

Research Article

Fuzzy Annihilator Ideals of C -Algebra

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In this paper, we introduce the concept of relative fuzzy annihilator ideals in C -algebras and investigate some its properties. We characterize relative fuzzy annihilators in terms of fuzzy points. It is proved that the class of fuzzy ideals of C -algebras forms Heything algebra. We observe that the class of all fuzzy annihilator ideals can be made as a complete Boolean algebra. Moreover, we study the concept of fuzzy annihilator preserving homomorphism. We provide a sufficient condition for a homomorphism to be a fuzzy annihilator preserving.

1. Introduction

Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate, or useful notions in describing the real-life problems because every object encountered in this real physical world carries some degree of fuzziness. A lot of work on fuzzy sets has come into being with many applications to various fields such as computer science, artificial intelligence, expert systems, control systems, decision making, medical diagnosis, management science, operations research, pattern recognition, neural network, and others (see [1–4]). Many papers on fuzzy algebras have been published since Rosenfeld [5] introduced the concept of fuzzy group in 1971. In particular, fuzzy subgroups of a group (see [6–8]), fuzzy ideals of lattices and MS-algebra (see [9–16]), fuzzy ideals of C -algebras (see [17, 18]), and intuitionistic fuzzy ideals of BCK-algebra, BG-algebra, and BCI-algebra (see [19–21]).

On the contrary, Guzman and Squier, in [22], introduced the variety of C -algebras as the variety generated by the three-element algebra $C = \{T, F, U\}$ with the operations $\wedge, \vee, '$ of type $(2, 2, 1)$, which is the algebraic form of the three-valued conditional logic. They proved that the two-

element Boolean algebras B and C are the only subdirectly irreducible C -algebras and that the variety of C -algebras is a minimal cover of the variety of Boolean algebras. In [23], U. M. Swamy et al. studied the center $B(A)$ of a C -algebra A and proved that the center of a C -algebra is a Boolean algebra. In [24], Rao and Sundarayya studied the concept of C -algebra as a poset. In a series of papers (see [25–28]), Vali et al. studied the concept of ideals, principal ideals, and prime ideals of C -algebras as well as the concept of prime spectrum, ideal congruences, and annihilators of C -algebras. Later, Rao carried out a study on annihilator ideals of C -algebras [29].

In this paper, we study the concept of relative fuzzy annihilator ideals in C -algebras. We characterize relative fuzzy annihilators in terms of fuzzy points. Using the concept of the relative fuzzy annihilator, we prove that the class of fuzzy ideals of C -algebras forms the Heything algebra. We also study fuzzy annihilator ideals. Basic properties of fuzzy annihilator ideals are also studied. It is shown that the class of all fuzzy annihilator ideals forms a complete Boolean algebra. Moreover, we study the concept of fuzzy annihilator preserving homomorphism and derived a sufficient condition for a homomorphism to be a fuzzy annihilator preserving. Finally, we prove

that the image and preimage of fuzzy annihilator ideals are again fuzzy annihilator ideals.

2. Preliminaries

In this section, we recall some definitions and basic results on c -algebras.

Definition 1 (see [22]). An algebra $(A, \vee, \wedge, ')$ of type $(2, 2, 1)$ is called a c -algebra if it satisfies the following axioms:

- (1) $a'' = a$
- (2) $(a \wedge b)' = a' \vee b'$
- (3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- (4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- (5) $(a \vee b) \wedge c = (a \wedge c) \vee (a' \wedge b \wedge c)$
- (6) $a \vee (a \wedge b) = a$
- (7) $(a \wedge b) \vee (b \wedge a) = (b \wedge a) \vee (a \wedge b)$, for all $a, b, c \in A$

Example 1. The three-element algebra $C = \{T, F, U\}$ with the operations given by the following tables is a C -algebra.

\vee	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U
\wedge	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U
	x	x'	
	T	F	
	F	T	
	U	U	

Note: the identities 2.1 (a) and 2.1 (b) imply that the variety of C -algebras satisfies all the dual statements of 2.1 (2) to 2.1 (7) in this view.

Lemma 1 (see [22]). *Every C -algebra satisfies the following identities:*

- (1) $x \wedge x = x$
- (2) $x \wedge x' = x' \wedge x$
- (3) $x \wedge y \wedge x = x \wedge y$
- (4) $x \wedge x' \wedge y = x \wedge x'$
- (5) $x \wedge y = (x' \vee y) \wedge x$
- (6) $x \wedge y = x \wedge (y \vee x')$
- (7) $x \wedge y = x \wedge (x' \vee y)$
- (8) $x \wedge y \wedge x' = x \wedge y \wedge y'$
- (9) $(x \vee y) \wedge x = x \vee (y \wedge x)$
- (10) $x \wedge (x' \vee x) = (x' \vee x) \wedge x = (x \vee x') \wedge x$

The dual statements of the above identities are also valid in a C -algebra.

Definition 2 (see [22]). An element z of a C -algebra A is called a left zero for \wedge if $z \wedge x = z$, for all $x \in A$.

Definition 3 (see [26]). A nonempty subset I of a C -algebra A is called an ideal of A if

- (1) $a, b \in I \Rightarrow a \vee b, a \wedge x \in I$
- (2) $a \in I \Rightarrow x \wedge a \in I$, for each $x \in A$.

It can also be observed that $a \wedge x \in I$, for all $a \in I$ and all $x \in A$. For any subset $S \subseteq A$, the smallest ideal of A containing S is called the ideal of A generated by S and is denoted by $\langle S \rangle$. Note that $\langle S \rangle = \{\vee (y_i \wedge x_i) : y_i \in A, x_i \in S, i = 1, \dots, n \text{ for some } n \in \mathbb{Z}_+\}$

If $S = \{a\}$, then we write $\langle a \rangle$ for $\langle S \rangle$. In this case, $\langle a \rangle = \{x \wedge a : x \in A\}$. Moreover, it is observed in [26] that the set $I_0 = \{x \wedge x' : x \in A\}$ is the smallest ideal in A .

Definition 4. Let $(A, \vee, \wedge, ', I_0)$ and $(A', \vee, \wedge, ', I'_0)$ be two C -algebras. Then, a mapping $f: A \rightarrow A'$ is called a homomorphism if it satisfies the following conditions:

- (1) $f(a \vee b) = f(a) \vee f(b)$
- (2) $f(a \wedge b) = f(a) \wedge f(b)$
- (3) $f(a') = f(a)'$, for all $a, b \in A$

Here, $I_0 = \{z \in A : z \text{ is a left zero for } \wedge\}$ and $I'_0 = \{y \in A' : y \text{ is a left zero for } \wedge\}$.

I_0 and I'_0 are the smallest ideals of the C -algebras A and A' , respectively. The kernel of the homomorphism is defined as $\text{Ker} f = \{x \in A : f(x) \in I'_0\}$.

Remember that, for any set A , a function $\mu: A \rightarrow [0, 1]$ is called a fuzzy subset of A . For each $t \in [0, 1]$, the set,

$$\mu_t = \{x \in A : \mu(x) \geq t\}, \quad (1)$$

is called the level subset of μ at t [30]. For numbers α and β in $[0, 1]$, we write $\alpha \wedge \beta$ for $\min\{\alpha, \beta\}$ and $\alpha \vee \beta$ for $\max\{\alpha, \beta\}$.

Definition 5 (see [17]). A fuzzy subset λ of A is called a fuzzy ideal of A if

- (1) $\lambda(z) = 1$, for all $z \in I_0$
- (2) $\lambda(a \vee b) \geq \lambda(a) \wedge \lambda(b)$
- (3) $\lambda(a \vee b) \geq \lambda(b)$, for all $a, b \in A$

We denote the class of all fuzzy ideals of A by $\text{FI}(A)$.

Lemma 2 (see [17]). *Let λ be a fuzzy ideal of A . Then, the following hold, for all $a, b \in A$:*

- (1) $\lambda(a \wedge b) \geq \lambda(a)$
- (2) $\lambda(a \wedge b) \geq \lambda(b \wedge a)$
- (3) $\lambda(a \wedge x \wedge b) \geq \lambda(a \wedge b)$, for each $x \in A$
- (4) $\lambda(a) \geq \lambda(a \vee b)$; hence, $\lambda(a) \wedge \lambda(b) \wedge \lambda(b \vee a)$
- (5) If $x \in \langle a \rangle$, then $\lambda(x) \geq \lambda(a)$

Let μ be a fuzzy subset of A . Then, the fuzzy ideal generated by μ is denoted by (μ) .

Theorem 1 (see [17]). *If λ and ν are fuzzy ideals of a C -algebra, then their supremum is given by*

$$(\lambda \vee \nu)(x) = \text{Sup}\{\wedge_{i=1}^n [\lambda(b_i) \vee \nu(b_i)]: x = \vee_{i=1}^n b_i, \quad b_i \in A\}. \quad (2)$$

We define the binary operations “+” and “.” on the set of all fuzzy subsets of A as

$$\begin{aligned} (\lambda + \nu)(x) &= \text{Sup}\{\lambda(y) \wedge \nu(z): y, z \in A, y \vee z = x\}, \\ (\lambda \cdot \nu)(x) &= \text{Sup}\{\lambda(y) \wedge \nu(z): y, z \in A, y \wedge z = x\}. \end{aligned} \quad (3)$$

If λ and ν are fuzzy ideals of A , then $\lambda \cdot \nu$ is a fuzzy ideal and $\lambda \cdot \nu = \lambda \cap \nu$. However, in a general case, $\lambda + \nu$ is not a fuzzy ideal.

Definition 6 (see [5]). Let f be a function from X to Y , μ be a fuzzy subset of X , and θ be a fuzzy subset of Y .

- (1) The image of μ under f , denoted by $f(\mu)$, is a fuzzy subset of Y defined, for each $y \in Y$, by

$$f(\mu)(y) = \begin{cases} \text{Sup}\{\mu(x): x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

- (2) The preimage of θ under f , denoted by $f^{-1}(\theta)$, is a fuzzy subset of X defined, for each $x \in X$, by

$$f^{-1}(\theta)(x) = \theta(f(x)). \quad (5)$$

Theorem 2 (see [31]). *Let f be a function from X to Y . Then, the following assertions hold:*

- (1) For all fuzzy subset μ_i of X , $i \in I$, $f(\cup_{i \in I} \mu_i) = \cup_{i \in I} f(\mu_i)$, so $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2)$.
- (2) For all fuzzy subset θ_j of Y , $j \in J$, $f^{-1}(\cup_{j \in J} \theta_j) = \cup_{j \in J} f^{-1}(\theta_j)$, $f^{-1}(\cap_{j \in J} \theta_j) = \cap_{j \in J} f^{-1}(\theta_j)$, and therefore, $\theta_1 \subseteq \theta_2 \Rightarrow f^{-1}(\theta_1) \subseteq f^{-1}(\theta_2)$.
- (3) $\mu \subseteq f^{-1}(f(\mu))$. In particular, if f is an injection, then $\mu = f^{-1}(f(\mu))$, for all fuzzy subset μ of X .
- (4) $f(f^{-1}(\theta)) \subseteq \theta$. In particular, if f is a surjection, then $f(f^{-1}(\theta)) = \theta$, for all fuzzy subset θ of Y .
- (5) $f(\mu) \subseteq \theta \Leftrightarrow \mu \subseteq f^{-1}(\theta)$, for all fuzzy subsets μ and θ of X and Y , respectively.

The class of fuzzy ideals of a C -algebra is denoted by $\text{FI}(A)$.

Note: throughout the rest of this paper, A stands for a C -algebra.

3. Relative Fuzzy Annihilator

In this section, we study the concept of relative fuzzy annihilator ideals in a C -algebra. Basic properties of relative fuzzy annihilator ideals are also studied. We characterize relative fuzzy annihilator in terms of fuzzy points. Finally, we prove that the class of fuzzy ideals of a C -algebra forms the Heyting algebra.

Definition 7. For any fuzzy subset λ of A and a fuzzy ideal ν , we define

$$(\lambda : \nu) = \bigcup \{ \eta : \eta \in [0, 1]^A, \eta \cdot \lambda \subseteq \nu \}. \quad (6)$$

A fuzzy subset $(\lambda : \nu)$ is called fuzzy annihilator of λ relative to ν .

For any $x \in A$,

$$(\lambda : \nu)(x) = \text{Sup}\{ \eta(x) : \eta \in [0, 1]^A, \eta \cdot \lambda \subseteq \nu \}. \quad (7)$$

For simplicity, we write

$$(\lambda : \nu) = \text{Sup}\{ \eta : \eta \in [0, 1]^A, \eta \cdot \lambda \subseteq \nu \}. \quad (8)$$

Lemma 3. *For any two fuzzy subsets λ and ν of a C -algebra A , we have*

$$(\lambda \cdot \nu) = (\lambda) \wedge (\nu). \quad (9)$$

Now, we prove the following lemma.

Lemma 4. *For any fuzzy subset λ of A and a fuzzy ideal ν , we have*

$$(\lambda : \nu) = \bigcup \{ \eta : \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu \}. \quad (10)$$

Proof. Clearly, $\bigcup \{ \eta : \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu \} \subseteq \bigcup \{ \delta : \delta \in [0, 1]^A, \delta \cdot \lambda \subseteq \nu \}$. Since $(\eta \cdot \lambda) = (\eta) \wedge (\lambda)$, we can easily show that the other inclusion holds. Thus,

$$(\lambda : \nu) = \text{Sup}\{ \eta : \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu \}. \quad (11) \quad \square$$

Theorem 3. *For any fuzzy subset λ of A and a fuzzy ideal ν , $(\lambda : \nu)$ is a fuzzy ideal of A .*

Proof. Since $\lambda \cdot \nu \subseteq \nu$ and ν is a fuzzy ideal, we get $\nu(z) = 1$, for all left zero element z for λ .

Let $x, y \in A$. Then,

$$\begin{aligned}
 (\lambda: \nu)(x) \wedge (\lambda: \nu)(y) &= \text{Sup}\{\eta(x): \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu\} \wedge \text{Sup}\{\sigma(y): \sigma \in \text{FI}(A), \sigma \cdot \lambda \subseteq \nu\} \\
 &= \text{Sup}\{\eta(x) \wedge \sigma(y): \eta, \sigma \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu, \sigma \cdot \lambda \subseteq \nu\} \\
 &\leq \text{Sup}\{(\eta \vee \sigma)(x) \wedge (\eta \vee \sigma)(y): \eta, \sigma \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu, \sigma \cdot \lambda \subseteq \nu\}.
 \end{aligned}
 \tag{12}$$

Since $\eta, \sigma \in \text{FI}(A)$, $\eta \cdot \lambda \subseteq \nu$, and $\sigma \cdot \lambda \subseteq \nu$, we get that $\eta \vee \sigma \in \text{FI}(A)$ and $(\eta \vee \sigma) \cdot \lambda \subseteq \nu$. Then,

$$\begin{aligned}
 (\lambda: \nu)(x) \wedge (\lambda: \nu)(y) &\leq \text{Sup}\{\gamma(x) \wedge \gamma(y): \gamma \in \text{FI}(A), \gamma \cdot \lambda \subseteq \nu\} \\
 &= \text{Sup}\{\gamma(x \vee y): \gamma \in \text{FI}(A), \gamma \cdot \lambda \subseteq \nu\} \\
 &= (\lambda: \nu)(x \vee y).
 \end{aligned}
 \tag{13}$$

Thus, $(\lambda: \nu)(x \vee y) \geq (\lambda: \nu)(x) \wedge (\lambda: \nu)(y)$.

On the contrary, let $x \in A$. Then,

$$\begin{aligned}
 (\lambda: \nu)(x) &= \text{Sup}\{\eta(x): \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu\} \\
 &\leq \text{Sup}\{\eta(x \wedge y): \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \nu\} \\
 &= (\lambda: \nu)(x \wedge y).
 \end{aligned}
 \tag{14}$$

Similarly, $(\lambda: \nu)(y) \leq (\lambda: \nu)(x \wedge y)$. So, $(\lambda: \nu)(x \wedge y) \geq (\lambda: \nu)(x) \vee (\lambda: \nu)(y)$. Hence, $(\lambda: \nu)$ is a fuzzy ideal of A .

In the following theorem, we characterize relative fuzzy annihilators in terms of fuzzy points. \square

Theorem 4. Let λ be a fuzzy subset of A and ν be a fuzzy ideal. Then, for each $x \in A$,

$$(\lambda: \nu)(x) = \text{Sup}\{\alpha \in [0, 1]: x_\alpha \cdot \lambda \subseteq \nu\}.
 \tag{15}$$

Proof. For each $x \in A$, let us define two sets C_x and B_x as follows:

$$\begin{aligned}
 C_x &= \{\eta(x): \eta \in [0, 1]^A, \eta \cdot \lambda \subseteq \nu\}, \\
 B_x &= \{\alpha \in [0, 1]: x_\alpha \cdot \lambda \subseteq \nu\}.
 \end{aligned}
 \tag{16}$$

Since $\lambda \cdot \nu \subseteq \nu$, then both C_x and B_x are nonempty subsets of $[0, 1]$. Now, we proceed to show that $\vee C_x = \vee B_x$. Let $\alpha \in C_x$. Then, $\alpha = \eta(x)$ for some fuzzy subset η of A satisfying $\eta \cdot \lambda \subseteq \nu$. If $\alpha = 0$, then we can find $\beta \in B_x$ such that $\alpha \leq \beta$. On the contrary, suppose that $\alpha \neq 0$. Then, x_α is a fuzzy point of A such that $x_\alpha \subseteq \eta$, which implies $x_\alpha \cdot \lambda \subseteq \eta \cdot \lambda$ and $\alpha \in B_x$. Thus, $C_x \subseteq B_x$. So, $\vee C_x \leq \vee B_x$.

To show $\vee B_x \leq \vee C_x$, let $\beta \in B_x$. Then, x_β is a fuzzy point of A such that $x_\beta \cdot \lambda \subseteq \nu$. This shows that $\beta \in C_x$. Thus, $B_x \subseteq C_x$. So, $\vee B_x \leq \vee C_x$. So, $\vee B_x = \vee C_x$. Hence, the result is obtained. \square

Example 2. Let $A = \{T, F, U\}$, and define \vee, \wedge , and $'$ on A as follows.

\wedge	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

\vee	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U
x	x'		
T	F		
F	T		
U	U		

Now, consider the C -algebra $C = A \times A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$, where $a_1 = (T, U), a_2 = (F, U), a_3 = (U, T), a_4 = (U, F), a_5 = (U, U), a_6 = (T, T), a_7 = (F, F), a_8 = (T, F)$, and $a_9 = (F, T)$. Then, the set of left zero for \wedge is $I_0 = \{a_2, a_4, a_5, a_7\}$.

If we define two fuzzy subsets θ and μ of C as

$$\begin{aligned}
 \theta(a_2) &= \theta(a_4) = \theta(a_5) = \theta(a_7) = 1, \\
 \theta(a_1) &= \theta(a_8) = 0.6, \\
 \theta(a_3) &= \theta(a_6) = \theta(a_9) = 0.5, \\
 \mu(a_1) &= \mu(a_2) = \mu(a_3) = 0.8, \\
 \mu(a_4) &= \mu(a_5) = \mu(a_6) = 0.7, \\
 \mu(a_7) &= \mu(a_8) = \mu(a_9) = 0.4,
 \end{aligned}
 \tag{17}$$

then θ is a fuzzy ideal of C and $(\mu: \theta)(a_2) = (\mu: \theta)(a_4) = (\mu: \theta)(a_5) = (\mu: \theta)(a_7) = 1, (\mu: \theta)(a_1) = (\mu: \theta)(a_8) = 0.6$ and $(\mu: \theta)(a_3) = (\mu: \theta)(a_6) = (\mu: \theta)(a_9) = 0.5$. Thus, $(\mu: \theta)$ is a fuzzy ideal of C .

In the following lemma, some basic properties of relative fuzzy annihilators can be observed.

Lemma 5. Let η and ξ be fuzzy subsets and λ, ν , and γ be fuzzy ideals of A . Then,

- (1) $(\eta: \lambda) = \chi_A \Leftrightarrow \eta \subseteq \lambda$
- (2) $\nu \subseteq (\eta: \nu)$
- (3) $\eta \subseteq \xi \Rightarrow (\xi: \lambda) \subseteq (\eta: \lambda)$
- (4) $\lambda \subseteq \nu \Rightarrow (\xi: \lambda) \subseteq (\xi: \nu)$
- (5) $(\eta: \lambda \cap \nu) = (\eta: \lambda) \cap (\eta: \nu)$
- (6) $((\eta): \lambda) = (\eta: \lambda)$
- (7) $(\eta \cup \xi: \lambda) = (\eta: \lambda) \cap (\xi: \lambda)$
- (8) $(\lambda \vee \nu: \gamma) = (\lambda: \gamma) \cap (\nu: \gamma)$
- (9) $(\lambda: \nu) = (\lambda \vee \nu: \nu) = (\lambda: \lambda \wedge \nu)$
- (10) $(\eta) \cap \nu \subseteq \lambda \Leftrightarrow \nu \subseteq (\eta: \lambda)$

Proof. The proof of (3) and (4) is straightforward. Now, we proceed to prove the following.

(1) Let $(\eta: \lambda) = \chi_A$. To show $\eta \subseteq \lambda$, assume that $\eta \not\subseteq \lambda$. Then, there is $x \in A$ such that $\eta(x) > \lambda(x)$. This implies that $\delta(x) \leq \lambda(x)$, for each δ such that $\delta \cdot \eta \subseteq \lambda$. Thus, $\lambda(x)$ is an upper bound of $\{\delta(x): \delta \cdot \eta \subseteq \lambda\}$. This shows that $\lambda(x) \geq 1$. Thus,

$\lambda(x) \geq \eta(x)$; it is a contradiction. So, $\eta \subseteq \lambda$. The converse part is trivial.

(2) Since $\nu \wedge (\eta) \subseteq \nu$, we get that $\nu \cdot \eta \subseteq \nu$. Thus, $\nu \subseteq (\eta: \nu)$.

(3)

$$\begin{aligned} (\eta: \lambda) \cap (\eta: \nu) &= \text{Sup}\{\gamma_1 \in \text{FI}(A), \gamma_1 \cdot \eta \subseteq \lambda\} \wedge \text{Sup}\{\gamma_2 \in \text{FI}(A), \gamma_2 \cdot \eta \subseteq \nu\} \\ &= \text{Sup}\{\gamma_1 \wedge \gamma_2: \gamma_1 \cdot \eta \subseteq \lambda, \gamma_2 \cdot \eta \subseteq \nu\} \\ &= \text{Sup}\{\gamma: \gamma_1 \cdot \eta \subseteq \lambda, \gamma_2 \cdot \eta \subseteq \nu\}, \quad \text{Where } \gamma = \gamma_1 \wedge \gamma_2 \\ &= \text{Sup}\{\gamma: \gamma \in \text{FI}(A), \gamma \cdot \eta \subseteq \lambda \cap \nu\} \\ &= (\eta: \lambda \cap \nu). \end{aligned} \tag{18}$$

(4) Since $(\gamma \cdot \eta) = \gamma \wedge (\eta)$, for every $\gamma \in \text{FI}(A)$, we can easily verified that $(\eta: \lambda) = t((\eta): \lambda)$.

(5) By property (3), we have that $(\eta \cup \xi, \lambda) \subseteq (\eta, \lambda) \cap (\xi, \lambda)$. On the contrary,

$$\begin{aligned} (\eta: \lambda) \cap (\xi: \lambda) &= ((\eta): \lambda) \cap ((\xi): \lambda) \\ &= \text{Sup}\{\gamma_1 \in \text{FI}(A): \gamma_1 \wedge (\eta) \subseteq \lambda\} \wedge \text{Sup}\{\gamma_2 \in \text{FI}(A), \gamma_2 \wedge (\xi) \subseteq \lambda\} \\ &= \text{Sup}\{\gamma_1 \wedge \gamma_2: \gamma_1, \gamma_2 \in \text{FI}(L), \gamma_1 \wedge (\eta) \subseteq \lambda, \gamma_2 \wedge (\xi) \subseteq \lambda\}. \end{aligned} \tag{19}$$

Since $\gamma_1 \wedge (\eta) \subseteq \lambda$ and $\gamma_2 \wedge (\xi) \subseteq \lambda$, there exists a fuzzy ideal γ of A contained in γ_1 and γ_2 such that $\gamma \wedge (\eta) \subseteq \lambda$ and $\gamma \wedge (\xi) \subseteq \lambda$. This implies that $(\gamma \wedge ((\eta) \vee (\xi))) \subseteq \lambda$. This shows that

$$\begin{aligned} (\eta: \lambda) \cap (\xi: \lambda) &\subseteq \text{Sup}\{\gamma: \gamma \wedge ((\eta) \vee (\xi)) \subseteq \lambda\} \\ &\subseteq \text{Sup}\{\gamma: \gamma \cdot (\eta \cup \xi) \subseteq \lambda\} \\ &= (\eta \cup \xi: \lambda). \end{aligned} \tag{20}$$

(6) Since $\lambda \vee \nu = (\lambda \cup \nu)$, by (6), we get $(\lambda \vee \nu: \gamma) = (\lambda \cup \nu: \gamma)$. Thus,

$$(\lambda \vee \nu: \gamma) = (\lambda: \gamma) \cap (\nu: \gamma). \tag{21}$$

(7) Since $(\theta: \theta) = \chi_A$, by (8), we get $(\lambda \vee \nu: \nu) = (\lambda: \nu)$. On the contrary, let $\gamma \wedge \lambda \subseteq \nu$ for some fuzzy ideal γ of A . Since $\gamma \wedge \lambda \subseteq \lambda$, we get that $\gamma \wedge \lambda \subseteq \lambda \wedge \nu$. Thus, $(\lambda: \nu) \subseteq (\lambda: \lambda \wedge \nu)$. Since $\lambda \wedge \nu \subseteq \nu$, by (4), we have $(\lambda: \lambda \wedge \nu) \subseteq (\lambda: \nu)$. Hence, $(\lambda: \nu) = (\lambda: \lambda \wedge \nu)$. So, $(\lambda: \nu) = (\lambda \vee \nu: \nu) = (\lambda: \lambda \wedge \nu)$.

(8) If $(\eta) \cap \nu \subseteq \lambda$, then, by (6) and by the definition of relative fuzzy annihilator, $\nu \subseteq (\eta: \lambda)$. Conversely, suppose $\nu \subseteq (\eta: \lambda)$. Since ν is a fuzzy ideal, we can express ν as follows:

$$\nu = \bigvee_{x_\alpha \subseteq \nu} (x_\alpha). \tag{22}$$

Let x_α be a fuzzy point of A such that $x_\alpha \subseteq \nu$. Since $\nu \subseteq (\eta: \lambda)$, we get $x_\alpha \subseteq (\eta: \lambda)$. Thus, $x_\alpha \cdot \eta \subseteq \lambda$. Now,

$$\begin{aligned} (\eta) \cap \nu &= \left(\bigvee_{x_\alpha \subseteq \nu} (x_\alpha) \right) \cap (\eta) \\ &= \bigvee_{x_\alpha \subseteq \nu} (x_\alpha \cdot \eta) \text{ by Lemma 3} \\ &\subseteq \nu. \end{aligned} \tag{23}$$

Thus, $(\eta) \cap \nu \subseteq \lambda$. □

Theorem 5. Let ν be a fuzzy ideal of A . If $\{\lambda_\alpha\}_{\alpha \in \Delta}$ is a class of fuzzy ideals of A , then

$$\left(\bigcup_{\alpha \in \Delta} \lambda_\alpha: \nu \right) = \bigcap_{\alpha \in \Delta} (\lambda_\alpha: \nu). \tag{24}$$

Proof. We know that $\lambda_\alpha \subseteq \bigcup_{\alpha \in \Delta} \lambda_\alpha$ for each $\alpha \in \Delta$. Thus, by Lemma 5 (3), we get $(\bigcup_{\alpha \in \Delta} \lambda_\alpha: \nu) \subseteq (\lambda_\alpha: \nu)$ for each $\alpha \in \Delta$. Thus,

$$\left(\bigcup_{\alpha \in \Delta} \lambda_\alpha: \nu \right) \subseteq \bigcap_{\alpha \in \Delta} (\lambda_\alpha: \nu). \tag{25}$$

On the contrary, put $\eta = \bigcap_{\alpha \in \Delta} (\lambda_\alpha: \nu)$. Then, $\eta \subseteq (\lambda_\alpha: \nu)$, for each $\alpha \in \Delta$. By Lemma 5 (10), we have $\lambda_\alpha \cap \eta \subseteq \nu$, for each $\alpha \in \Delta$. This implies

$$\left(\bigvee_{\alpha \in \Delta} \lambda_\alpha \right) \cap \eta = \bigvee_{\alpha \in \Delta} (\lambda_\alpha \cap \eta) \subseteq \nu. \tag{26}$$

So, by Lemma 5 (10), we have $\eta \subseteq (\bigvee_{\alpha \in \Delta} \lambda_\alpha: \nu)$. Thus,

$$\bigcap_{\alpha \in \Delta} (\lambda_\alpha: \nu) \subseteq \left(\bigcup_{\alpha \in \Delta} \lambda_\alpha: \nu \right). \tag{27}$$

So,

$$\left(\bigcup_{\alpha \in \Delta} \lambda_{\alpha} : \nu \right) = \bigcap_{\alpha \in \Delta} (\lambda_{\alpha} : \nu). \quad (28)$$

In the following theorem, we prove that $(\lambda : \nu)$ is a relative pseudocomplement of λ and ν in the class of $\text{FI}(A)$. \square

Theorem 6. *Let η be a fuzzy subset and λ and ν fuzzy ideals of A . Then,*

- (1) $(\eta : \lambda)$ is the largest fuzzy ideal such that $(\eta) \cap (\eta : \lambda) \subseteq \lambda$
- (2) $(\lambda : \nu)$ is the largest fuzzy ideal such that $\lambda \cap (\lambda : \nu) \subseteq \nu$

Proof. First, we have to show that $(\eta) \cap (\eta : \lambda) \subseteq \lambda$. For any $x \in A$,

$$\begin{aligned} ((\eta) \cap (\eta : \lambda))(x) &= (\eta)(x) \wedge \text{Sup}\{\gamma(x) : \gamma \in \text{FI}(A), \gamma \wedge (\eta) \subseteq \lambda\} \\ &= \text{Sup}\{(\eta)(x) \wedge \gamma(x) : \gamma \wedge (\eta) \subseteq \lambda\} \\ &= \text{Sup}\{((\eta) \cap \gamma)(x) : \gamma \cap (\eta) \subseteq \lambda\} \\ &\leq \lambda(x). \end{aligned} \quad (29)$$

Thus, $(\eta) \cap (\eta : \lambda) \subseteq \lambda$.

Now, we show that $(\eta : \lambda)$ is the largest fuzzy ideal satisfying $(\eta) \cap (\eta : \lambda) \subseteq \lambda$. Suppose not. Then, there exists a fuzzy ideal γ properly containing $(\eta : \lambda)$ such that $(\eta) \cap \gamma \subseteq \lambda$. Then, by Lemma 5 (10), we get that $\gamma \subseteq (\eta : \lambda)$, which is a contradiction. Therefore, $(\eta : \lambda)$ is the largest fuzzy ideal, satisfying $(\eta) \cap (\eta : \lambda) \subseteq \lambda$.

In [18], Alaba and Addis introduced the concept of fuzzy ideals of C -algebra, and they proved that the class of all fuzzy ideals of a C -algebra is a complete distributive lattice. In the following theorem, using the concept of relative fuzzy annihilator ideals of a C -algebra, we prove that the class of fuzzy ideals of a C -algebra forms the Heyting algebra. \square

Theorem 7. *The set $\text{FI}(A)$ of all fuzzy ideals of A is the Heyting algebra.*

Proof. We know that the set $(\text{FI}(A), \vee, \cap, \chi_{I_0}, \chi_A)$ of all fuzzy ideals of A is a complete distributive lattice. For any fuzzy ideals λ and ν of A , by Theorem 6, $(\lambda : \nu)$ is the largest fuzzy ideal of $\{\gamma \in \text{FI}(A) : \gamma \cap \lambda \subseteq \nu\}$. Thus,

$$\lambda \longrightarrow \nu = (\lambda : \nu). \quad (30)$$

So, $(\text{FI}(A), \vee, \cap, \longrightarrow, \chi_{I_0}, \chi_A)$ is the Heyting algebra. \square

4. Fuzzy Annihilator Ideals

In this section, we study fuzzy annihilator ideals in C -algebras. Some basic properties of fuzzy annihilator ideals are also studied. It is proved that the set of all fuzzy annihilator ideals forms a complete Boolean algebra.

Definition 8. For any fuzzy subset λ of A , the fuzzy subset $(\lambda : \chi_{I_0})$ is a fuzzy ideal denoted by λ^* and λ^* is called a fuzzy annihilator of λ .

Lemma 6. *Let λ be a fuzzy subset of A . Then,*

- (1) $\chi_{I_0} \subseteq \lambda^*$
- (2) $\lambda \cdot \lambda^* \subseteq \chi_{I_0}$
- (3) $\lambda \cdot \lambda^* = \chi_{I_0}$, whenever $\lambda(z) = 1$, where $z \in I_0$
- (4) $\lambda^* \wedge \lambda^{**} = \chi_{I_0}$

Proof. Here, it is enough to prove property (3). Let λ be any fuzzy subset of A and $x \in A$. Then,

$$\begin{aligned} (\lambda \cdot \lambda^*)(x) &= \text{Sup}\{\lambda(a) \wedge \lambda^*(b) : x = a \wedge b\} \\ &= \text{Sup}\{\lambda(a) \wedge \text{Sup}\{\eta(b) : \eta \in \text{FI}(A), \eta \cdot \lambda \subseteq \chi_{I_0}\} : x = a \wedge b\} \\ &= \text{Sup}\{\text{Sup}\{\lambda(a) \wedge \eta(b) : \eta \cdot \lambda \subseteq \chi_{I_0}\} : x = a \wedge b\} \\ &= \text{Sup}\{\text{Sup}\{\lambda(a) \wedge \eta(b) : x = a \wedge b\} : \eta \cdot \lambda \subseteq \chi_{I_0}\} \\ &= \text{Sup}\{(\lambda \cdot \eta)(x) : \eta \cdot \lambda \subseteq \chi_{I_0}\} \\ &\leq \chi_{I_0}(x). \end{aligned} \quad (31)$$

This shows that $\lambda \cdot \lambda^* \subseteq \chi_{I_0}$. If $\lambda(z) = 1$, for $z \in I_0$, then $(\lambda \cdot \lambda^*)(z) = 1$ and $\chi_{I_0} = \lambda \cdot \lambda^*$. \square

Lemma 7. *Let λ and ν be fuzzy subsets of A . Then,*

- (1) $\lambda \subseteq \nu \Rightarrow \nu^* \subseteq \lambda^*$
- (2) $\nu \cdot \lambda \subseteq \chi_{I_0} \Leftrightarrow \nu \subseteq \lambda^*$
- (3) $\nu \cdot \lambda = \chi_{I_0} \Leftrightarrow \nu \subseteq \lambda^*$, whenever $\lambda(z) = 1 = \nu(z)$ for $z \in I_0$

- (4) $\lambda \subseteq \lambda^{**}$
- (5) $\lambda^* = \lambda^{***}$
- (6) $((\lambda])^* = \lambda^*$
- (7) $(\lambda \cup \nu)^* = \lambda^* \cap \nu^*$

Lemma 8. *For any fuzzy ideals λ and ν of A , we have*

- (1) $(\lambda \vee \nu)^* = \lambda^* \cap \nu^*$

$$\begin{aligned}
 (2) \quad & (\lambda \vee \lambda^*)^* = \chi_{I_0} & \left(\bigcup_{i \in I} \lambda_i \right)^* &= \bigcap_{i \in I} \lambda_i^*. & (32) \\
 (3) \quad & (\chi_{I_0})^* = \chi_A \\
 (4) \quad & (\chi_A)^* = \chi_{I_0}
 \end{aligned}$$

Theorem 8. *The set $FI(A)$ of all fuzzy ideals of A is a pseudocomplemented lattice.*

Proof. Let λ be a fuzzy ideal of A . Then, it is clear that λ^* is a fuzzy ideal of A and that $\lambda \cap \lambda^* = \chi_{I_0}$. Suppose now $\theta \in FI(A)$ such that $\lambda \cap \theta = \chi_{I_0}$. Then, by Lemma 7 (2), $\theta \subseteq \lambda^*$, and consequently, λ^* is the pseudocomplement of λ . \square

Lemma 9. *If $\lambda_i \in [0, 1]^A$, for every $i \in I$, then*

Proof. Let $\{\lambda_i: i \in I\}$ be family of fuzzy subsets of L . Since $\lambda_i \subseteq (\cup_{i \in I} \lambda_i)$ for each $i \in I$, by Lemma 7 (1), we have $(\cup_{i \in I} \lambda_i)^* \subseteq \lambda_i^*$. Thus,

$$\left(\bigcup_{i \in I} \lambda_i \right)^* \subseteq \bigcap_{i \in I} \lambda_i^*. \tag{33}$$

To prove $\cap_{i \in I} \lambda_i^* \subseteq (\cup_{i \in I} \lambda_i)^*$, it is enough to show that $(\cap_{i \in I} \lambda_i^*) \cdot (\cup_{j \in I} \lambda_j) \subseteq \chi_{I_0}$. For any $x \in A$,

$$\begin{aligned}
 \left(\left(\bigcap_{i \in I} \lambda_i^* \right) \cdot \left(\bigcup_{j \in I} \lambda_j \right) \right) (x) &= \text{Sup} \left\{ \left(\bigcap_{i \in I} \lambda_i^* \right) (a) \wedge \left(\bigcup_{j \in I} \lambda_j \right) (b) : a \wedge b = x \right\} \\
 &= \text{Sup} \left\{ \left(\bigwedge_{i \in I} \lambda_i^* (a) \right) \wedge \left(\bigvee_{j \in I} \lambda_j (b) \right) : a \wedge b = x \right\} \\
 &= \text{Sup} \left\{ \bigvee_{j \in I} \left(\left(\bigwedge_{i \in I} \lambda_i^* (a) \right) \wedge \lambda_j (b) \right) : a \wedge b = x \right\} \\
 &\leq \text{Sup} \left\{ \bigvee_{j \in I} \left(\lambda_j^* (a) \wedge \lambda_j (b) \right) : a \wedge b = x \right\} & (34) \\
 &\leq \text{Sup} \left\{ \bigvee_{j \in I} \left(\lambda_j^* \cdot \lambda_j \right) (x) : a \wedge b = x \right\} \\
 &\leq \text{Sup} \left\{ \bigvee_{j \in I} \left(\chi_{I_0} \right) (x) \right\} \\
 &= \chi_{I_0} (x).
 \end{aligned}$$

Thus, by Lemma 7 (2), we get that $(\cap_{i \in I} \lambda_i^*) \subseteq (\cup_{i \in I} \lambda_i)^*$. So,

$$\left(\bigcap_{i \in I} \lambda_i^* \right) = \left(\bigcup_{i \in I} \lambda_i \right)^*. \tag{35}$$

Now, we define the fuzzy annihilator ideal. \square

Definition 9. A fuzzy ideal λ of A is called a fuzzy annihilator ideal if $\lambda = \nu^*$, for some fuzzy subset ν of A , or equivalently, if $\lambda = \lambda^{**}$.

We denote the class of all fuzzy annihilator ideals of A by $FI^*(A)$.

Example 3. Consider the three-element C -algebra $A = \{T, F, U\}$ and $C = A \times A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ given in Example 2. If we define a fuzzy subset λ of C as

$$\begin{aligned}
 \lambda(a_1) &= \lambda(a_2) = \lambda(a_4) = \lambda(a_5) = \lambda(a_7) = \lambda(a_8) = 1, \\
 \lambda(a_3) &= \lambda(a_6) = \lambda(a_9) = 0.4,
 \end{aligned} \tag{36}$$

then λ is a fuzzy ideal of C and $\lambda = \lambda^{**}$. Thus, λ is a fuzzy annihilator ideal of C .

Lemma 10. *Let $\lambda, \nu \in FI^*(A)$. Then,*

- (1) $\lambda \cap \nu = (\lambda^* \vee \nu^*)^*$
- (2) $\lambda \cap \nu = (\lambda \cap \nu)^{**}$

The result (2) of the above lemma can be generalized as given in the following.

Corollary 1. *If $\{\lambda_i: i \in \Delta\}$ is a family of fuzzy annihilator ideals of A , then*

$$\left(\bigcap_{i \in \Delta} \lambda_i \right)^{**} = \bigcap_{i \in \Delta} \lambda_i. \tag{37}$$

Theorem 9. *A map $\alpha: FI(A) \rightarrow FI(A)$ defined by $\alpha(\lambda) = \lambda^{**}, \forall \lambda \in FI(A)$ is a closure operator on $FI(A)$. That is,*

- (1) $\lambda \subseteq \alpha(\lambda)$
- (2) $\alpha(\alpha(\lambda)) = \alpha(\lambda)$
- (3) $\lambda \subseteq \nu \Rightarrow \alpha(\lambda) \subseteq \alpha(\nu)$, for any two fuzzy ideals λ, ν of A

Fuzzy annihilator ideals are simply the closed elements with respect to the closure operator.

Lemma 11. *If $\lambda, \nu \in FI^*(L)$, the supremum of λ and ν is given by*

$$\lambda \sqcup \nu = (\lambda^* \cap \nu^*)^*. \quad (38)$$

Proof. First, we need to show $\lambda \sqcup \nu$ is a fuzzy annihilator ideal. Clearly $\lambda \sqcup \nu$ is a fuzzy ideal of A . Since $\lambda^* \cap \nu^* \subseteq \lambda^*$, we get $\lambda = \lambda^{**} \subseteq (\nu^* \subseteq \lambda^*)^* = \lambda \sqcup \nu$. Similarly, $\nu \subseteq \lambda \sqcup \nu$. This implies $\lambda \sqcup \nu$ is an upper bound of λ and ν . Suppose that γ is a fuzzy annihilator ideal of A such that $\lambda \subseteq \gamma$ and $\nu \subseteq \gamma$. Then, we get $\gamma^* \subseteq \lambda^*$ and $\gamma^* \subseteq \nu^*$ and $\gamma \subseteq \lambda^* \cap \nu^*$. This implies $\lambda \sqcup \nu = (\lambda^* \cap \nu^*)^* \subseteq \gamma^{**} = \gamma$. Hence, $\lambda \sqcup \nu$ is the smallest fuzzy annihilator ideal containing λ and ν . \square

Corollary 2. *Let $(\lambda_i)_{i \in I}$ be a family of fuzzy annihilator ideals of A . Then, $\sqcup_{i \in I} \lambda_i$ is the smallest fuzzy annihilator ideal containing each λ_i .*

In the following theorem, we prove that the class of all fuzzy annihilator ideals forms a complete Boolean algebra.

Theorem 10. *The set $FI^*(A)$ of all fuzzy annihilator ideals of A forms a complete Boolean algebra.*

Proof. Clearly, $(FI^*(A), \cap, \sqcup, \chi_{I_0}, \chi_A)$ is a complete bounded lattice. To show the distributivity, let $\lambda, \nu, \eta \in FI^*(A)$. Then,

$$\begin{aligned} \lambda \sqcup (\nu \cap \eta) &= (\lambda^* \cap (\nu \cap \eta)^*)^* \\ &= (\lambda^{***} \cap (\nu^{**} \cap \eta^{**})^*)^* \\ &= (\lambda^* \cap (\nu^* \vee \eta^*))^{***} \\ &= ((\lambda^* \cap \nu^*) \vee (\lambda^* \cap \eta^*))^* \\ &= (\lambda^* \cap \nu^*)^* \cap (\lambda^* \cap \eta^*)^* \\ &= (\lambda \sqcup \nu) \cap (\lambda \sqcup \eta). \end{aligned} \quad (39)$$

Hence, $(FI^*(A), \cap, \sqcup, \chi_{I_0}, \chi_A)$ is a complete distributive lattice. For any $\lambda \in FI^*(A)$, we have $\lambda \cap \lambda^* = \chi_{I_0}$ and $\lambda \wedge \lambda^* = (\lambda \sqcup \lambda^*)^* = \chi_A$. Hence, λ^* is the complement of λ in $FI^*(A)$. Therefore, $(FI^*(A), \cap, \sqcup, \chi_{I_0}, \chi_A)$ is a complete Boolean algebra. \square

Definition 10. A fuzzy ideal μ of A is called dense fuzzy ideal if $\mu^* = \chi_{I_0}$.

5. Fuzzy Annihilator Preserving Homomorphism

In this section, we study some basic properties of fuzzy annihilator preserving homomorphisms. We give a sufficient condition for a homomorphism to be fuzzy annihilator preserving. Finally, we show that the images and inverse images of fuzzy annihilator ideals are again fuzzy annihilator ideals.

Throughout this section, A and A' denote C -algebras with the smallest ideals I_0 and I'_0 , respectively, and $f: A \rightarrow A'$ denotes a C -algebra homomorphism.

Lemma 12. *In A , the following conditions hold:*

- (1) $\chi_{I_0} \subseteq \chi_{Kerf}$
- (2) $f(\chi_{I_0}) \subseteq \chi_{I'_0}$
- (3) χ_{Kerf} is a fuzzy ideal of A

Lemma 13. *If λ is any fuzzy subset of A and ν is a fuzzy ideal of A , then*

$$f((\lambda: \nu)) \subseteq (f(\lambda): f(\nu)). \quad (40)$$

In particular, if $\nu = \chi_{I_0}$, then $f(\lambda^*) \subseteq (f(\lambda))^*$.

Proof. Let λ be any fuzzy subset of A and ν be a fuzzy ideal of A . For any $y \in A'$,

$$\begin{aligned} f((\lambda: \nu))(y) &= \text{Sup}\{(\lambda: \nu)(x): x \in f^{-1}(y)\} \\ &= \text{Sup}\{\text{Sup}\{\eta(x): \eta \in FI(A), \eta \cdot \lambda \subseteq \nu\}: x \in f^{-1}(y)\} \\ &= \text{Sup}\{f(\eta)(y): \eta \cdot \lambda \subseteq \nu\} \\ &\leq \text{Sup}\{\gamma(y): \gamma \in FI(A'), \lambda \cdot f(\lambda) \subseteq f(\nu)\} \\ &= (f(\lambda): f(\nu))(y). \end{aligned} \quad (41)$$

Then, $f((\lambda: \nu)) \subseteq (f(\lambda): f(\nu))$. \square

Definition 11. For any fuzzy subset λ of L , f is said to be a fuzzy annihilator preserving if $f(\lambda^*) = (f(\lambda))^*$.

In the following theorem, we give a sufficient condition for a homomorphism to be fuzzy annihilator preserving.

Theorem 11. *If $Kerf = I_0$ and f is onto, then f is a fuzzy annihilator preserving.*

Proof. Let λ be any fuzzy subset of A . Then, $f(\lambda^*) \subseteq (f(\lambda))^*$. Since $Kerf = I_0$ and f is onto, $f^{-1}(\chi_{I'_0}) = \chi_{I_0}$ and $\nu = f(f^{-1}(\nu))$, for all $\nu \in [0, 1]^{A'}$. Let $y \in A'$. Then,

$$\begin{aligned} (f(\lambda))^*(y) &= \text{Sup}\{\nu(y): \nu \in FI(A'), \nu \cdot f(\lambda) \subseteq \chi_{I'_0}\} \\ &= \text{Sup}\{\nu(y): f(f^{-1}(\nu)) \cdot f(\lambda) \subseteq \chi_{I'_0}\} \\ &= \text{Sup}\{\nu(y): f(f^{-1}(\nu) \cdot \lambda) \subseteq \chi_{I'_0}\} \\ &= \text{Sup}\{\nu(y): f^{-1}(\nu) \cdot \lambda \subseteq f^{-1}(\chi_{I'_0})\} \\ &= \text{Sup}\{\nu(f(x)): f^{-1}(\nu) \cdot \lambda \subseteq \chi_{I_0}, f(x) = y\} \\ &\leq \text{Sup}\{\text{Sup}\{f^{-1}(\nu)(x): f^{-1}(\nu) \cdot \lambda \subseteq \chi_{I_0}\}: x \in f^{-1}(y)\} \\ &\leq \text{Sup}\{\text{Sup}\{\eta(x): \eta \in FI(A), \eta \cdot \lambda \subseteq \chi_{I_0}\}: x \in f^{-1}(y)\} \\ &= f(\lambda^*)(y). \end{aligned} \quad (42)$$

Thus, $(f(\lambda))^* \subseteq f(\lambda^*)$. So, f preserves fuzzy annihilator. \square

Theorem 12. If $\text{Ker}f = I_0$ and f is onto, then f^{-1} preserves the fuzzy annihilator.

Proof. Let λ be any fuzzy subset of A' and $x \in A$. Then,

$$\begin{aligned} f^{-1}(\lambda^*)(x) &= (\lambda^*)(f(x)) \\ &= \text{Sup}\{\nu(f(x)): \nu \in \text{FI}(A'), \nu \cdot \lambda \subseteq \chi_{I_0}\} \\ &= \text{Sup}\{\nu(f(x)): f(f^{-1}(\nu)) \cdot f(f^{-1}(\lambda)) \subseteq \chi_{I_0}\} \\ &= \text{Sup}\{f^{-1}(\nu)(x): f^{-1}(\nu) \cdot f^{-1}(\lambda) \subseteq \chi_{I_0}\} \\ &\leq \text{Sup}\{\eta(x): \eta \in \text{FI}(A), \eta \cdot f^{-1}(\lambda) \subseteq \chi_{I_0}\} \\ &= (f^{-1}(\lambda))^*(x). \end{aligned} \tag{43}$$

Thus, $f^{-1}(\lambda^*) \subseteq (f^{-1}(\lambda))^*$. Similarly, $(f^{-1}(\lambda))^* \subseteq f^{-1}(\lambda^*)$. Therefore, $f^{-1}(\lambda^*) = (f^{-1}(\lambda))^*$. \square

Theorem 13. For any C -algebra A , the following conditions hold:

- (1) If f is fuzzy annihilator preserving and onto, then $f(\lambda)$ is a fuzzy annihilator ideal of A' , for every fuzzy annihilator ideal λ of A
- (2) If f^{-1} preserves the annihilator, then $f^{-1}(\lambda)$ is a fuzzy annihilator ideal of A , for every fuzzy annihilator ideal λ of A'
- (3) $\chi_{\text{Ker}f}$ is a fuzzy annihilator ideal of A

6. Conclusion

In this work, we studied the concept of relative fuzzy annihilator ideals of C -algebras. We characterized relative fuzzy annihilators in terms of fuzzy points. We proved that the class of fuzzy ideals of C -algebras forms the Heything algebra. We also studied fuzzy annihilator ideals and investigate some its properties. It is shown that the class of all fuzzy annihilator ideals forms a complete Boolean algebra. Moreover, we study the concept of fuzzy annihilator preserving homomorphism. Our future work will focus on fuzzy congruence relation on C -algebras.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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