decision making, etc. Shannon [1] was the first to argue that measure of information is essentially a measure of uncertainty, and he called it as measure of entropy. Based on probability distribution $P = (p_1, p_2, \ldots, p_n)$ in an experiment, he found that the information given by this experiment is $H(P) = -\sum_{i=1}^{n} p_i \ln p_i$ [1]. This formula for entropy given by Shannon is known as Shannon's entropy. Kullback and Leibler [2] developed the corresponding divergence measure from probability distribution $P = (p_1, p_2, \ldots, p_n)$ to $Q = (q_1, q_2, \ldots, q_n)$ as $D(P;Q) = \sum_{i=1}^{n} p_i \ln(p_i/q_i)$. After that, various other generalized measures of entropy were developed taking Shannon's entropy as the base. Later on, in 1965, Zadeh [3] introduced the concept of fuzzy sets. Corresponding to Kullback and Leibler's divergence measure [2], Bhandari and Pal [4] defined measures of fuzzy entropy and measure of fuzzy divergence. Corresponding to Harvada and Charvat [5], Hooda [6] presented a fuzzy divergence measure. Parkash et al. [7] proposed a fuzzy divergence measure corresponding to Ferreri's probabilistic

measure [8] of divergence. Thereafter, Hooda and Jain [9] offered a generalized fuzzy divergence measure. In recent years, Tomar and Ohlan [10–13] have extended fuzzy measures of information, divergence, and their generalizations. After this, Atanassov [14] proposed the concept of intuitionistic fuzzy set (IFS) which has broadened the idea of fuzzy set with the addition of degree of non-membership. As IFS theory is more efficient in taking care of the uncertainties in information, it is much better than the crisp set theory.

For this, Szmidt and Kacprzyk [15] proposed the set of axioms for the entropy under the IFS environment. Corresponding to De Luca and Termini's fuzzy entropy measure [16], Vlachos and Sergiadis [17] extended their measure in the IFS environment. Thereafter, many researchers like Junjun et al. [18], Xia and Xu [19], Wei and Ye [20], Zhang and Jiang [21], and Hung and Yang [22] have defined the divergence measures for IFS, and the findings are applied in variety of fields. However, Wei and Ye [20] pointed out the downside of Vlachos and Sergiadis' measure [17] and modified the measure of Vlachos and Sergiadis. Later on, Verma and Sharma [23] improvised the divergence measure of Wei and Ye. Harish Garg et al. [24] proposed parametric version of intuitionistic fuzzy divergence measure given by Verma and Sharma [23]. In 2016, Maheshwari and

Srivastava [25] pointed that that the measures given by Vlachos and Sergiadis [17], Zhang and Jiang [21], Junjun et al. [18] do not satisfy the basic requirement of non-negativity of the intuitionistic fuzzy divergence measure. Ohlan [26] extended the idea of Fan and Xie [27] and proposed an intuitionistic fuzzy exponential divergence measure.

In the present study, we suggest a new intuitionistic fuzzy divergence measure. Afterwards, we examine its properties and provide an illustration of how it can help in choosing the best medical treatment for the patients.

2. Preliminaries

We start by reviewing some well-known concepts and definitions related to fuzzy set theory and intuitionistic fuzzy set theory. Since we use the proposed notions in applications, we provide all concepts for the finite universe X.

Definition 1. Fuzzy set [3]: a fuzzy set defined on a finite universe of discourse $X = \{x_1 x_2, \ldots, x_n\}$ is given as

$$R = \{(x, \mu_R(x)): x \in X\}, \tag{1}$$

where $\mu_R: X \longrightarrow [0, 1]$ is the membership function of R. The membership value $\mu_R(x)$ gives the degree of the belongingness of $x \in X$ in R when $\mu_R(x)$ is valued in [0, 1].

Atanassov [14] introduced the concept of intuitionistic fuzzy set (IFS) as the generalization of the concept of fuzzy set.

Definition 2. Intuitionistic fuzzy set (IFS) [14]: an intuitionistic fuzzy set (IFS) R on a finite universe of discourse $X = \{x_1 x_2, \ldots, x_n\}$ is defined as

$$R = \{(x, \mu_R(x), \nu_R(x)): x \in X\}, \tag{2}$$

where $\mu_R: X \longrightarrow [0, 1]$, $\nu_R: X \longrightarrow [0, 1]$, and $0 \le \mu_R + \nu_R \le 1, \forall x \in X$.

$\mu_R(x)$ denotes the degree of membership and $\nu_R(x)$ denotes the degree of non-membership of x in R. For each IFS R in X, we will call $\pi_R(x) = 1 - \mu_R(x) - \nu_R(x) = 1 - \mu_R(x) - \nu_R(x)$, the hesitation degree of x in R. It is obvious that $0 \le \pi_R(x) \le 1$ for each $x \in X$.

Atanassov [14] further introduced the set operations on IFSs as follows.

Let R and $S \in \text{IFS}(X)$ be the family of all IFSs in the universe X, given by

$$\begin{aligned} R &= \{(x, \mu_R(x), \nu_R(x)): x \in X\}, \\ S &= \{(x, \mu_S(x), \nu_S(x)): x \in X\}. \end{aligned} \tag{3}$$

(i) $R \subseteq S$ iff $\mu_R(x) \le \mu_S(x)$ and $\nu_R(x) \ge \nu_S(x), \forall x \in X$.

(ii) $R = S$ iff $R \subseteq S$ and $S \subseteq R$.

(iii) $R^c = \{(x, \nu_R(x), \mu_R(x))|, x \in X\}$.

(iv) $R \cup S = \{(x, \max(\mu_R(x), \mu_S(x)), \min(\nu_R(x), \nu_S(x)))|x \in X\}$.

(v) $R \cap S = \{(x, \min(\mu_R(x), \mu_S(x)), \max(\nu_R(x), \nu_S(x)))|x \in X\}$.

Definition 3. Let R and S be two intuitionistic fuzzy sets in X. A mapping D: $\text{IFS}(X) \times \text{IFS}(X) \longrightarrow R$ is a divergence measure for IFS if it satisfies the following axioms [17]:

(D1) $D(R:S) \ge 0$.

(D2) $D(R:S) = 0$ if and only if $R = S$.

(D3) $D(R:S) = D(R^c:S^c)$.

In 1967, Harvda and Charvat [5] defined the measure of divergence of a probability distribution $P = (p_1 p_2, \ldots, p_n)$ from another probability distribution $Q = (q_1 q_2, \ldots, q_n)$ as

$$D_\alpha(P; Q) = \frac{1}{\alpha - 1} \sum_{i=1}^{n} \left(p_i^\alpha q_i^{1-\alpha} - 1\right), \quad \alpha > 0, \alpha \ne 1. \tag{4}$$

It is known as the generalized divergence of degree α. Then, in 1959, Kullback and Leibler [2] proposed the following measure of symmetric divergence:

$$J_\alpha(P: Q) = \frac{1}{\alpha - 1} \sum_{i=1}^{n} \left(p_i^\alpha q_i^{1-\alpha} + q_i^\alpha p_i^{1-\alpha} - 2\right), \quad \alpha > 0, \alpha \ne 1, \tag{5}$$

which is also known as distance measure of degree α.

Corresponding to measures (4) and (5), Hooda [6] suggested the following measures of fuzzy divergence:

$$I_\alpha(R: S) = \frac{1}{\alpha - 1} \sum_{i=1}^{n} \left[\mu_R^\alpha(x_i) \mu_S^{1-\alpha}(x_i) + (1 - \mu_R(x_i))^\alpha (1 - \mu_S(x_i))^{1-\alpha} - 1\right], \tag{6}$$

$$J_\alpha(R: S) = I_\alpha(R: S) + I_\alpha(S: R).$$

Now, corresponding to Hooda [6], we define a new measure of intuitionistic fuzzy divergence.

3. New Parametric Divergence Measure on IFS

Let R and S be two IFSs defined on universal set $X = \{x_1 x_2, \ldots, x_n\}$; then, a parametric intuitionistic fuzzy divergence measure based on parameter α is defined as follows:

$$
D_\alpha^{\text{IFS}}(R\colon S) = \frac{1}{\alpha - 1} \sum_{i=1}^n \left[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \right.
$$
$$
\left. + \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} - 1 \right],
$$

(7)

where $\alpha > 0$, $\alpha \neq 1$.

Since $D_\alpha^{\text{IFS}}(R\colon S) \neq D_\alpha^{\text{IFS}}(S\colon R)$, the above proposed measure is not symmetric. In order to imbue the symmetry property, we define $J_\alpha^{\text{IFS}}(R\colon S) = D_\alpha^{\text{IFS}}(R\colon S) + D_\alpha^{\text{IFS}}(S\colon R)$.

$$
J_\alpha^{\text{IFS}}(R\colon S) = \frac{1}{\alpha - 1} \sum_{i=1}^n \left[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \right.
$$
$$
+ \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha}
$$
$$
+ \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha}
$$
$$
\left. + \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} - 2 \right],
$$

(8)

where $\alpha > 0$, $\alpha \neq 1$.

3.1. Validity Proof of the Defined Measure of Divergence

Theorem 1. $J_\alpha^{IFS}(R\colon S)$ *is a valid measure of directed divergence on IFS, i.e., it satisfies*

$$
J_\alpha^{\text{IFS}}(R\colon S) \geq 0,
$$
$$
J_\alpha^{\text{IFS}}(R\colon S) = 0, \quad \text{iff } R = S,
$$
$$
J_\alpha^{\text{IFS}}(R\colon S) = J_\alpha^{\text{IFS}}(R^c\colon S^c).
$$

(9)

Proof. In [6], it is proved that $I_\alpha(R\colon S) \geq 0$ and vanishes only when $R = S$.

As

$$
0 \leq \frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \leq 1,
$$

(10)

the result holds for D_α^{IFS} also. Therefore, $D_\alpha^{\text{IFS}}(R\colon S) \geq 0$, and hence $J_\alpha^{\text{IFS}}(R\colon S) \geq 0$.

Similarly, $J_\alpha^{\text{IFS}}(R\colon S) = 0$ iff $R = S$.

$$
J_\alpha^{\text{IFS}}(R^c\colon S^c) = \frac{1}{\alpha - 1} \sum_{i=1}^n \left[\left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} \right.
$$
$$
+ \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha}
$$
$$
+ \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha}
$$
$$
\left. + \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha} - 2 \right]
$$
$$
= J_\alpha^{\text{IFS}}(R\colon S).
$$

(11)

3.2. Properties of the Proposed Intuitionistic Fuzzy Divergence Measure. For the proofs of the properties, we need to separate X into X_1 and X_2 such that

$$
\begin{aligned}
X_1 &= \{x_i \mid x_i \in X, R(x_i) \subseteq S(x_i)\}, \\
X_2 &= \{x_i \mid x_i \in X, R(x_i) \supseteq S(x_i)\}.
\end{aligned}
\tag{12}
$$

Therefore, for all $x_i \in X_1$, we have

$$
\begin{aligned}
\mu_R(x_i) &\le \mu_S(x_i), \\
\nu_R(x_i) &\ge \nu_S(x_i),
\end{aligned}
\tag{13}
$$

and for all $x_i \in X_2$, we have

$$
\begin{aligned}
\mu_R(x_i) &\ge \mu_S(x_i), \\
\nu_R(x_i) &\le \nu_S(x_i).
\end{aligned}
\tag{14}
$$

Theorem 2. *Let R, S, T be an IFS on universal set* $X = X = \{x_1, x_2, \ldots, x_n\}$*; then, the proposed measure given by (8) satisfies the following properties:*

(a) $J_\alpha^{\text{IFS}}(R \cup S : R \cap S) = J_\alpha^{\text{IFS}}(R : S)$.

(b) $J_\alpha^{\text{IFS}}(R : R \cup S) = J_\alpha^{\text{IFS}}(S : R \cap S)$.

(c) $J_\alpha^{\text{IFS}}(R : R \cap S) = J_\alpha^{\text{IFS}}(S : R \cup S)$.

(d) $J_\alpha^{\text{IFS}}(R : R \cup S) + J_\alpha^{\text{IFS}}(R : R \cap S) = J_\alpha^{\text{IFS}}(R : S)$.

(e) $J_\alpha^{\text{IFS}}(R \cup S : T) \le J_\alpha^{\text{IFS}}(R : T) + J_\alpha^{\text{IFS}}(S : T)$.

(f) $J_\alpha^{\text{IFS}}(R \cap S : T) \le J_\alpha^{\text{IFS}}(R : T) + J_\alpha^{\text{IFS}}(S : T)$.

(g) $J_\alpha^{\text{IFS}}(R \cap S : T) + J_\alpha^{\text{IFS}}(R \cup S : T) = J_\alpha^{\text{IFS}}(R : T) + J_\alpha^{\text{IFS}}(S : T)$.

(h) $J_\alpha^{\text{IFS}}(R : S^c) = J_\alpha^{\text{IFS}}(R^c : S)$.

Proof. (a)

$$
\begin{aligned}
J_\alpha^{\text{IFS}}(R \cup S : R \cap S) = \frac{1}{\alpha-1} &\left[\sum_{i=1}^{n} \left[\left(\frac{\mu_{R \cup S}(x_i) + 1 - \nu_{R \cup S}(x_i)}{2} \right)^\alpha \left(\frac{\mu_{R \cap S}(x_i) + 1 - \nu_{R \cap S}(x_i)}{2} \right)^{1-\alpha} \right. \right. \\
&\left. + \left(\frac{\nu_{R \cup S}(x_i) + 1 - \mu_{R \cup S}(x_i)}{2} \right)^\alpha \left(\frac{\nu_{R \cap S}(x_i) + 1 - \mu_{R \cap S}(x_i)}{2} \right)^{1-\alpha} \right. \\
&\left. + \left(\frac{\mu_{R \cap S}(x_i) + 1 - \nu_{R \cap S}(x_i)}{2} \right)^\alpha \left(\frac{\mu_{R \cup S}(x_i) + 1 - \nu_{R \cup S}(x_i)}{2} \right)^{1-\alpha} \right. \\
&\left. \left. + \left(\frac{\nu_{R \cap S}(x_i) + 1 - \mu_{R \cap S}(x_i)}{2} \right)^\alpha \left(\frac{\nu_{R \cup S}(x_i) + 1 - \mu_{R \cup S}(x_i)}{2} \right)^{1-\alpha} \right] \right],
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{\alpha-1} &\left[\sum_{x_i \in X_1} \left[\left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha} \right. \right. \\
&+ \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} \\
&+ \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \\
&\left. + \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} - 2 \right] \\
&+ \sum_{x_i \in X_2} \left[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \right. \\
&+ \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} \\
&+ \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha} \\
&\left. \left. + \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} - 2 \right] \right]
\end{aligned}
\tag{15}
$$

$$
= J_\alpha^{\text{IFS}}(R : S).
$$

(b)

$$
\begin{aligned}
J_\alpha^{\mathrm{IFS}}(R:R\cup S) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\Bigg[& \left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^\alpha\left(\frac{\mu_{R\cup S}(x_i)+1-\nu_{R\cup S}(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^\alpha\left(\frac{\nu_{R\cup S}(x_i)+1-\mu_{R\cup S}(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\mu_{R\cup S}(x_i)+1-\nu_{R\cup S}(x_i)}{2}\right)^\alpha\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\nu_{R\cup S}(x_i)+1-\mu_{R\cup S}(x_i)}{2}\right)^\alpha\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^{1-\alpha} -2\Bigg],
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
= \frac{1}{\alpha-1}\Bigg[& \sum_{x_i\in X_1}\Bigg[\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^\alpha\left(\frac{\mu_S(x_i)+1-\nu_S(x_i)}{2}\right)^{1-\alpha}\Bigg] \\
& +\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^\alpha\left(\frac{\nu_S(x_i)+1-\mu_S(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\mu_S(x_i)+1-\nu_S(x_i)}{2}\right)^\alpha\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\nu_S(x_i)+1-\mu_S(x_i)}{2}\right)^\alpha\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^{1-\alpha}-2\Bigg]+\sum_{x_i\in X_2}[0]\Bigg],
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
J_\alpha^{\mathrm{IFS}}(S:R\cap S) = \frac{1}{\alpha-1}\Bigg[& \sum_{x_i\in X_1}\Bigg[\left(\frac{\mu_S(x_i)+1-\nu_S(x_i)}{2}\right)^\alpha\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^{1-\alpha}\Bigg] \\
& +\left(\frac{\nu_S(x_i)+1-\mu_S(x_i)}{2}\right)^\alpha\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^\alpha\left(\frac{\mu_S(x_i)+1-\nu_S(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^\alpha\left(\frac{\nu_S(x_i)+1-\mu_S(x_i)}{2}\right)^{1-\alpha}-2\Bigg]+\sum_{x_i\in X_2}(0)\Bigg].
\end{aligned}
\tag{18}
$$

From (17) and (18),

$$
J_\alpha^{\mathrm{IFS}}(R:R\cup S)=J_\alpha^{\mathrm{IFS}}(S:R\cap S).
\tag{19}
$$

(c)

$$
\begin{aligned}
J_\alpha^{\mathrm{IFS}}(R:R\cap S) = \frac{1}{\alpha-1}\Bigg[& \sum_{x_i\in X_1}[0]+\sum_{x_i\in X_2}\Bigg[\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^\alpha\left(\frac{\mu_S(x_i)+1-\nu_S(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^\alpha\left(\frac{\nu_S(x_i)+1-\mu_S(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\mu_S(x_i)+1-\nu_S(x_i)}{2}\right)^\alpha\left(\frac{\mu_R(x_i)+1-\nu_R(x_i)}{2}\right)^{1-\alpha} \\
& +\left(\frac{\nu_S(x_i)+1-\mu_S(x_i)}{2}\right)^\alpha\left(\frac{\nu_R(x_i)+1-\mu_R(x_i)}{2}\right)^{1-\alpha}-2\Bigg]\Bigg] \\
= & \; J_\alpha^{IFS}(S:R\cup S).
\end{aligned}
\tag{20}
$$

(d) Adding (17) and (20), we get   (e) It suffices to show that

$$J_\alpha^{\mathrm{IFS}}(R: R \cup S) + J_\alpha^{\mathrm{IFS}}(R: R \cap S) = J_\alpha^{\mathrm{IFS}}(R: S). \qquad (21)$$

$$J_\alpha^{\mathrm{IFS}}(R: T) + J_\alpha^{\mathrm{IFS}}(S: T) - J_\alpha^{\mathrm{IFS}}(R \cup S: T) \geq 0,$$

$$J_\alpha^{\mathrm{IFS}}(R: T) = \frac{1}{\alpha - 1} \sum_{i=1}^{n} \left[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^{1-\alpha} \right.$$

$$+ \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^{1-\alpha}$$

$$+ \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha}$$

$$\left. + \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} - 2 \right],$$

$$J_\alpha^{\mathrm{IFS}}(S: T) = \frac{1}{\alpha - 1} \sum_{i=1}^{n} \left[\left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^{1-\alpha} \right.$$

$$+ \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^{1-\alpha}$$

$$+ \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha}$$

$$\left. + \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} - 2 \right],$$

$$J_\alpha^{\mathrm{IFS}}(R: T) + J_\alpha^{\mathrm{IFS}}(S: T) - J_\alpha^{\mathrm{IFS}}(R \cup S: T) \qquad (22)$$

$$= \frac{1}{\alpha - 1} \left[\sum_{x_i \in X_2} \left[\left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^{1-\alpha} \right. \right.$$

$$+ \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^{1-\alpha}$$

$$+ \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha}$$

$$\left. \cdot \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} - 2 \right]$$

$$+ \sum_{x_i \in} \left[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^{1-\alpha} \right.$$

$$+ \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^{1-\alpha}$$

$$= \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha}$$

$$\left. \left. + \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} - 2 \right] \right]$$

$$\geq 0.$$

(f) The proof of (f) is similar to (e). (g)

$$
\begin{aligned}
J_\alpha^{\text{IFS}} &(R \cap S\colon T) \\
&= \frac{1}{\alpha - 1} \Bigg[\sum_{x_i \in X_1} \Bigg[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_T(x_{ii}) + 1 - \mu_T(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} - 2 \Bigg] \\
&\quad + \sum_{x_i \in X_2} \Bigg[\left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\mu_T(x_i) + 1 - \nu_T(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_T(x_i) + 1 - \mu_T(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} - 2 \Bigg] \Bigg].
\end{aligned}
\tag{23}
$$

Add (h)

$$
J_\alpha^{\text{IFS}} (R \cap S\colon T) + J_\alpha^{\text{IFS}} (R \cup S\colon T) = J_\alpha^{\text{IFS}} (R\colon T) + J_\alpha^{\text{IFS}} (S\colon T).
\tag{24}
$$

$$
\begin{aligned}
J_\alpha^{\text{IFS}} (R\colon S^c) &= \frac{1}{\alpha - 1} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} - 2 \Bigg], \\
J_\alpha^{\text{IFS}} (R^c\colon S) &= \frac{1}{\alpha - 1} \sum_{i=1}^{n} \Bigg[\left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^\alpha \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^\alpha \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\mu_S(x_i) + 1 - \nu_S(x_i)}{2} \right)^\alpha \left(\frac{\nu_R(x_i) + 1 - \mu_R(x_i)}{2} \right)^{1-\alpha} \\
&\quad + \left(\frac{\nu_S(x_i) + 1 - \mu_S(x_i)}{2} \right)^\alpha \left(\frac{\mu_R(x_i) + 1 - \nu_R(x_i)}{2} \right)^{1-\alpha} - 2 \Bigg] \\
\therefore J_\alpha^{\text{IFS}} &(R\colon S^c) = J_\alpha^{\text{IFS}} (R^c\colon S).
\end{aligned}
\tag{25}
$$

4. Drawbacks of Other Measures of Divergence for IFS

Vlachos and Sergiadis [17] defined the following measure of divergence:

$$D_1(R\colon S) = \sum_{i=1}^{n} \mu_R(x_i)\ln\left(\frac{\mu_R(x_i)}{(1/2)(\mu_R(x_i) + \mu_S(x_i))}\right) + \nu_R(x_i)\ln\left(\frac{\nu_R(x_i)}{(1/2)(\nu_R(x_i) + \nu_S(x_i))}\right). \tag{26}$$

Junjun et al. [18] defined

$$D_2(R\colon S) = \sum_{i=1}^{n} \pi_R(x_i)\ln\left(\frac{\pi_R(x_i)}{(1/2)(\pi_R(x_i) + \pi_S(x_i))}\right) + \Delta_R(x_i)\ln\left(\frac{\Delta_R(x_i)}{(1/2)(\Delta_R(x_i) + \Delta_S(x_i))}\right), \tag{27}$$

where $\Delta_R(x_i) = |\mu_R(x_i) - \nu_R(x_i)|$.

Example 1. Consider the IFS given by

$$R = \{\langle x_1, 0.44, 0.385\rangle, \langle x_2, 0.43, 0.39\rangle, \langle x_3, 0.42, 0.38\rangle\},$$
$$S = \{\langle x_1, 0.34, 0.48\rangle, \langle x_2, 0.37, 0.46\rangle, \langle x_3, 0.38, 0.45\rangle\}. \tag{28}$$

Corresponding to R and S, we get

$$\begin{aligned} D_1(R\colon S) &= -0.00712, \\ D_2(R\colon S) &= -0.04547. \end{aligned} \tag{29}$$

Therefore, the measures given by (26) and (27) do not satisfy the axiom of non-negativity.

5. Application

A lot of uncertainty and fuzziness is associated with most of the decisions in medical science. There are various diseases like migraine, depression, and many viral diseases for which vaccine is not available or 100 percent cure is not available. One of the current viral diseases which is known to affect more than 50,00,000 lives in the whole world is COVID-19. On 11th March 2020, the WHO officially declared the outbreak of COVID-19 as pandemic. Still, no vaccine or 100 percent effective medicine is invented for its cure. In such cases, doctors have to choose the medicines already available in the market. For example, in the case of COVID-19, medicines already available are remdesivir, favipiravir, and hydroxychloroquine. Now, the following question arises: which medicine or treatment is most effective among the available ones? This kind of decision-making problem can be solved by IFS theory. IFS theory makes it possible to define the medical information in terms of intuitionistic fuzzy sets and thereby apply the decision-making process discussed below to choose the best treatment or medicine.

Now, we describe the decision-making process in intuitionistic fuzzy sets.

Let $M = \{M_1, M_2, M_3.....M_p\}$ be the set of p alternatives to be examined under the set of q criterion set $N = \{N_1, N_2, N_3......N_q\}$. Then, we take following steps to choose the best alternative.

Step 1. Constructing the decision-making matrix: let μ_{ij} denote the degree of the alternative M_i satisfying the criteria N_j and ν_{ij} denote the degree of the alternative M_i not satisfying the criteria N_j. Then, the decision-making matrix is a $p \times q$ matrix whose entries $\langle x_{ij}\rangle$ are

$$\langle \mu_{ij}, \nu_{ij}\rangle, \quad i = 1, 2, \ldots, p,$$
$$j = 1, 2, \ldots, q, \quad \text{where } \mu_{ij}\nu_{ij} \in [0, 1] \text{ and } \mu_{ij} + \nu_{ij} \le 1. \tag{30}$$

Step 2. Computing the ideal alternative: find the ideal alternative M^* as

$$M^* = \left\{\langle \mu_1^*, \nu_1^*\rangle, \langle \mu_2^*, \nu_2^*\rangle, \ldots, \langle \mu_q^*, \nu_q^*\rangle\right\}, \tag{31}$$

where $\mu_j^* = \max_{1 \le i \le p}\mu_{ij}$ and $\nu_j^* = \min_{1 \le i \le q}\nu_{ij}$.

Step 3. Calculating the value of proposed divergence measure: find $J_\alpha^{IFS}(M_i\colon M^*)$ using the formula

TABLE 1: Normalized intuitionistic fuzzy decision matrix.

	A_1	A_2	A_3	A_4
T_1	<0.7, 0.2>	<0.8, 0.1>	<0.5, 0.3>	<0.6, 0.3>
T_2	<0.9, 0.1>	<0.8, 0.2>	<0.5, 0.3>	<0.8, 0.2>
T_3	<0.6, 0.4>	<0.5, 0.3>	<0.6, 0.3>	<0.5, 0.4>
T_4	<0.8, 0.1>	<0.7, 0.2>	<0.7, 0.2>	<0.8, 0.2>

TABLE 2: Values of $J_\alpha^{\text{IFS}}(T_i: M^*)$ for different values of α.

	$\alpha = 0.5$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$	Rank
$J_\alpha^{\text{IFS}}(T_1: M^*)$	0.1903	0.8233	1.4254	2.3755	6.606	3rd
$J_\alpha^{\text{IFS}}(T_2: M^*)$	0.0605	0.2490	0.3928	0.5651	0.7817	2nd
$J_\alpha^{\text{IFS}}(T_3: M^*)$	0.5948	2.8656	6.0696	13.8161	35.305	4th
$J_\alpha^{\text{IFS}}(T_4: M^*)$	0.0432	0.1791	0.2866	0.4209	0.5980	1st

$$
\begin{aligned}
J_\alpha^{IFS}(M_i: M^*) = {} & \frac{1}{\alpha - 1} \sum_{j=1}^{q} \left(\frac{\mu_{ij}(x_{ij}) + 1 - \nu_{ij}(x_{ij})}{2} \right)^\alpha \left(\frac{\mu_{j^*}(x_{ij}) + 1 - \nu_{j^*}(x_{ij})}{2} \right)^{1-\alpha} \Bigg] \\
& + \left(\frac{\nu_{ij}(x_{ij}) + 1 - \mu_{ij}(x_{ij})}{2} \right)^\alpha \left(\frac{\nu_{j^*}(x_{ij}) + 1 - \mu_{j^*}(x_{ij})}{2} \right)^{1-\alpha} \\
& + \left(\frac{\mu_{j^*}(x_{ij}) + 1 - \nu_{j^*}(x_{ij})}{2} \right)^\alpha \left(\frac{\mu_{ij}(x_{ij}) + 1 - \nu_{ij}(x_{ij})}{2} \right)^{1-\alpha} \\
& + \left(\frac{\nu_{j^*}(x_{ij}) + 1 - \mu_{j^*}(x_{ij})}{2} \right)^\alpha \left(\frac{\nu_{ij}(x_{ij}) + 1 - \mu_{ij}(x_{ij})}{2} \right)^{1-\alpha} - 2 \Bigg], \quad \forall i = 1, 2, \ldots, p.
\end{aligned}
\tag{32}
$$

Step 4. Ranking the alternatives: give ranking to all the alternatives. The alternative whose $J_\alpha^{IFS}(M_i: M^*)$ is minimum will be considered as the best alternative with rank 1.

We demonstrate the application of proposed divergence method in choosing the best medical treatment for the patients.

Suppose a person is suffering from anxiety. A doctor tried different treatments $T_1, T_2, T_3, T_4 T_1, T_2, T_3, T_4$ on 4 different subjects A_1, A_2, A_3, A_4 and noted the results of clinical trials (see Table 1).

Based on this normalized intuitionistic fuzzy decision matrix, we find the ideal alternative

$$
M^* = \{\langle 0.9, 0.1 \rangle, \langle 0.8, 0.1 \rangle, \langle 0.7, 0.2 \rangle, \langle 0.8, 0.2 \rangle\}. \tag{33}
$$

Now, we calculate the value of divergence of each treatment T_i from the ideal alternative M^* for different values of α.

According to Table 2, T_4 is the best treatment for the patients of anxiety. The order of preference of treatments is as follows:

$$
T_4 \succ T_2 \succ T_1 \succ T_3. \tag{34}
$$

6. Conclusion

In this paper, we have proposed a new parametric intuitionistic fuzzy divergence measure for IFS with its proof of validity. The proposed measure is found to satisfy various properties and does not assume any negative value as in case of many existing divergence measures. The proposed divergence measure also has application in decision making and thereby deciding the best medical treatment for the patients. Furthermore, the parameter provides flexibility in criteria for decision making.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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