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Research Article

Approximate Solution of Intuitionistic Fuzzy Differential Equations with the Linear Differential Operator by the Homotopy Analysis Method

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In this work, the purpose is to discuss the homotopy analysis method (HAM) for the use of intuitionistic fuzzy differential equations with the linear differential operator. Furthermore, a numerical example is presented to shed light on the capability of the present method, and the numerical results illustrated by adopting the homotopy perturbation method (HPM) are compared with the exact solution to ensure the validity of our outcomes.

1. Introduction

Intuitionistic fuzzy sets theory plays a major key role in different domains such as industry, audiovisual systems, robotics, the control of complex processes, the transmission of energy in a material medium in various forms, and the evolution of certain populations and organisms. The notion of intuitionistic fuzzy theory (IFT) was first mentioned by Atanassov [1–3] as a generalization of Zadeh's fuzzy sets [4]. The concept of the intuitionistic fuzzy metric space was introduced by Melliani et al. [5]. The authors in [6] constructed the existence and uniqueness theorem of a solution to the nonlocal intuitionistic fuzzy differential equation. The study of numerical methods for solving intuitionistic fuzzy differential equations has been rapidly growing in recent years. It is difficult to obtain exact solutions for intuitionistic fuzzy DEs, and hence, some numerical methods presented in [7–11]. In [12–14], the authors gave a thorough and systematic introduction to the latest research achievement on the theories of interval-valued intuitionistic fuzzy sets and their applications to multiattribute group decision-making (MAGDM). Also, some arithmetic aggregation operators for the triangular Atanassov intuitionistic fuzzy number (TAIFN) are defined in [15]. The first publication on intuitionistic fuzzy partial differential equations was [23].

In this paper, we well resolve the intuitionistic fuzzy differential equation using an analytical method called homotopy analysis method (HAM). This approach was first set by Liao in 1992 [18, 19]. Numerous authors used this method to resolve different linear and nonlinear differential equations for the benefit of many practical use cases in scientific and engineering problems [19–23], and the homotopy analysis method rapidly converges in many linear and nonlinear problems. The principal benefit of the homotopy analysis method (HAM) is the applicability to give an approximate and exact solution to linear and nonlinear problems, without the necessity of discretization and linearization as in the numerical methods. The structure of this paper is organized as follows.

After discussing the motivation behind this research in the introduction section, Section 2 is intended to give the basic notion of intuitionistic fuzzy sets (IFS) and intuitionistic fuzzy numbers (IFN). Section 3 is dedicated to present some basic notions about the homotopy method. For the sake of clarity, the homotopy perturbation method for the use of resolving the intuitionistic fuzzy differential equations with the linear differential operator is presented. In Section 4, we give an example to illustrate the capability and flexibility of the proposed method, and finally, conclusion is given in Section 5.

2. Preliminaries

In this section, we present the necessary definitions and notations that will be used in this work as follows.

2.1. Intuitionistic Fuzzy Sets. An intuitionistic fuzzy set $A \in X$ is given by

$$A = \{(x, \kappa_A(x), \omega_A(x)) | x \in X\}, \tag{1}$$

where the function $\kappa_A(x), \omega_A(x)\colon X \longrightarrow [0,1]$ defines, respectively, the degree of membership and degree of nonmembership of the element $x \in X$ to set A, which is a subset of X, which satisfies for every $x \in X$, $0 \leq \kappa(x) + \omega(x) \leq 1$.

For the sake of clarity, every fuzzy set has the form

$$\{(x, \kappa_A(x), \kappa_{A^c}(x)) | x \in X\}. \tag{2}$$

For each intuitionistic fuzzy set $A \in X$, we will call

$$\pi_A(x) = 1 - \kappa_A(x) - \omega_A(x). \tag{3}$$

The intuitionistic fuzzy index of $x \in A$ verifies that $0 \leq \pi_A(x) \leq 1$.

2.2. Intuitionistic Fuzzy Numbers. An element $\langle \kappa, \omega \rangle$ of $\mathbb{F}_1$ is said to be an intuitionistic fuzzy number if it satisfies the following conditions:

(i) $\langle \kappa, \omega \rangle$ is normal, i.e., there exist $x_0, x_1 \in \mathbb{R}$ such that $\kappa(x_0) = 1$ and $\omega(x_1) = 1$

(ii) The membership function κ is fuzzy convex, i.e., $\kappa(\eta x_1 + (1-\eta)x_2) \geq \min(\kappa(x_1), \kappa(x_2))$

(iii) The nonmembership function ω is fuzzy concave, i.e., $\omega(\eta x_1 + (1-\eta)x_2) \leq \max(\omega(x_1), \omega(x_2))$

(iv) κ is upper semicontinuous, and ω is lower semicontinuous

(v) $\mathrm{Supp}\langle \kappa, \omega \rangle = \mathrm{cl}\{x \in \mathbb{R}\colon |\omega(x) < 1\}$ is bounded

So, we denote the collection of all intuitionistic fuzzy numbers by $\mathbb{F}_1$.

For $\alpha \in [0,1]$ and $\langle \kappa, \omega \rangle \in \mathbb{F}^1$, the upper and lower α-cuts of $\langle \kappa, \omega \rangle$ are defined by

$$\begin{aligned} [\langle \kappa, \omega \rangle]^\alpha &= \{x \in \mathbb{R}\colon \omega(x) \leq 1 - \alpha\}, \\ [\langle \kappa, \omega \rangle]_\alpha &= \{x \in \mathbb{R}\colon \kappa(x) \geq \alpha\}. \end{aligned} \tag{4}$$

Remark 1. If $\langle \kappa, \omega \rangle \in \mathbb{F}_1$, we can see $[\langle \kappa, \omega \rangle]_\alpha$ as $[\kappa]^\alpha$ and $[\langle \kappa, \omega \rangle]^\alpha$ as $[1-\omega]^\alpha$ in the fuzzy case.

We define $0_{(1,0)} \in \mathbb{F}_1$ as

$$0_{(1,0)}(t) = \begin{cases} (1,0), & t = 0, \\ (0,1), & t \neq 0. \end{cases} \tag{5}$$

Let $\langle \kappa, \omega \rangle, \langle \kappa', \omega' \rangle \in \mathbb{F}_1$ and $\eta \in \mathbb{R}$; we define the following operations by

$$\begin{aligned} (\langle \kappa, \omega \rangle \oplus \langle \kappa', \omega' \rangle)(z) &= \sup_{z=x+y} \min(\kappa(x), \kappa'(y)), \inf_{z=x+y} \max(\omega(x), \omega'(y)), \\ \eta \langle \kappa, \omega \rangle &= \begin{cases} \langle \eta\kappa, \eta\omega \rangle & \text{if } \eta \neq 0, \\ 0_{(1,0)} & \text{if } \eta = 0. \end{cases} \end{aligned} \tag{6}$$

For $\langle \kappa, \omega \rangle$, $\langle \chi, \Psi \rangle \in \mathbb{F}_1$ and $\eta \in \mathbb{R}$, the addition and scalar multiplication are defined as follows:

$$\begin{aligned} [\langle \kappa, \omega \rangle \oplus \langle \chi, \Psi \rangle]^\alpha &= [\langle \kappa, \omega \rangle]^\alpha + [\langle \chi, \Psi \rangle]^\alpha, \\ [\eta \langle \chi, \Psi \rangle]^\alpha &= \eta [\langle \chi, \Psi \rangle]^\alpha, \\ [\langle \kappa, \omega \rangle \oplus \langle \chi, \Psi \rangle]_\alpha &= [\langle \kappa, \omega \rangle]_\alpha + [\langle \chi, \Psi \rangle]_\alpha, \\ [\eta \langle \chi, \Psi \rangle]_\alpha &= \eta [\langle \chi, \Psi \rangle]_\alpha. \end{aligned} \tag{7}$$

Definition 1. Let $\langle \kappa, \omega \rangle$ be an element of $\mathbb{F}_1$ and $\alpha \in [0.1]$; we define the following sets:

$$\begin{aligned} &[\langle \kappa, \omega \rangle]_l^+(\alpha) = \inf\{x \in \mathbb{R} | \kappa(x) \geq \alpha\}, \\ &[\langle \kappa, \omega \rangle]_r^+(\alpha) = \sup\{x \in \mathbb{R} | \kappa(x) \geq \alpha\} [\langle \kappa, \omega \rangle]_l^-(\alpha) = \inf\{x \in \mathbb{R} | \omega(x) \leq 1 - \alpha\}, \\ &[\langle \kappa, \omega \rangle]_r^-(\alpha) = \sup\{x \in \mathbb{R} | \omega(x) \leq 1 - \alpha\}. \end{aligned} \tag{8}$$

Remark 2. $[\langle \kappa, \omega \rangle]_\alpha = [[\langle \kappa, \omega \rangle]_l^+(\alpha), [\langle \kappa, \omega \rangle]_r^+(\alpha)]$ and $[\langle \kappa, \omega \rangle]^\alpha = [[\langle \kappa, \omega \rangle]_l^-(\alpha), [\langle \kappa, \omega \rangle]_r^-(\alpha)]$.

Proposition 1. *For all* $\alpha, \beta \in [0,1]$ *and* $\langle \kappa, \omega \rangle \in \mathbb{F}_1$,

(i) $[\langle \kappa, \omega \rangle]_\alpha \subset [\langle \kappa, \omega \rangle]^\alpha$

(ii) $[\langle\kappa,\omega\rangle]_\alpha$ and $[\langle\kappa,\omega\rangle]^\alpha$ are nonempty compact convex sets in $\mathbb{R}$

(iii) If $\alpha \le \beta$, then $[\langle\kappa,\omega\rangle]_\beta \subset [\langle\kappa,\omega\rangle]_\alpha$ and $[\langle\kappa,\omega\rangle]^\beta \subset [\langle\kappa,\omega\rangle]^\alpha$

(iv) If $\alpha_n \nearrow \alpha$, then $[\langle\kappa,\omega\rangle]_\alpha = \cap_n [\langle\kappa,\omega\rangle]_{\alpha_n}$ and $[\langle\kappa,\omega\rangle]^\alpha = \cap_n [\langle\kappa,\omega\rangle]^{\alpha_n}$

Let H be any set and $\alpha \in [0,1]$; we denote by

$$\begin{aligned} H_\alpha &= \{x \in \mathbb{R}: \kappa(x) \ge \alpha\}, \\ H^\alpha &= \{x \in \mathbb{R}: \omega(x) \le 1-\alpha\}. \end{aligned} \tag{9}$$

Lemma 1. *Let $\{H_\alpha, \alpha \in [0,1]\}$ and $\{H^\alpha, \alpha \in [0,1]\}$ be two families of $\mathbb{R}$ satisfying (i)–(iv) in Proposition 1; if κ and ω are defined by*

$$\begin{aligned} \kappa(x) &= \begin{cases} 0, & \text{if } x \notin H_0, \\ \sup\{\alpha \in [0,1]: x \in H_\alpha\}, & \text{if } x \in H_0, \end{cases} \\ \omega(x) &= \begin{cases} 1, & \text{if } x \notin H_0, \\ 1-\sup\{\alpha \in [0,1]: x \in H^\alpha\}, & \text{if } x \in H_0, \end{cases} \end{aligned} \tag{10}$$

then $\langle\kappa,\omega\rangle \in \mathbb{F}_1$.

Lemma 2. *A mapping $d: \mathbb{F}_1 \times \mathbb{F}_1 \longrightarrow \mathbb{R}$ is said to be an intuitionistic fuzzy metric on $\mathbb{F}_1$ if it satisfies the following conditions:*

(1) $d(\langle\kappa_1,\omega_1\rangle,\langle\kappa_2,\omega_2\rangle) \ge 0,\ \forall \langle\kappa_1,\omega_1\rangle,\langle\kappa_2,\omega_2\rangle \in \mathbb{F}_1$

(2) $d(\langle\kappa_1,\omega_1\rangle,\langle\kappa_2,\omega_2\rangle) = 0$ iff $\langle\kappa_1,\omega_1\rangle = \langle\kappa_2,\omega_2\rangle$

(3) $d(\langle\kappa_1,\omega_1\rangle,\langle\kappa_2,\omega_2\rangle) = d(\langle\kappa_2,\omega_2\rangle,\langle\kappa_1,\omega_1\rangle)\forall \langle\kappa_1,\omega_1\rangle,\ \langle\kappa_2,\omega_2\rangle \in \mathbb{F}_1$

(4) $d(\langle\kappa_1,\omega_1\rangle,\langle\kappa_3,\omega_3\rangle) \le d(\langle\kappa_1,\omega_1\rangle,\langle\kappa_2,\omega_2\rangle) + d(\langle\kappa_2,\omega_2\rangle,\langle\kappa_3,\omega_3\rangle),\ \forall \langle\kappa_1,\omega_1\rangle,\ \langle\kappa_2,\omega_2\rangle,\langle\kappa_3,\omega_3\rangle$

On the space $\mathbb{F}_1$, we will consider the following metric:

$$\begin{aligned} d_\infty(\langle\kappa,\omega\rangle,\langle\chi,\Psi\rangle) &= \frac{1}{4}\sup_{0<\alpha\le 1}\left\|\langle\kappa,\omega\rangle_r^+(\alpha) - \langle\chi,\Psi\rangle_r^+(\alpha)\right\| \\ &\quad + \frac{1}{4}\sup_{0<\alpha\le 1}\left\|\langle\kappa,\omega\rangle_l^+(\alpha) - \langle\chi,\Psi\rangle_l^+(\alpha)\right\| \\ &\quad + \frac{1}{4}\sup_{0<\alpha\le 1}\left\|\langle\kappa,\omega\rangle_r^-(\alpha) - \langle\chi,\Psi\rangle_r^-(\alpha)\right\| \\ &\quad + \frac{1}{4}\sup_{0<\alpha\le 1}\left\|\langle\kappa,\omega\rangle_l^-(\alpha) - \langle\chi,\Psi\rangle_l^-(\alpha)\right\|, \end{aligned} \tag{11}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Proposition 2 (see [24]). *$(\mathbb{F}_1, d_p)$ is a metric space.*

Definition 2. The generalized Hukuhara difference of two fuzzy numbers $\langle\kappa,\omega\rangle$, $\langle\kappa',\omega'\rangle \in \mathbb{F}_1$ is defined as follows:

$$\begin{aligned} &\langle\kappa,\omega\rangle \ominus_{gH} \langle\kappa',\omega'\rangle = \langle\Psi,\chi\rangle \\ &\Leftrightarrow \begin{cases} \langle\kappa,\omega\rangle = \langle\kappa',\omega'\rangle + \langle\Psi,\chi\rangle \\ \text{or } \langle\kappa',\omega'\rangle = \langle\kappa,\omega\rangle + (-1)\langle\Psi,\chi\rangle. \end{cases} \end{aligned} \tag{12}$$

Definition 3 (see [25]). Let $F: (a,b) \longrightarrow W^1$ and $x_0 \in (a,b)$. It is said that F is strongly generalized differentiable on x_0 if $\exists F'^+(x_0)$, $\exists F'^-(x_0) \in E^1$ such that

(i) For all $h > 0$ sufficiently small, $\exists F^+(x_0+h) - F^+(x_0)$, $F^+(x_0) - F^+(x_0-h)$, and the limits (in metric D).

$$\lim_{h\longrightarrow 0}\frac{F^+(x_0+h) - F^+(x_0)}{h} = \lim_{h\longrightarrow 0}\frac{F^+(x_0) - F^+(x_0-h)}{h} = F'^+(x_0), \tag{13}$$

or

(ii) For all $h > 0$ sufficiently small, $\exists F^+(x_0) - F^+(x_0+h)$, $F^+(x_0-h) - F^+(x_0)$, and the limits

$$\lim_{h\longrightarrow 0}\frac{F^+(x_0) - F^+(x_0+h)}{-h} = \lim_{h\longrightarrow 0}\frac{F^+(x_0-h) - F^+(x_0)}{-h} = F'^+(x_0), \tag{14}$$

or

(iii) For all $h > 0$ sufficiently small, $\exists F^+(x_0+h) - F^+(x_0)$, $F^+(x_0-h) - F^+(x_0)$, and the limits

$$\lim_{h\longrightarrow 0}\frac{F^+(x_0) - F^+(x_0-h)}{h} = \lim_{h\longrightarrow 0}\frac{F^+(x_0-h) - F^+(x_0)}{-h} = F'^+(x_0). \tag{15}$$

(iv) For all $h>0$ sufficiently small, $\exists F^{+}(x_0)-F^{+}(x_0+h)$, $F^{+}(x_0)-F^{+}(x_0-h)$, and the limits

$$\lim_{h\longrightarrow 0}\frac{F^{+}(x_0)-F^{+}(x_0+h)}{-h}=\frac{F^{+}(x_0)-F^{+}(x_0-h)}{-h}=F'^{+}(x_0). \tag{16}$$

3. Homotopy Analysis Method

In this section, we are interested to resolve the partial differential equations with the intuitionistic fuzzy approach by adopting the homotopy analysis method. Therefore, we describe the basic idea of the homotopy analysis method by considering the following differential equation.

We consider the following differential equation:

$$\mathbb{N}[\phi(x,t)]=f(x,t), \tag{17}$$

where ϕ is an unknown intuitionistic fuzzy function, $\mathbb{N}$ is an intuitionistic fuzzy linear or nonlinear differential operator, and $f(x,t)$ is an intuitionistic fuzzy function.

Here,

$$\begin{aligned}
\phi_{\alpha,\beta}(x,t)&=\langle\phi_{\alpha}(x,t),\phi_{\beta}(x,t)\rangle\\
&=\langle[\phi_{\alpha l}(x,t),\phi_{\alpha r}(x,t)],\left[\phi_{\beta l}(x,t),\phi_{\beta r}(x,t)\right]\rangle,\\
\mathbb{N}_{\alpha,\beta}(x,t)&=\mathbb{N}\left(x,t,\langle[\phi_{\alpha l}(x,t,\alpha),\phi_{\alpha r}(x,t,\alpha)],\left[\phi_{\beta l}(x,t,\beta),\phi_{\beta r}(x,t,\beta)\right]\rangle\right)\\
&=\langle[\mathbb{N}_{\alpha l}(x,t,\phi_{\alpha l},\phi_{\alpha r})],[\mathbb{N}_{\alpha r}(x,t,\phi_{\alpha l},\phi_{\alpha r})],\left[\mathbb{N}_{\beta l}\left(x,t,\phi_{\beta l},\phi_{\beta r}\right)\right],\left[\mathbb{N}_{\beta r}\left(x,t,\phi_{\beta l},\phi_{\beta r}\right)\right]\rangle,\\
f_{\alpha,\beta}(x,t)&=\langle f_{\alpha}(x,t),f_{\beta}(x,t)\rangle\\
&=\langle[f_{\alpha l}(x,t),f_{\alpha r}(x,t)],\left[f_{\beta l}(x,t),f_{\beta r}(x,t)\right]\rangle.
\end{aligned} \tag{18}$$

Now, from (17), we get

$$\begin{aligned}
&\mathbb{N}\left(x,t,\langle[\phi_{\alpha l}(x,t,\alpha),\phi_{\alpha r}(x,t,\alpha)],\left[\phi_{\beta l}(x,t,\beta),\phi_{\beta r}(x,t,\beta)\right]\rangle\right)\\
&=\langle[f_{\alpha l}(x,t),f_{\alpha r}(x,t)],\left[f_{\beta l}(x,t),f_{\beta r}(x,t)\right]\rangle.
\end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned}
\mathbb{N}(x,t,[\phi_{\alpha l}(x,t,\alpha),\phi_{\alpha r}(x,t,\alpha)])&=[f_{\alpha l}(x,t),f_{\alpha r}(x,t)],\\
\mathbb{N}\left(x,t,\left[\phi_{\beta l}(x,t,\beta),\phi_{\beta r}(x,t,\beta)\right]\right)&=\left[f_{\beta l}(x,t),f_{\beta r}(x,t)\right].
\end{aligned} \tag{20}$$

In order to generalize the traditional method of homotopy, we will treat the original differential equation for the objective to construct a family of zero-order deformation equations as follows:

$$(1-q)\Gamma(\phi_{\alpha l}(x,t,q)-u_{\alpha l0}(x,t))=CH(x,t)q\mathbb{N}(\phi_{\alpha l}(x,t;q)), \tag{21}$$

$$(1-q)\Gamma(\phi_{\alpha r}(x,t,q)-u_{\alpha r0}(x,t))=CH(x,t)q\mathbb{N}(\phi_{\alpha r}(x,t;q)), \tag{22}$$

$$(1-q)\Gamma\left(\phi_{\beta l}(x,t,q)-u_{\beta l0}(x,t)\right)=CH(x,t)q\mathbb{N}\left(\phi_{\beta l}(x,t;q)\right), \tag{23}$$

$$(1-q)\Gamma\left(\phi_{\beta r}(x,t,q)-u_{\beta r0}(x,t)\right)=CH(x,t)q\mathbb{N}\left(\phi_{\beta r}(x,t;q)\right). \tag{24}$$

The last equations are called the zero-order deformation equations whose solutions vary continuously with respect to the parameter $q \in [0,1]$, where q is the deformation parameter, C is the nonzero convergence control parameter, Γ is the linear operator, $H(x)$ is the nonzero auxiliary function, and $u_0(x)$ is the initial approximation of the desired solution.

It is well known that if $q=0$, then $\Gamma(\phi_{\alpha\beta}(x,t,0)-u_{\alpha\beta,0}(x,t))=0$ since Γ is linear; therefore, $\phi_{\alpha,\beta}(x,t,0)=u_{\alpha\beta 0}(x,t)$; this is the initial condition of the problem $\mathbb{N}[\phi(x,t)]=f(x,t)$.

And if $q=1$, then $CH(x,t)\mathbb{N}(\phi_{\alpha,\beta}(x,t,1))=0$ since $C\neq 0$ and $H(x,t)\neq 0$; thus, $\mathbb{N}(\phi_{\alpha,\beta}(x,t,1))=0$ such that $\phi_{\alpha,\beta}(x,t,1)$ is a solution of the problem $\mathbb{N}[\phi_{\alpha\beta}(x,t)]=f_{\alpha,\beta}(x,t)$.

As q increases from 0 to 1, the solution $\phi_{\alpha,\beta}(x,t,q)$ will vary from the initial condition $u_{\alpha\beta,0}(x,t)$ to the solution $u_{\alpha\beta}(x,t)$.

Using Taylor's development for $\phi_{\alpha,\beta}(x,q)$ with respect to q, we have

$$\phi_{\alpha,\beta}(x,t,q)=u_{\alpha,\beta 0}(x,t)+\sum_{m=1}^{\infty}u_{\alpha,\beta m}(x,t)q^m, \tag{25}$$

where

$$u_{\alpha,\beta m}(x,t)=\frac{1}{m!}\frac{\partial^m\phi_{\alpha,\beta}}{\partial q^m}(x,t,q)|_{q=0}, \tag{26}$$

and when the linear operator, the initial approximation, the auxiliary function, and the convergence control parameter are well selected, therefore, (21)–(24) converge for $q=1$ and

$$\phi_{\alpha,\beta}(x,t,q)=u_{\alpha,\beta 0}(x,t)+\sum_{m=1}^{\infty}u_{\alpha,\beta m}(x,t). \tag{27}$$

For $H(x,t)=1$ and $C=-1$, equations (21)–(24) turn into

$$\begin{aligned}
&(1-q)\Gamma\left(\phi_{\alpha_l}(x,t,q)-u_0(x)\right)+q\mathbb{N}\left(\phi_{\alpha_l}(x,t,q)\right)=0,\\
&(1-q)\Gamma\left(\phi_{\alpha_r}(x,t,q)-u_0(x)\right)+q\mathbb{N}\left(\phi_{\alpha_r}(x,t,q)\right)=0,\\
&(1-q)\Gamma\left(\phi_{\beta_l}(x,t,q)-u_0(x)\right)+q\mathbb{N}\left(\phi_{\beta_l}(x,t,q)\right)=0,\\
&(1-q)\Gamma\left(\phi_{\beta_r}(x,t,q)-u_0(x)\right)+q\mathbb{N}\left(\phi_{\beta_r}(x,t,q)\right)=0,
\end{aligned} \tag{28}$$

which are mainly used in the homotopy perturbation method (HPM), proving that this method is a special case of the homotopy analysis method (HAM).

Differentiating equations (21)–(24) m times with respect to the integrated parameter q, then setting $q=0$, and finally dividing them by $m!$, we have the mth order of deformation equations:

$$\Gamma\left[u_{\alpha_l m}(x,t)-\chi_m u_{\alpha_l m-1}(x,t)\right]=CH(x,t)R_m\left(\vec{u}_{\alpha_l m-1}\right), \tag{29}$$

$$\Gamma\left[u_{\alpha_r m}(x,t)-\chi_m u_{\alpha_r m-1}(x,t)\right]=CH(x,t)R_m\left(\vec{u}_{\alpha_r m-1}\right), \tag{30}$$

$$\Gamma\left[u_{\beta_l m}(x,t)-\chi_m u_{\beta_l m-1}(x,t)\right]=CH(x,t)R_m\left(\vec{u}_{\beta_l m-1}\right), \tag{31}$$

$$\Gamma\left[u_{\beta_r m}(x,t)-\chi_m u_{\beta_r m-1}(x,t)\right]=CH(x,t)R_m\left(\vec{u}_{\beta_r m-1}\right), \tag{32}$$

where

$$\chi_m=\begin{cases}0, & m\leq 1,\\ 1, & m>1,\end{cases} \tag{33}$$

$$\begin{aligned}
R_m\left(\vec{u}_{\alpha_l m-1}\right)&=\frac{1}{m-1}\frac{\partial^{m-1}}{\partial q^{m-1}}\mathbb{N}\left[\phi_{\alpha_l}(x,t,q)\right]|_{q=0},\\
R_m\left(\vec{u}_{\alpha_r m-1}\right)&=\frac{1}{m-1}\frac{\partial^{m-1}}{\partial q^{m-1}}\mathbb{N}\left[\phi_{\alpha_r}(x,t,q)\right]|_{q=0},\\
R_m\left(\vec{u}_{\beta_l m-1}\right)&=\frac{1}{m-1}\frac{\partial^{m-1}}{\partial q^{m-1}}\mathbb{N}\left[\phi_{\beta_l}(x,t,q)\right]|_{q=0},\\
R_m\left(\vec{u}_{\beta_r m-1}\right)&=\frac{1}{m-1}\frac{\partial^{m-1}}{\partial q^{m-1}}\mathbb{N}\left[\phi_{\beta_r}(x,t,q)\right]|_{q=0}.
\end{aligned} \tag{34}$$

Theorem 1. *The series $\phi_{\alpha,\beta}(x,t)=\sum_{m=0}^{\infty}u_{\alpha,\beta m}(x,t)$ converge to $u_{\alpha,\beta}(x,t)$ or $u_{\alpha,\beta m}(x,t)$ which is ruled by high-order deformation equations (29)–(32) with definitions (33) and (34), and it must be the exact solution of equation (17).*

Proof. Liao [18]. □

4. Numerical Application

Let us consider the fuzzy intuitionistic equation of the following form:

$$\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=ae^x, \tag{35}$$

where $a=\langle(-1,0,1),(-1,0,2)\rangle$, i.e., $a_{\alpha,\beta}=\langle[\alpha-1,1-\alpha][-\beta,2\beta]\rangle$ with initial conditions

$$\begin{aligned}
u_{\alpha,\beta}(0)&=\langle[1+\alpha,3-\alpha][2-2\beta,2+2\beta]\rangle,\\
u'_{\alpha,\beta}(0)&=\langle[\alpha,2-\alpha][1-2\beta,1+\beta]\rangle.
\end{aligned} \tag{36}$$

The exact solution, given by the classical solution method, is

$$\begin{aligned}
u_{\alpha_l}(x,t,\alpha)&=\left[(2-t)e^t+(\alpha-1)te^{-t}+(\alpha-1)\right]e^x,\\
u_{\alpha_r}(x,t,\alpha)&=\left[(2-t)e^t-(\alpha-1)te^{-t}-(\alpha-1)\right]e^x,\\
u_{\beta_l}(x,t,\alpha)&=\left[\frac{1}{2}(4-\beta-2t)e^t-(\beta+2\beta t)e^{-t}-\beta\right]e^x,\\
u_{\beta_r}(x,t,\alpha)&=\left[\frac{1}{2}(4-\beta-2t)e^t+(\beta+2\beta t)e^{-t}+2\beta\right]e^x.
\end{aligned} \tag{37}$$

According to the homotopy analysis method (HPM), we are looking for $\phi(x,t,q)$ that has the form

$$\phi_{\alpha,\beta}(x,t,q) = \sum_{m\geq 0} u_{\alpha,\beta_m}(x,t)q^m. \tag{38}$$

Thus, according to the deformation equations of order m, (29)–(32), with

$$\begin{aligned} R_{m-1}\left(\vec{u}_{\alpha_l m-1}\right) &= u_{t_{m-1}} - u_{xx_{m-1}} - (\alpha-1)e^x, \\ R_{m-1}\left(\vec{u}_{\alpha_r m-1}\right) &= u_{t_{m-1}} - u_{xx_{m-1}} + (\alpha-1)e^x, \\ R_m\left(\vec{u}_{\beta_l m-1}\right) &= u_{t_{m-1}} - u_{xx_{m-1}} + \beta e^x, \\ R_m\left(\vec{u}_{\beta_r m-1}\right) &= u_{t_{m-1}} - u_{xx_{m-1}} - 2\beta e^x, \end{aligned} \tag{39}$$

we find that

$$\begin{aligned} \Gamma\left[u_{\alpha_l m}(x,t) - \chi_m u_{\alpha_l m-1}(x,t)\right] &= CH(x,t)\left[u_{t_{m-1}} - u_{xx_{m-1}} - (\alpha-1)e^x\right], \\ \Gamma\left[u_{\alpha_r m}(x,t) - \chi_m u_{\alpha_r m-1}(x,t)\right] &= CH(x,t)\left[u_{t_{m-1}} - u_{xx_{m-1}} + (\alpha-1)e^x\right], \\ \Gamma\left[u_{\beta_l m}(x,t) - \chi_m u_{\beta_l m-1}(x,t)\right] &= CH(x,t)\left[u_{t_{m-1}} - u_{xx_{m-1}} + \beta e^x\right], \\ \Gamma\left[u_{\beta_r m}(x,t) - \chi_m u_{\beta_r m-1}(x,t)\right] &= CH(x,t)\left[u_{t_{m-1}} - u_{xx_{m-1}} - 2\beta e^x\right]. \end{aligned} \tag{40}$$

If we take $C=-1$ and $H(x,t)=1$, we have

$$\begin{aligned} u_{\alpha_l m}(x,t) - \chi_m u_{\alpha_l m-1}(x,t) &= -\Gamma^{-1}\left[u_{t_{m-1}} - u_{xx_{m-1}} - (\alpha-1)e^x\right], \\ u_{\alpha_r m}(x,t) - \chi_m u_{\alpha_r m-1}(x,t) &= -\Gamma^{-1}\left[u_{t_{m-1}} - u_{xx_{m-1}} + (\alpha-1)e^x\right], \\ u_{\beta_l m}(x,t) - \chi_m u_{\beta_l m-1}(x,t) &= -\Gamma^{-1}\left[u_{t_{m-1}} - u_{xx_{m-1}} + \beta e^x\right], \\ u_{\beta_r m}(x,t) - \chi_m u_{\beta_r m-1}(x,t) &= -\Gamma^{-1}\left[u_{t_{m-1}} - u_{xx_{m-1}} - 2\beta e^x\right], \end{aligned} \tag{41}$$

and we choose the operator $\Gamma = (d/dt)$, then $\Gamma^{-1}(\ast) = \int(\ast)dt$, and by using the initial values, we get

(i) $u_{\alpha_l m}(x,t)$.

For $m=1$,

$$\begin{aligned} u_{\alpha_l 1}(x,t) &= -\Gamma^{-1}\left[u_{\alpha_l t_0} - u_{\alpha_l xx_0} - (\alpha-1)e^x\right] + \chi_1 u_{\alpha_l 0}(x,t) \\ &= -\Gamma^{-1}\left[-(1+\alpha)e^x - (\alpha-1)e^x\right] \\ &= -\Gamma^{-1}\left[-2\alpha e^x\right] = 2\alpha t e^x. \end{aligned} \tag{42}$$

For $m=2$,

$$\begin{aligned} u_{\alpha_l 2}(x,t) &= -\Gamma^{-1}\left[u_{\alpha_l t_1} - u_{\alpha_l xx_1} - (\alpha-1)e^x\right] + \chi_2 u_{\alpha_l 1}(x,t) \\ &= -\Gamma^{-1}\left[2\alpha e^x - 2\alpha t e^x - \alpha e^x + e^x\right] + 2\alpha t e^x \\ &= -\Gamma^{-1}\left[(\alpha+1)e^x - 2\alpha t e^x\right] + 2\alpha t e^x \\ &= (\alpha-1)te^x + \alpha t^2 e^x. \end{aligned} \tag{43}$$

For $m=3$,

$$\begin{aligned} u_{\alpha_l 3}(x,t) &= -\Gamma^{-1}\left[u_{\alpha_l t_2} - u_{\alpha_l xx_2} - (\alpha-1)e^x\right] + \chi_3 u_{\alpha_l 2}(x,t) \\ &= -\Gamma^{-1}\left[(\alpha-1)e^x + 2\alpha t e^x - (\alpha-1)te^x - \alpha t^2 e^x - (\alpha-1)e^x\right] + (\alpha-1)te^x + \alpha t^2 e^x \\ &= -\Gamma^{-1}\left[(\alpha+1)te^x - \alpha t^2 e^x\right] + (\alpha-1)te^x + \alpha t^2 e^x \\ &= (\alpha-1)te^x + (\alpha+1)\frac{t^2}{2}e^x + \alpha\frac{t^3}{3}e^x. \end{aligned} \tag{44}$$

For $m=4$,

$$\begin{aligned}u_{\alpha_l 4}(x,t) &= -\Gamma^{-1}\left[u_{\alpha_l t_3} - u_{\alpha_l x x_3} - (\alpha-1)e^x\right] + \chi_4 u_{\alpha_l 3}(x,t)\\ &= -\Gamma^{-1}\left[\alpha t^2 e^x + (\alpha+1)te^x + (\alpha-1)e^x - (\alpha-1)te^x - (\alpha+1)\frac{t^2}{2}e^x - \alpha\frac{t^3}{3}e^x - (\alpha-1)e^x\right]\\ &\quad + (\alpha-1)te^x + (\alpha+1)\frac{t^2}{2}e^x + \alpha\frac{t^3}{3}e^x\\ &= -\Gamma^{-1}\left[-\alpha\frac{t^3}{3}e^x + (\alpha-1)\frac{t^2}{2}e^x + 2te^x\right] + (\alpha-1)te^x + (\alpha+1)\frac{t^2}{2}e^x + \alpha\frac{t^3}{3}e^x\\ &= (\alpha-1)te^x + (\alpha-1)\frac{t^2}{2!}e^x + (\alpha+1)\frac{t^3}{3!}e^x + 2\alpha\frac{t^4}{4!}.\end{aligned} \tag{45}$$

After five iterations, we get

$$\begin{aligned}&u_{\alpha_l 0}(x,t) + u_{\alpha_l 1}(x,t) + u_{\alpha_l 2}(x,t) + u_{\alpha_l 3}(x,t) + u_{\alpha_l 4}(x,t)\\ &= (\alpha+1)e^x + (5\alpha-3)te^x + 4\alpha\frac{t^2}{2!}e^x + (3\alpha+1)\frac{t^3}{3!}e^x + 2\alpha\frac{t^4}{4!}.\end{aligned} \tag{46}$$

(ii) $u_{\alpha_r m}(x,t)$.

For $m=1$,

$$\begin{aligned}u_{\alpha_r 1}(x,t) &= -\Gamma^{-1}\left[u_{t_0} - u_{xx_0} - (1-\alpha)e^x\right] + \chi_1 u_{\alpha_r 0}(x,t)\\ &= -\Gamma^{-1}\left[-(3-\alpha)e^x - (1-\alpha)e^x\right]\\ &= -\Gamma^{-1}\left[-4e^x + 2\alpha e^x\right] = (4-2\alpha)te^x.\end{aligned} \tag{47}$$

For $m=2$,

$$\begin{aligned}u_{\alpha_r 2}(x,t) &= -\Gamma^{-1}\left[u_{t_1} - u_{xx_1} - (1-\alpha)e^x\right] + \chi_2 u_{\alpha_r 1}(x,t)\\ &= -\Gamma^{-1}\left[(4-2\alpha)e^x - (4-2\alpha)te^x - (1-\alpha)e^x\right]\\ &\quad + (4-2\alpha)te^x\\ &= -\Gamma^{-1}\left[(3-\alpha)e^x - 2(2-\alpha)te^x\right] + (4-2\alpha)te^x\\ &= (1-\alpha)te^x + (2-\alpha)t^2e^x.\end{aligned} \tag{48}$$

For $m=3$,

$$\begin{aligned}u_{\alpha_r 3}(x,t) &= -\Gamma^{-1}\left[u_{t_2} - u_{xx_2} - (1-\alpha)e^x\right] + \chi_3 u_{\alpha_r 2}(x,t)\\ &= -\Gamma^{-1}\left[(1-\alpha)e^x + 2(2-\alpha)te^x - (1-\alpha)te^x - (2-\alpha)t^2e^x - (1-\alpha)e^x\right] + (1-\alpha)te^x + (2-\alpha)t^2e^x\\ &= -\Gamma^{-1}\left[-(2-\alpha)t^2e^x + (3-\alpha)te^x\right] + (1-\alpha)te^x + (2-\alpha)t^2e^x\\ &= (1-\alpha)te^x + (7-3\alpha)\frac{t^2}{2}e^x + (2-\alpha)\frac{t^3}{3}e^x.\end{aligned} \tag{49}$$

For $m=4$,

$$u_{\alpha_r 4}(x,t) = -\Gamma^{-1}\left[u_{t_3} - u_{xx_3} - (1-\alpha)e^x\right] + \chi_4 u_{\alpha_r 3}(x,t)$$
$$= -\Gamma^{-1}\left[(1-\alpha)e^x + (7-3\alpha)te^x + (2-\alpha)t^2e^x - (1-\alpha)te^x - (7-3\alpha)\frac{t^2}{2}e^x - (2-\alpha)\frac{t^3}{3}e^x - (1-\alpha)e^x\right]$$
$$+ (1-\alpha)te^x + (7-3\alpha)\frac{t^2}{2}e^x + (2-\alpha)\frac{t^3}{3}e^x \tag{50}$$
$$= -\Gamma^{-1}\left[(2-\alpha)\frac{t^3}{3}e^x + (\alpha-3)\frac{t^2}{2}e^x + (6-2\alpha)te^x\right] + (1-\alpha)te^x + (7-3\alpha)\frac{t^2}{2}e^x + (2-\alpha)\frac{t^3}{3}e^x$$
$$= (1-\alpha)te^x + (13-5\alpha)\frac{t^2}{2}e^x + (7-3\alpha)\frac{t^3}{3!}e^x + (2-\alpha)\frac{t^4}{4!}e^x.$$

After five iterations, we get

$$u_{\alpha_r 0}(x,t) + u_{\alpha_r 1}(x,t) + u_{\alpha_r 2}(x,t) + u_{\alpha_r 3}(x,t) + u_{\alpha_l 4}(x,t)$$
$$= (3-\alpha)e^x + (7-5\alpha)te^x + (24-10\alpha)\frac{t^2}{2}e^x + (11-5\alpha)\frac{t^3}{3!}e^x + 2(2-\alpha)\frac{t^4}{4!}e^x. \tag{51}$$

In the same way for $u_{\beta_l m}(x,t)$ and $u_{\beta_r m}(x,t)$, we find that

$$u_{\beta_l 1}(x,t) = (2-3\beta)te^x,$$
$$u_{\beta_l 2}(x,t) = -\beta te^x + (2-3\beta)\frac{t^2}{2}e^x,$$
$$u_{\beta_l 3}(x,t) = -\beta te^x - \beta\frac{t^2}{2}e^x + (2-3\beta)\frac{t^3}{3!}e^x,$$
$$u_{\beta_l 4}(x,t) = -\beta te^x - \beta\frac{t^2}{2}e^x + (4-5\beta)\frac{t^3}{3!}e^x + (2-3\beta)\frac{t^4}{4!}e^x,$$
and (52)
$$u_{\beta_r 1}(x,t) = (2+4\beta)te^x,$$
$$u_{\beta_r 2}(x,t) = 2\beta te^x + (2+4\beta)\frac{t^2}{2}e^x,$$
$$u_{\beta_r 3}(x,t) = 2\beta te^x + \beta t^2e^x + (2+4\beta)\frac{t^3}{3!}e^x,$$
$$u_{\beta_r 4}(x,t) = 2\beta te^x + 2\beta\frac{t^2}{2}e^x + 2\beta\frac{t^3}{3!}e^x - (2+4\beta)\frac{t^4}{4!}e^x,$$

so after five iterations, we get

$$u_{\beta_r 1}(x,t) + u_{\beta_r 2}(x,t) + u_{\beta_r 3}(x,t) + u_{\beta_r 4}(x,t)$$
$$= (2+2\beta)e^x + (2+10\beta)te^x + (2+8\beta)\frac{t^2}{2}e^x + (2+6\beta)\frac{t^3}{3!}e^x - (2+4\beta)\frac{t^4}{4!}e^x. \tag{53}$$

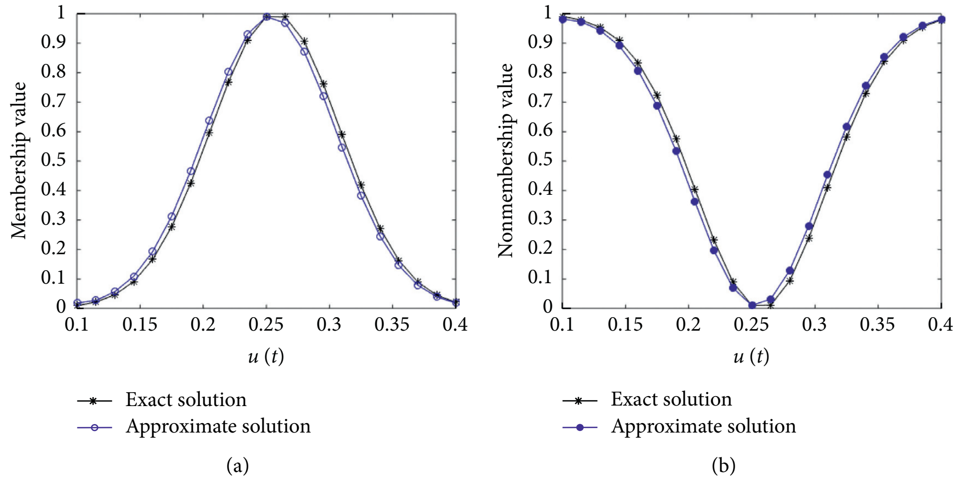

FIGURE 1: Exact and approximate solutions for membership and nonmembership functions at $t = 1$ and $m = 4$.

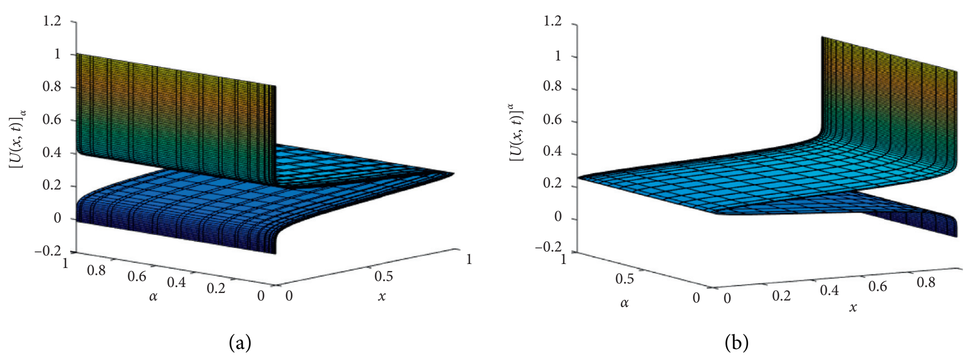

FIGURE 2: The surface of intuitionistic fuzzy solutions.

To ensure the validity of the present model, we illustrate in Figures 1 and 2 the comparison of the numerical solutions with the exact ones for the membership and nonmembership functions at $t = 1$ and $m = 4$ for $\alpha \in [0, 1]$; we have calculated all the data by using MATLAB.

From the figures, we can see that the results of the homotopy perturbation method (HPM) are close to the exact solution which confirms the validity of our method.

5. Conclusion

In this work, we have presented the procedure for simulating and computing an approximate solution for intuitionistic fuzzy differential equations with the linear differential operator by using the homotopy analysis method, which can also be used to solve some linear and nonlinear problems that cannot be solved by classical methods. Moreover, in the homotopy analysis method, we can choose h appropriately to ensure the convergence of the series solution for highly nonlinear problems. The basic ideas of this approach should be used to solve other intuitionistic fuzzy problems in many practical domains such as fluid mechanics.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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