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Research Article

Generalized Ideals of BCK/BCI-Algebras Based on Fuzzy Soft Set Theory

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In the present paper, using Lukaswize triple-valued logic, we introduce the notion of (α, β)-intuitionistic fuzzy soft ideal of BCK/BCI-algebras, where α and β are the membership values between an intuitionistic fuzzy soft point and intuitionistic fuzzy set. Moreover, intuitionistic fuzzy soft ideals with thresholds are introduced, and their related properties are investigated.

1. Introduction

In the real world, there are several difficult problems that cannot be solved by usual mathematical methods. As a result, several theories were introduced to solve the existing problems. One of them is the theory of fuzzy sets proposed by Zadeh [1]. After the introduction of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields up to now (see, for example, [2, 3]), and many generalizations of fuzzy sets have been defined and studied.

Atanassov defined a new generalization of the fuzzy set, namely, intuitionistic fuzzy set [4]. Kim considered intuitionistic fuzziness on subalgebras and ideals in BCK-algebras [5]. All these theories have their weaknesses as pointed out in [6]; thus, Molodstov introduced the idea of soft sets which have been a useful tool for dealing with uncertainties. Maji et al. defined soft binary operations [7]. Also, Maji et al. presented the concept of fuzzy soft set [8]. Later on, Maji et al. introduced and studied intuitionistic fuzzy soft sets (see [9–11]), and more specifically, Akram et al. studied intuitionistic fuzzy soft K-algebras (see [12]). In recent years, a number of research papers have been devoted to the study of soft set theory applied to different algebraic structures (see, for example, [13–18]). Jun et al. applied soft set and fuzzy soft set theories to BCK/BCI-algebras in [19, 20], respectively, and Akram et al. applied the same theories on K-algebras in [21]. Larimi and Jun introduced the concepts of $(\in, \in\vee q)$-intuitionistic fuzzy h-ideals of hemiring [22]. Various attributes of BCK/BCI-algebra are considered in [23–31].

Based on fuzzy points, Jana et al. studied different types of ideals [32, 33]. Moreover, the same authors studied generalized intuitionistic fuzzy ideals of BCK/BCI-algebras and Lukaswize intuitionistic fuzzy BCK/BCI-subalgebras based on 3-valued logic (see [34, 35]).

This motivated us to study intuitionistic fuzzy soft ideals of BCK/BCI-algebra X using cut sets and the degree of existence of fuzzy soft point (x, a) in an intuitionistic fuzzy soft set $(\widetilde{R}, tA)$ of X. Also, an (α, β)-intuitionistic fuzzy soft ideal of X is introduced by applying the Lukaswize triple-valued logic, where $\alpha, \beta \in \{\in, q, \in\wedge q, \in\vee q\}$, with $\alpha \neq \in\wedge q$. Moreover, intuitionistic fuzzy soft ideals of BCK/BCI-algebras with thresholds are investigated, and related results are obtained.

2. Preliminaries

The algebraic structures of BCK- and BCI-algebras were introduced by K. Iseki.

An algebra $(X, *, 0)$ of type $(2, 0)$ with 0 as the identity element is called a BCI-algebra if for every $x, y, z \in X$, the following conditions are satisfied:

$$\begin{aligned}
&(K_1)((x * y) * (x * z)) * (z * y) = 0,\\
&(K_2)(x * (x * y)) * y = 0,\\
&(K_3) x * x = 0,\\
&(K_4) x * y = 0,\\
&y * x = 0 \Longrightarrow x = y.
\end{aligned} \tag{1}$$

The partial ordering is defined as $x \leq y \Longleftrightarrow x * y = 0$.

If BCI-algebra X satisfies $0 * x = 0$ for every $x \in X$, then X is a BCK-algebra.

A nonempty subset I of X is called an ideal of X if it satisfies the following:

(K_{11}) $0 \in I$

(K_{12}) for every $x, y \in X$, $x * y \in I$, and $y \in I \Rightarrow x \in I$

Unless or otherwise mentioned, X denotes a BCK/BCI-algebra.

Definition 1 (see [12]). For an initial set X and a set of parameters A, a pair (R, A) is said to be a soft set over $X \Leftrightarrow R$ which is a mapping of A into the set of all subsets of the set X.

Definition 2 (see [14]). Let E be a collection of parameters, and let $\mathbb{R}(X)$ denote the collection of all fuzzy sets in X. Then, $(\widetilde{R}, tA)$ is called a fuzzy soft set over X, where A is a subset of E and $\widetilde{R}$ is a mapping given by $\widetilde{R}\colon A \longrightarrow \mathbb{R}(X)$.

It is easy to see that every classical soft set may be considered as a fuzzy soft set. In general, for every $\delta \in A$, $\widetilde{R}[\delta]$ is a fuzzy subset of X, and it is called a fuzzy value set. If for every $\delta \in A$, $\widetilde{R}[\delta]$ is a crisp subset of X, then $(\widetilde{R}, tA)$ is generated as a standard soft set. Let $\varphi_{\widetilde{R}[\delta]}(x)$ denote the degree of existence function; then, $\widetilde{R}[\delta]$ can be written as a fuzzy set such that $\widetilde{R}[\delta] = \left\{(x, \varphi_{\widetilde{R}[\delta]}(x)) | x \in X \text{ and } \delta \in A\right\}$.

Definition 3 (see [15]). Let E be a collection of parameters, and let $\mathbb{IR}(X)$ denote the set of all intuitionistic fuzzy sets in X. Then, $(\widetilde{R}, tA)$ is called an intuitionistic fuzzy soft set (IFSS) over X, where A is a subset of E and $\widetilde{R}$ is a mapping given by $\widetilde{R}\colon A \longrightarrow \mathbb{IR}(X)$.

In general, for every $\delta \in A$, $\widetilde{R}[\delta]$ is an intuitionistic fuzzy subset of X, and it is called an intuitionistic fuzzy value set. Clearly, $\widetilde{R}[\delta]$ can be written as an intuitionistic fuzzy set (IFS) such that $\widetilde{R}[\delta] = \left\{(x, \varphi_{\widetilde{R}[\delta]}(x), \psi_{\widetilde{R}[\delta]}(x)) | \quad x \in X \text{ and } \delta \in A\right\}$, where $\varphi_{\widetilde{R}[\delta]}(x)$ and $\psi_{\widetilde{R}[\delta]}(x)$ represent the degree of existence and nonexistence functions, respectively. If for every $x \in X$, $\varphi_{\widetilde{R}[\delta]}(x) = 1 - \psi_{\widetilde{R}[\delta]}(x)$, then $\widetilde{R}[\delta]$ will be generated as a standard fuzzy set, and then $(\widetilde{R}, tA)$ will be generated as a traditional fuzzy soft set.

3. (α, β)-Intuitionistic Fuzzy Soft Ideals of BCK/BCI-Algebras

Definition 4. An IFSS $(\widetilde{R}, tA)$ in X is called an intuitionistic fuzzy soft ideal (IFSID) of X if $\widetilde{R}[\delta] = \left\{(x, \varphi_{\widetilde{R}[\delta]}(x), \psi_{\widetilde{R}[\delta]}(x)) | x \in X \text{ and } \delta \in A\right\}$ satisfies the following:

(1) $\varphi_{\widetilde{R}[\delta]}(0) \geq \varphi_{\widetilde{R}[\delta]}(x)$ and $\psi_{\widetilde{R}[\delta]}(0) \leq \psi_{\widetilde{R}[\delta]}(x)$,

(2) $\varphi_{\widetilde{R}[\delta]}(x) \geq \varphi_{\widetilde{R}[\delta]}(x * y) \vee \varphi_{\widetilde{R}[\delta]}(y)$ and $\psi_{\widetilde{R}[\delta]}(x) \leq \psi_{\widetilde{R}[\delta]}(x * y) \wedge \psi_{\widetilde{R}[\delta]}(y)$,

for each $x, y \in X$ and $\delta \in A$.

Definition 5. Let $(\widetilde{R}, tA)$ be an IFSS of X and $a \in [0, 1]$.

(1) The sets

$$\widetilde{R}[\delta]_a(x) = \begin{cases} 1, & \text{if } \varphi_{\widetilde{R}[\delta]}(x) \geq a,\\ \dfrac{1}{2}, & \text{if } \varphi_{\widetilde{R}[\delta]}(x) < a \leq 1 - \psi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a > 1 - \psi_{\widetilde{R}[\delta]}(x), \end{cases} \tag{2}$$

$$\widetilde{R}[\delta]_{\underline{a}}(x) = \begin{cases} 1, & \text{if } \varphi_{\widetilde{R}[\delta]}(x) > a,\\ \dfrac{1}{2}, & \text{if } \varphi_{\widetilde{R}[\delta]}(x) \leq a < 1 - \psi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a \geq 1 - \psi_{\widetilde{R}[\delta]}(x), \end{cases} \tag{3}$$

are called the a-upper cut and a-stronger upper cut of the IFSS $(\widetilde{R}, tA)$, respectively.

(2) The sets

$$\widetilde{R}[\delta]^a(x) = \begin{cases} 1, & \text{if } \psi_{\widetilde{R}[\delta]}(x) \geq a,\\ \dfrac{1}{2}, & \text{if } \psi_{\widetilde{R}[\delta]}(x) < a \leq 1 - \varphi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a > 1 - \varphi_{\widetilde{R}[\delta]}(x), \end{cases} \tag{4}$$

$$\widetilde{R}[\delta]^{\underline{a}}(x) = \begin{cases} 1, & \text{if } \psi_{\widetilde{R}[\delta]}(x) > a,\\ \dfrac{1}{2}, & \text{if } \psi_{\widetilde{R}[\delta]}(x) \leq a < 1 - \varphi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a \geq 1 - \varphi_{\widetilde{R}[\delta]}(x), \end{cases} \tag{5}$$

are called the a-lower cut and a-stronger lower cut of the IFSS $(\widetilde{R}, tA)$, respectively.

(3) The sets

$$\widetilde{R}[\delta]_{[a]}(x)=\begin{cases}1, & \text{if } \varphi_{\widetilde{R}[\delta]}(x)+a\geq 1,\\ \frac{1}{2}, & \text{if } \psi_{\widetilde{R}[\delta]}(x)\leq a<1-\varphi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a<\psi_{\widetilde{R}[\delta]}(x),\end{cases} \quad (6)$$

$$\widetilde{R}[\delta]_{[\underline{a}]}(x)=\begin{cases}1, & \text{if } \varphi_{\widetilde{R}[\delta]}(x)+a>1,\\ \frac{1}{2}, & \text{if } \psi_{\widetilde{R}[\delta]}(x)<a\leq 1-\varphi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a\leq\psi_{\widetilde{R}[\delta]}(x),\end{cases} \quad (7)$$

are called the a-upper Q-cut and a-stronger upper Q-cut of the IFSS $(\widetilde{R}, tA)$, respectively.

(4) The sets

$$\widetilde{R}[\delta]^{[a]}(x)=\begin{cases}1, & \text{if } \psi_{\widetilde{R}[\delta]}(x)+a\geq 1,\\ \frac{1}{2}, & \text{if } \varphi_{\widetilde{R}[\delta]}(x)\leq a<1-\psi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a<\varphi_{\widetilde{R}[\delta]}(x),\end{cases} \quad (8)$$

$$\widetilde{R}[\delta]^{[\underline{a}]}(x)=\begin{cases}1, & \text{if } \psi_{\widetilde{R}[\delta]}(x)+a>1,\\ \frac{1}{2}, & \text{if } \varphi_{\widetilde{R}[\delta]}(x)<a\leq 1-\psi_{\widetilde{R}[\delta]}(x),\\ 0, & \text{if } a\leq\varphi_{\widetilde{R}[\delta]}(x),\end{cases} \quad (9)$$

are called the a-lower Q-cut and a-stronger lower Q-cut of the IFSS $(\widetilde{R}, tA)$, respectively.

Definition 6

(1) The degree of existence of $(x,a)\in\widetilde{R}[\delta]$ is $[(x,a)\in\widetilde{R}[\delta]]$, and the degree of existence of $(x,a)q\widetilde{R}[\delta]$ is $[(x,a)q\widetilde{R}[\delta]]$ if it satisfies the following relations: $[(x,a)\in\widetilde{R}[\delta]]=\widetilde{R}[\delta]_a(x)$ and $[(x,a)q\widetilde{R}[\delta]]=\widetilde{R}[\delta]_{[\underline{a}]}(x)$

(2) The degree of existence of $(x,a)\in\widetilde{R}[\delta]$ and $(x,a)q\widetilde{R}[\delta]$ is $[(x,a)\in\wedge q\widetilde{R}[\delta]]$, and the degree of existence of $(x,a)\in\widetilde{R}[\delta]$ or $(x,a)q\widetilde{R}[\delta]$ is $[(x,a)\in\vee q\widetilde{R}[\delta]]$ if it satisfies the following relations: $[(x,a)\in\wedge q\widetilde{R}[\delta]]=[(x,a)\in\widetilde{R}[\delta]]\wedge[(x,a)q\widetilde{R}[\delta]]=\widetilde{R}[\delta]_a(x)\wedge\widetilde{R}[\delta]_{[\underline{a}]}(x)$ and $[(x,a)\in\vee q\widetilde{R}[\delta]]=[(x,a)\in\widetilde{R}[\delta]]\vee[(x,a)q\widetilde{R}[\delta]]=\widetilde{R}[\delta]_a(x)\vee\widetilde{R}[\delta]_{[\underline{a}]}(x)$

(3) The degree of nonexistence of $(x,a)\in\widetilde{R}[\delta]$ is $[(x,a)\overline{\in}\widetilde{R}[\delta]]$, and the degree of nonexistence of $(x,a)q\widetilde{R}[\delta]$ is $[(x,a)\overline{q}\widetilde{R}[\delta]]$ if it satisfies the following relations: $[(x,a)\overline{\in}\widetilde{R}[\delta]]=\widetilde{R}[\delta]^a(x)$ and $[(x,a)\overline{q}\widetilde{R}[\delta]]=\widetilde{R}[\delta]^{[\underline{a}]}(x)$

(4) The degree of nonexistence of $(x,a)\in\widetilde{R}[\delta]$ and $(x,a)q\widetilde{R}[\delta]$ is $[(x,a)\overline{\in\wedge q}\widetilde{R}[\delta]]$, and the degree of nonexistence of $(x,a)\in\widetilde{R}[\delta]$ or $(x,a)q\widetilde{R}[\delta]$ is $[(x,a)\overline{\in\vee q}\widetilde{R}[\delta]]$ if it satisfies the following relations: $[(x,a)\overline{\in\wedge q}\widetilde{R}[\delta]]=[(x,a)\overline{\in}\vee\overline{q}\widetilde{R}[\delta]]=[(x,a)\overline{\in}\widetilde{R}[\delta]]\vee[(x,a)\overline{q}\widetilde{R}[\delta]]=\widetilde{R}[\delta]^a(x)\vee\widetilde{R}[\delta]^{[\underline{a}]}(x)$ and $[(x,a)\overline{\in\vee q}\widetilde{R}[\delta]]=[(x,a)\overline{\in}\widetilde{R}[\delta]\wedge(x,a)\overline{q}\widetilde{R}[\delta]]=[(x,a)\overline{\in}\widetilde{R}[\delta]]\wedge[(x,a)\overline{q}\widetilde{R}[\delta]]=\widetilde{R}[\delta]^a(x)\wedge\widetilde{R}[\delta]^{[\underline{a}]}(x)$

Let $\longrightarrow$ denote the implication of Lukaswize triple-valued logic. Lukaswize truth table is presented in Table 1.

Let $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$. Then, for $a\in[0,1], x\in X$, (x,a) is a fuzzy soft point, and $[(x,a)\alpha\widetilde{R}[\delta]]$, $[(x,a)\beta\widetilde{R}[\delta]]\in\{0,1/2,1\}$.

Definition 7. Let $(\widetilde{R}, tA)$ be an IFSS in X. If for every $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$ and $s,t\in(0,1]$ such that $(\widetilde{R}, tA)$ satisfies for every $x,y\in X$ and $\delta\in A$,

(1) $[(x,s)\alpha\widetilde{R}[\delta]]\longrightarrow[(0,s)\beta\widetilde{R}[\delta]]=1$,

(2) $[(x*y,s)\alpha\widetilde{R}[\delta]]\wedge[(y,t)\alpha\widetilde{R}[\delta]]\longrightarrow[(x,s\wedge t)\beta\widetilde{R}[\delta]]=1$,

then $(\widetilde{R}, tA)$ is called an (α,β)-intuitionistic fuzzy soft ideal of X.

Let X be a set. Then, define a triple-valued fuzzy set mapping $\widetilde{R}[\delta]: X\longrightarrow\{0,1/2,1\}$.

Definition 8. Let $(\widetilde{R}, tA)$ be an IFSS in X. If for every $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$ and $s,t\in(0,1]$, the IFS $\widetilde{R}[\delta]$ satisfies for every $x,y\in X$,

(1) $[(0,s)\beta\widetilde{R}[\delta]]\geq[(x,s)\alpha\widetilde{R}[\delta]]$,

(2) $[(x,s\wedge t)\beta\widetilde{R}[\delta]]\geq[(x*y,s)\alpha\widetilde{R}[\delta]]\wedge[(y,t)\alpha\widetilde{R}[\delta]]$,

then $(\widetilde{R}, tA)$ is an (α,β)-intuitionistic fuzzy soft ideal of X.

Example 1. Consider a BCK/BCI-algebra $X=\{0,1,2,3,4\}$ with Table 2.

Let us consider the following IFS of X:

$$\varphi_{\widetilde{R}[\delta]}(x)=\begin{cases}0.5, & \text{if } x=0,\\ 0.3, & \text{if } x=1,3,\\ 0.2, & \text{if } x=2,4,\end{cases}$$
$$\psi_{\widetilde{R}[\delta]}(x)=\begin{cases}0.3, & \text{if } x=0,\\ 0.5, & \text{if } x=1,3,\\ 0.6, & \text{if } x=2,4.\end{cases} \quad (10)$$

Then, it can be easily shown that $\widetilde{R}[\delta]=(x,\varphi_{\widetilde{R}[\delta]}(x),\psi_{\widetilde{R}[\delta]}(x))$ is an $(\in,\in\vee q)$–IFID of X. Hence, $(\widetilde{R}, tA)$ is an $(\in,\in\vee q)$–IFSID of X.

Also, if we consider the IFS of X,

Table 1: Lukaswize truth table.

⟶	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

Table 2: : Cayley table of the binary operation *.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

$$\varphi_{\widetilde{R}[\delta]}(x)=\begin{cases}0.6 & \text{if } x=0,\\ 0.2 & \text{if } x=1,3,\\ 0.1 & \text{if } x=2,4,\end{cases}\qquad \psi_{\widetilde{R}[\delta]}(x)=\begin{cases}0.1 & \text{if } x=0,\\ 0.2 & \text{if } x=1,3,\\ 0.5 & \text{if } x=2,4,\end{cases} \tag{11}$$

then it can be easily proved that $\widetilde{R}[\delta]=(x,\varphi_{\widetilde{R}[\delta]}(x),\psi_{\widetilde{R}[\delta]}(x))$ is an $(\in\wedge q,\in)$-IFID of X. Hence, $(\widetilde{R},tA)$ is an $(\in,\in\wedge q)$–IFSID of X.

Theorem 1. *Let $(\widetilde{R},tA)$ be an (α,β)-intuitionistic fuzzy soft ideal of X. If $\alpha\notin\wedge q$, then $(\widetilde{R},tA)_{\underline{0}}$ is a fuzzy soft ideal of X.*

Proof. We shall prove that

(1) $\widetilde{R}[\delta]_{\underline{0}}(0)\geq\widetilde{R}[\delta]_{\underline{0}}(x)$

(2) $\widetilde{R}[\delta]_{\underline{0}}(x)\geq\widetilde{R}[\delta]_{\underline{0}}(x*y)\wedge\widetilde{R}[\delta]_{\underline{0}}(y)$ for every $x,y\in X$

(1) We have to show $\widetilde{R}[\delta]_{\underline{0}}(x)=1\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(0)=1$. If $\widetilde{R}[\delta]_{\underline{0}}(x)=1$, then $\varphi_{\widetilde{R}[\delta]}(x)>0$. Let $\varphi_{\widetilde{R}[\delta]}(x)>t$ and $t>0$ for all $x\in P$ and for all $t\in(0,1]$. Then, there exists $s\in(0,1]$ such that $0<s-1<t=\varphi_{\widetilde{R}[\delta]}(x)$. Thus, $[(x,t)\in\widetilde{R}[\delta]]=\widetilde{R}[\delta]_t(x)=1$ and $[(x,s)q\widetilde{R}[\delta]]=\widetilde{R}[\delta]_{[\underline{s}]}(x)=1$.

If $\alpha=\ \in$ or $\alpha=\ \in\vee q$, then $[(x,t)\alpha\widetilde{R}[\delta]]=1$, and $\beta\in\{\in,q,\wedge q,\vee q\}$. Now, from Definition 8, $1\geq[(0,t)\beta\widetilde{R}[\delta]]\geq[(x,t)\alpha\widetilde{R}[\delta]]=1$. Therefore, $[(0,t)\beta\widetilde{R}[\delta]]=1$

$\Rightarrow$ either $\widetilde{R}[\delta]_t(0)=1$ or $\widetilde{R}[\delta]_{[\underline{t}]}(0)=1$
$\Rightarrow$ either $\varphi_{\widetilde{R}[\delta]}(0)\geq t>0$ or $\varphi_{\widetilde{R}[\delta]}(0)>1-t\geq0$
$\Rightarrow\varphi_{\widetilde{R}[\delta]}(0)>0\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(0)=1$

If $\alpha=q$, then $[(x,s)\alpha\widetilde{R}[\delta]]=1$, and $\beta\in\{\in,q,\wedge q,\vee q\}$. Now, from Definition 8, $1\geq[(0,s)\beta\widetilde{R}[\delta]]\geq[(x,s)\alpha\widetilde{R}[\delta]]=1$. Therefore, $[(0,s)\beta\widetilde{R}[\delta]]=1$

$\Rightarrow$ either $\widetilde{R}[\delta]_s(0)=1$ or $\widetilde{R}[\delta]_{[\underline{s}]}(0)=1$
$\Rightarrow$ either $\varphi_{\widetilde{R}[\delta]}(0)\geq s>0$ or $\varphi_{\widetilde{R}[\delta]}(0)>1-s\geq0$
$\Rightarrow\varphi_{\widetilde{R}[\delta]}(0)>0\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(0)=1$

Next, we have to show that $\widetilde{R}[\delta]_{\underline{0}}(x)=1/2\Rightarrow\widetilde{R}[\delta]_{[\underline{0}]}(0)\geq1/2$. Suppose $\widetilde{R}[\delta]_{\underline{0}}(x)=1/2$. Then, $\psi_{\widetilde{R}[\delta]}(x)<1$. Then, there exist $s,t\in(0,1]$ such that $\psi_{\widetilde{R}[\delta]}(x)<1-t<s<1$. Then, $0<t<1-\psi_{\widetilde{R}[\delta]}(x)$, and so, $\widetilde{R}[\delta]_t(x)\geq1/2$. Thus, $[(x,t)\in\widetilde{R}[\delta]]\geq1/2$ and $\widetilde{R}[\delta]_{[\underline{s}]}(x)\geq1/2\Rightarrow[(x,s)q\widetilde{R}[\delta]]\geq1/2$.

If $\alpha=\ \in$ or $\alpha=\ \in\vee q$, then $[(x,t)\alpha\widetilde{R}[\delta]]\geq1/2$, and $\beta\in\{\in,q,\wedge q,\vee q\}$. Now, from Definition 8, $1\geq[(0,t)\beta\widetilde{R}[\delta]]\geq[(x,t)\alpha\widetilde{R}[\delta]]\geq1/2$. Therefore, $[(0,t)\beta\widetilde{R}[\delta]]\geq1/2$

$\Rightarrow$ either $\widetilde{R}[\delta]_t(0)\geq1/2$ or $\widetilde{R}[\delta]_{[\underline{t}]}(0)\geq1/2$
$\Rightarrow$ either $\psi_{\widetilde{R}[\delta]}(0)\leq1-t<1$ or $\psi_{\widetilde{R}[\delta]}(0)<t<1$
$\Rightarrow\psi_{\widetilde{R}[\delta]}(0)<1\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(0)\geq1/2$

If $\alpha=q$, then $[(x,s)\alpha\widetilde{R}[\delta]]\geq1/2$, and so, $\beta\in\{\in,q,\wedge q,\vee q\}$. Now, from Definition 8, $[(0,s)\beta\widetilde{R}[\delta]]\geq[(x,t)\alpha\widetilde{R}[\delta]]\geq1/2$. Therefore, $[(0,s)\beta\widetilde{R}[\delta]]\geq1/2$

$\Rightarrow$ either $\widetilde{R}[\delta]_s(0)\geq1/2$ or $\widetilde{R}[\delta]_{[\underline{s}]}(0)\geq1/2$
$\Rightarrow$ either $\psi_{\widetilde{R}[\delta]}(0)\leq1-s<1$ or $\psi_{\widetilde{R}[\delta]}(0)<s<1$
$\Rightarrow\psi_{\widetilde{R}[\delta]}(0)<1\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(0)\geq1/2$

Hence, $\widetilde{R}[\delta]_{\underline{0}}(0)\geq\widetilde{R}[\delta]_{\underline{0}}(x)$ for all $x\in X$ and $\delta\in A$.

(2) First, we show that $\widetilde{R}[\delta]_{\underline{0}}(x*y)\wedge\widetilde{R}[\delta]_{\underline{0}}(y)=1\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(x)=1$. If $\widetilde{R}[\delta]_{\underline{0}}(x*y)\wedge\widetilde{R}[\delta]_{\underline{0}}(y)=1$, then $\widetilde{R}[\delta]_{\underline{0}}(x*y)=1$ and $\widetilde{R}[\delta]_{\underline{0}}(y)=1$, and so, $\varphi_{\widetilde{R}[\delta]}(x*y)>0$ and $\varphi_{\widetilde{R}[\delta]}(y)>0$. Let $\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)=t>0$. Then, there exists $s\in(0,1)$ such that $0<1-s<t=\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$. Thus, $[(x*y,t)\in\widetilde{R}[\delta]]=\widetilde{R}[\delta]_t(x*y)=1$, $[(y,t)\in\widetilde{R}\ [\delta]]=\widetilde{R}[\delta]_t(y)=1$, $[(x*y,s)q\widetilde{R}\ [\delta]]=\widetilde{R}\delta_{[\underline{s}]}(x*y)=1$, and $[(y,s)q\widetilde{R}[\delta]]=\widetilde{R}[\delta]_{[\underline{s}]}(y)=1$.

If $\alpha=\ \in$ or $\alpha=\ \in\vee q$, then $[(x*y,t)\alpha\widetilde{R}[\delta]]=1$, $[(y,t)\alpha\widetilde{R}[\delta]]=1$, and $\beta\in\{\in,q,\wedge q,\vee q\}$. Now, from Definition 8, $1\geq[(x,t)\beta\widetilde{R}[\delta]]\geq[(x*y,\ t)\alpha\widetilde{R}[\delta]]\wedge[(y,t)\alpha\widetilde{R}[\delta]]=1$. Therefore, $[(x,t)\beta\widetilde{R}[\delta]]=1$

$\Rightarrow$ either $\widetilde{R}[\delta]_t(x)=1$ or $\widetilde{R}[\delta]_{[\underline{t}]}(x)=1$
$\Rightarrow$ either $\varphi_{\widetilde{R}[\delta]}(x)\geq t>0$ or $\varphi_{\widetilde{R}[\delta]}(x)>1-t\geq0$
$\Rightarrow\varphi_{\widetilde{R}[\delta]}(x)>0\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(x)=1$

If $\alpha=q$, then $[(x*y,s)\alpha\widetilde{R}[\delta]]=[(y,s)\alpha\widetilde{R}[\delta]]=1$, and $\beta\in\{\in,q,\wedge q,\vee q\}$. Now, from Definition 8, $1\geq[(x,s)\beta\widetilde{R}[\delta]]\geq[(x*y,s)\alpha\widetilde{R}[\delta]]\wedge[(y,s)\alpha\widetilde{R}[\delta]]=1$. Therefore, $[(x,s)\beta\widetilde{R}[\delta]]=1$

$\Rightarrow$ either $\widetilde{R}[\delta]_s(x)=1$ or $\widetilde{R}[\delta]_{[\underline{s}]}(x)=1$
$\Rightarrow$ either $\varphi_{\widetilde{R}[\delta]}(x)\geq s>0$ or $\varphi_{\widetilde{R}[\delta]}(x)>1-s\geq0$
$\Rightarrow\varphi_{\widetilde{R}[\delta]}(x)>0\Rightarrow\widetilde{R}[\delta]_{\underline{0}}(x)=1$

Next, we show that $\widetilde{R}[\delta]_{\underline{0}}(x*y)\wedge\widetilde{R}[\delta]_{\underline{0}}(y)=1/2\Rightarrow\widetilde{R}[\delta]_{[\underline{0}]}(x)\geq1/2$. Suppose $\widetilde{R}[\delta]_{\underline{0}}(x*y)\wedge\widetilde{R}[\delta]_{\underline{0}}(y)=1/2$. Then, $\widetilde{R}[\delta]_{\underline{0}}(x*y)\geq1/2$ and $\widetilde{R}[\delta]_{\underline{0}}(y)\geq1/2\Rightarrow\ \psi_{\widetilde{R}[\delta]}(x*y)<1$ and $\psi_{\widetilde{R}[\delta]}(y)<1\Rightarrow\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)<1$. Let $s,t\in(0,1]$ such that $\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)<1-t<s<1$.

Then, $0 < t < 1 - \psi_{\widetilde{R}[\delta]}(x*y) \vee \psi_{\widetilde{R}[\delta]}(y) = (1 - \psi_{\widetilde{R}[\delta]}(x*y)) \wedge (1 - \psi_{\widetilde{R}[\delta]}(y)) \Rightarrow 1 - \psi_{\widetilde{R}[\delta]}(x*y) > t$ and $1 - \psi_{\widetilde{R}[\delta]}(y) > t$. Thus, $[(x*y,t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_t(x*y) \geq 1/2$ and $[(y,t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_t(y) \geq 1/2$. Again, $\psi_{\widetilde{R}[\delta]}(x*y) \vee \psi_{\widetilde{R}[\delta]}(y) < s < 1$ such that $\psi_{\widetilde{R}[\delta]}(x*y) < s$ and $\psi_{\widetilde{R}[\delta]}(y) < s$. Thus, $[(x*y,s)\ q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_{[\underline{s}]}(x*y) \geq 1/2$ and $[(y,s)\ q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_{[\underline{s}]}(y) \geq 1/2$.

If $\alpha = \in$ or $\alpha = \in\vee q$, then $[(x*y,t)\alpha\widetilde{R}[\delta]] = 1/2$, $[(y,t)\alpha\widetilde{R}[\delta]] = 1/2$, and $\beta \in \{\in, q, \wedge q, \vee q\}$. Now, from Definition 8, $[(x,t)\beta\widetilde{R}[\delta]] \geq [(x*y,t)\alpha\widetilde{R}[\delta]] \wedge [(y,t)\alpha\widetilde{R}[\delta]] \geq 1/2$. Therefore, $[(x,t)\beta\widetilde{R}[\delta]] \geq 1/2$

$\Rightarrow [(x,t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_t(x) \geq 1/2$ or $[(x,t)q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_{[\underline{t}]}(x) \geq 1/2$

$\Rightarrow \psi_{\widetilde{R}[\delta]}(x) \leq 1 - t < 1$ or $\psi_{\widetilde{R}[\delta]}(x) < t < 1$

$\Rightarrow \psi_{\widetilde{R}[\delta]}(x) < 1 \Rightarrow \widetilde{R}[\delta]_{\underline{0}}(x) \geq 1/2$

If $\alpha = q$, then $[(x*y,s)\alpha\widetilde{R}[\delta]] \wedge [(y,s)\alpha\widetilde{R}[\delta]] \geq 1/2$, and $\beta \in \{\in, q, \wedge q, \vee q\}$. Now, from Definition 8, $[(x,s)\beta\widetilde{R}[\delta]] \geq 1/2 \Rightarrow [(x,s) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_s(x) \geq 1/2$ or $[(x,s)q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_{[\underline{s}]}(x) \geq 1/2$

$\Rightarrow \psi_{\widetilde{R}[\delta]}(x) \leq 1 - s < 1$ or $\psi_{\widetilde{R}[\delta]}(x) < s < 1$

$\Rightarrow \psi_{\widetilde{R}[\delta]}(x) < 1 \Rightarrow \widetilde{R}[\delta]_{\underline{0}}(x) \geq 1/2$

Hence, $\widetilde{R}[\delta]_{\underline{0}}(x) \geq \widetilde{R}[\delta]_{\underline{0}}(x*y) \wedge \widetilde{R}[\delta]_{\underline{0}}(y)$ for every $x, y \in X$ and $\delta \in A$. Therefore, $\widetilde{R}[\delta]_{\underline{0}}$ is a fuzzy ideal of X. Thus, $(\widetilde{R}, tA)_{\underline{0}}$ is a fuzzy soft ideal of X. □

4. Intuitionistic Fuzzy Soft Ideals of BCK/BCI-Algebras with Thresholds

Definition 9. Let $(\widetilde{R}, tA)$ be an IFSS of X. Then, $(\widetilde{R}, tA)$ is an intuitionistic fuzzy soft ideal (IFSID) of X with thresholds $(s,t) \Leftrightarrow \widetilde{R}[\delta] = (x, \varphi_{\widetilde{R}[\delta]}(x), \psi_{\widetilde{R}[\delta]}(x))$ is an intuitionistic fuzzy ideals (IFIDs) of X which satisfies the following for every $x, y \in X$ and $\delta \in A$:

(1) $\varphi_{\widetilde{R}[\delta]}(0) \vee s \geq \varphi_{\widetilde{R}[\delta]}(x) \wedge t$ and $\psi_{\widetilde{R}[\delta]}(0) \wedge (1-s) \leq \psi_{\widetilde{R}[\delta]}(x) \vee (1-t)$

(2) $\varphi_{\widetilde{R}[\delta]}(x) \vee s \geq \varphi_{\widetilde{R}[\delta]}(x*y) \wedge \varphi_{\widetilde{R}[\delta]}(y) \wedge t$ and $\psi_{\widetilde{R}[\delta]}(x) \wedge (1-s) \leq \psi_{\widetilde{R}[\delta]}(x*y) \vee \psi_{\widetilde{R}[\delta]}(y) \vee (1-t)$

Example 2. Let us consider a BCI-algebra $X = \{0,1,2,3\}$ with the Cayley table given in Table 3.

Let us suppose $\varphi_{\widetilde{R}[\delta]}(0) = \varphi_{\widetilde{R}[\delta]}(1) = m$, $\varphi_{\widetilde{R}[\delta]}(2) = \varphi_{\widetilde{R}[\delta]}(3) = s$, and $\psi_{\widetilde{R}[\delta]}(0) = \psi_{\widetilde{R}[\delta]}(1) = 1 - t$, $\psi_{\widetilde{R}[\delta]}(2) = \psi_{\widetilde{R}[\delta]}(3) = w$, where $0 < s < t < 1$, $m \in (0, s]$, and $w \in [0, 1-t]$. Henceforth, $\widetilde{R}[\delta]$ is an IFID of X with thresholds (s,t). Consequently, $(\widetilde{R}, tA)$ is an IFSID of X with thresholds (s,t).

Table 3: Cayley table of the binary operation $*$.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Theorem 2. *An IFSS $(\widetilde{R}, tA)$ of X is an $(\in, \in\vee q)$-IFSID of $X \Leftrightarrow (\widetilde{R}, tA)$ is an IFSID of X with thresholds $(0, 1/2)$.*

Proof. ($\Rightarrow$) Suppose $(\widetilde{R}, tA)$ is an $(\in, \in\vee q)$-IFSID of X. We have to show that $(\widetilde{R}, tA)$ is an IFSID of X with thresholds $(0, 1/2)$. It is enough to show that

(1) $\varphi_{\widetilde{R}[\delta]}(0) \geq \varphi_{\widetilde{R}[\delta]}(x) \wedge 1/2$ and $\varphi_{\widetilde{R}[\delta]}(0) \leq \varphi_{\widetilde{R}[\delta]}(x) \vee 1/2$

(2) $\varphi_{\widetilde{R}[\delta]}(x) \geq \varphi_{\widetilde{R}[\delta]}(x*y) \wedge \varphi_{\widetilde{R}[\delta]}(y) \wedge 1/2$ and $\psi_{\widetilde{R}[\delta]}(x) \leq \psi_{\widetilde{R}[\delta]}(x*y) \vee \psi_{\widetilde{R}[\delta]}(y) \vee 1/2$ for every $x, y \in X$ and $\delta \in A$

(1) Let $x, y \in X$, and let $t = \varphi_{\widetilde{R}[\delta]}(x) \wedge 1/2$. Then, $\varphi_{\widetilde{R}[\delta]}(x) \geq t$, and so, $[(x,t) \in \widetilde{R}[\delta]] = 1$. Therefore, from Definition 8,

$1 \geq [(0,t) \in\vee q\widetilde{R}[\delta]] \geq [(x,t) \in \widetilde{R}[\delta]] = 1$

$\Rightarrow [(0,t) \in\vee q\widetilde{R}[\delta]] = 1$

$\Rightarrow [(0,t) \in \widetilde{R}[\delta]] = 1$ or $[(0,t) \in q\widetilde{R}[\delta]] = 1$

$\Rightarrow \varphi_{\widetilde{R}[\delta]}(0) \geq t$ *or* $\varphi_{\widetilde{R}[\delta]}(0) + t > 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(0) \geq t$ or $\varphi_{\widetilde{R}[\delta]}(0) > 1 - t \geq 1/2 \geq t$

Thus, $\varphi_{\widetilde{R}[\delta]}(0) \geq t = \varphi_{\widetilde{R}[\delta]}(x) \wedge 1/2$ for all $x \in X$. Now, let $\psi_{\widetilde{R}[\delta]}(x) \vee 1/2 = 1 - s \Rightarrow (1 - \psi_{\widetilde{R}[\delta]}(x)) \wedge 1/2 = s$; then, $[(x,s) \in \widetilde{R}[\delta]] \geq 1/2$. Therefore, from Definition 8, $[(0,s) \in\vee q\widetilde{R}[\delta]] \geq [(x,s) \in \widetilde{R}[\delta]] \geq 1/2$

$\Rightarrow [(0,s) \in\vee q\widetilde{R}[\delta]] \geq 1/2$

$\Rightarrow [(0,s) \in \widetilde{R}[\delta]] \geq 1/2$ or $[(0,s)q\widetilde{R}[\delta]] \geq 1/2$

$\Rightarrow s \leq 1 - \psi_{\widetilde{R}[\delta]}(x)$ or $\psi_{\widetilde{R}[\delta]}(0) < 1 - s \leq s$

Since $1 - s \geq 1/2$, $s < 1/2$. Thus, $\psi_{\widetilde{R}[\delta]}(0) \leq 1 - s = \psi_{\widetilde{R}[\delta]}(x) \vee 1/2$.

(2) Let $x, y \in X$, and let $t = \varphi_{\widetilde{R}[\delta]}(x*y) \wedge \varphi_{\widetilde{R}[\delta]}(y) \wedge 1/2$. Then, $\varphi_{\widetilde{R}[\delta]}(x*y) \geq t$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq t$, and so, $[(x*y,t) \in \widetilde{R}[\delta]] = 1$ and $[(y,t) \in \widetilde{R}[\delta]] = 1$. Therefore, from Definition 8,

$1 \geq [(x,t) \in\vee q\widetilde{R}[\delta]] \geq [(x*y,t) \in \widetilde{R}[\delta]] \wedge [(y,t) \in \widetilde{R}[\delta]] = 1$

$\Rightarrow [(x,t) \in\vee q\widetilde{R}[\delta]] = 1$

$\Rightarrow [(x,t) \in \widetilde{R}[\delta]] \vee [(x,t)q\widetilde{R}[\delta]] = 1$

$\Rightarrow [(x,t) \in \widetilde{R}[\delta]] = 1$ or $[(x,t)q\widetilde{R}[\delta]] = 1$

$\Rightarrow \varphi_{\widetilde{R}[\delta]}(x) \geq t$ or $\varphi_{\widetilde{R}[\delta]}(x) + t > 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(x) \geq t$ or $\varphi_{\widetilde{R}[\delta]}(x) > 1 - t \geq 1/2 \geq t$

Thus, $\varphi_{\widetilde{R}[\delta]}(x) \geq t = \varphi_{\widetilde{R}[\delta]}(x*y) \wedge \varphi_{\widetilde{R}[\delta]}(y) \wedge 1/2$ for all $x, y \in X$, and let $\psi_{\widetilde{R}[\delta]}(x*y) \vee \psi_{\widetilde{R}[\delta]}(y) \vee 1/2 = 1 - s$; then, $\psi_{\widetilde{R}[\delta]}(x*y) \leq 1 - s$ and $\psi_{\widetilde{R}[\delta]}(y) \leq 1 - s \Rightarrow s \leq 1 - \psi_{\widetilde{R}[\delta]}(x*y)$ and $s \leq 1 - \psi_{\widetilde{R}[\delta]}(y) \Rightarrow [(x*y,s) \in \widetilde{R}[\delta]] \geq 1/2$ and $[(y,s) \in \widetilde{R}[\delta]] \geq 1/2$. Therefore, from Definition 8,

$[(x,s)\in\vee q\widetilde{R}[\delta]]\geq[(x*y,s)\in\widetilde{R}[\delta]]\vee[(y,s)\in\widetilde{R}[\delta]]\geq 1/2$
$\Rightarrow[(x,s)\in\vee q\widetilde{R}[\delta]]\geq 1/2$
$\Rightarrow[(x,s)\in\widetilde{R}[\delta]]\geq 1/2$ or $[(x,s)q\widetilde{R}[\delta]]\geq 1/2$
$\Rightarrow s\leq 1-\psi_{\widetilde{R}[\delta]}(x)$ or $\psi_{\widetilde{R}[\delta]}(x)<s\leq 1-s$

Since $1-s\geq 1/2$, $s<1/2$. Hence, $\psi_{\widetilde{R}[\delta]}(x)\leq 1-s=\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)\vee 1/2$.

($\Leftarrow$) Let $(\widetilde{R},tA)$ be an IFSID of X with thresholds $(0,1/2)$. Let $s,t\in(0,1]$, and for $x\in X$ and $\delta\in A$, we have to prove $[(0,s)\in\vee q\widetilde{R}[\delta]]\geq[(x,s)\in\widetilde{R}[\delta]]$.

(1) Let $a=[(x,s)\in\widetilde{R}[\delta]]$.

Case 1: when $a=1$, $[(x,s)\in\widetilde{R}[\delta]]=1\Rightarrow\varphi_{\widetilde{R}[\delta]}(x)\geq s$. If $[(0,s)\in\vee q\widetilde{R}[\delta]]\leq 1/2$, then $\varphi_{\widetilde{R}[\delta]}(0)<s$ and $\varphi_{\widetilde{R}[\delta]}(0)\leq 1-s$. Thus, $1/2>\varphi_{\widetilde{R}[\delta]}(0)\geq\varphi_{\widetilde{R}[\delta]}(x)\wedge 1/2$; so, $\varphi_{\widetilde{R}[\delta]}(0)\geq\varphi_{\widetilde{R}[\delta]}(x)\geq s$. This is a contradiction to $\varphi_{\widetilde{R}[\delta]}(x)<s$. Hence, $[(0,s)\in\vee q\widetilde{R}[\delta]]=1$.

Case 2: for $a=1/2$, we have $s\leq 1-\psi_{\widetilde{R}[\delta]}(x)$. If $[(0,s)\in\vee q\widetilde{R}[\delta]]=0$, then $[(0,s)\in\widetilde{R}[\delta]]=0$ and $[(0,s)q\widetilde{R}[\delta]]=0\Rightarrow s>1-\psi_{\widetilde{R}[\delta]}(0)$ and $s\leq\psi_{\widetilde{R}[\delta]}(0)\Rightarrow\psi_{\widetilde{R}[\delta]}(0)>1-s$ and $s\leq\psi_{\widetilde{R}[\delta]}(0)$. Thus, $1/2<\psi_{\widetilde{R}[\delta]}(0)\leq\psi_{\widetilde{R}[\delta]}(x)$; so, $\psi_{\widetilde{R}[\delta]}(0)\leq\psi_{\widetilde{R}[\delta]}(x)\Rightarrow 1-\psi_{\widetilde{R}[\delta]}(0)\geq 1-\psi_{\widetilde{R}[\delta]}(x)\geq s$, which is a contradiction to $s>1-\psi_{\widetilde{R}[\delta]}(0)$. Therefore, $[(0,s)\in\vee q\widetilde{R}[\delta]]\geq 1/2=[(x,s)\in\widetilde{R}[\delta]]$. Hence, $[(0,s)\in\vee q\widetilde{R}[\delta]]\geq[(x,s)\in\widetilde{R}[\delta]]$.

(2) Let $x,y\in X$, and let $b=[(x*y,s)\in\widetilde{R}[\delta]]\wedge[(y,t)\in\widetilde{R}[\delta]]$.

Case 1: at $b=1\Rightarrow[(x*y,s)\in\widetilde{R}[\delta]]=1$ and $[(y,t)\in\widetilde{R}[\delta]]=1\Rightarrow\varphi_{\widetilde{R}[\delta]}(x*y)\geq s$ and $\varphi_{\widetilde{R}[\delta]}(y)\geq t$, if $[(x,s\wedge t)\in\vee q\widetilde{R}[\delta]]\leq 1/2$, then $\varphi_{\widetilde{R}[\delta]}(x)<s\wedge t$ and $\varphi_{\widetilde{R}[\delta]}(x)\leq 1-s\wedge t$. Thus, $1/2>\varphi_{\widetilde{R}[\delta]}(x)\geq\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge 1/2$, and so, $\varphi_{\widetilde{R}[\delta]}(x)\geq\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\geq s\wedge t$. This is a contradiction to $\varphi_{\widetilde{R}[\delta]}(x)<s\wedge t$. Thus, $[(x,s\wedge t)\in\vee q\widetilde{R}[\delta]]=1$.

Case 2: when $b=1/2$, $[(x*y,s)\in\widetilde{R}[\delta]]\geq 1/2$ and $[(y,t)\in\widetilde{R}[\delta]]\geq 1/2\Rightarrow 1-\psi_{\widetilde{R}[\delta]}(x*y)\geq s$ and $1-\psi_{\widetilde{R}[\delta]}(y)\geq t$. Now, $1-\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}}[\delta](y)=(1-\psi_{\widetilde{R}[\delta]}(x*y))\wedge(1-\psi_{\widetilde{R}[\delta]}(y))\geq s\wedge t$.

If $[(x,s\wedge t)\in\vee q\widetilde{R}[\delta]]=0$, then $(1-\psi_{\widetilde{R}[\delta]}(x))<s\wedge t$ and $\psi_{\widetilde{R}[\delta]}(x)\geq s\wedge t$. Now, $1/2<\psi_{\widetilde{R}[\delta]}(x)\leq\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}}[\delta](y)\vee 1/2\Rightarrow\psi_{\widetilde{R}[\delta]}(x)\leq\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)\Rightarrow 1-\psi_{\widetilde{R}[\delta]}(x)\geq 1-(\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y))=(1-\psi_{\widetilde{R}[\delta]}(x*y))\wedge(1-\psi_{\widetilde{R}[\delta]}(y))\geq s\wedge t$ which is a contradiction to $(1-\psi_{\widetilde{R}[\delta]}(x))<s\wedge t$. Thus, $[(x,s\wedge t)\in\vee q\widetilde{R}[\delta]]\geq 1/2=[(x*y,s)\in\widetilde{R}[\delta]]\wedge[(y,t)\in\widetilde{R}[\delta]]$. Hence, $[(x,s\wedge t)\in\vee q\widetilde{R}[\delta]]\geq[(x*y,s)\in\widetilde{R}[\delta]]\wedge[(y,t)\in\widetilde{R}[\delta]]$. Therefore, $\widetilde{R}[\delta]$ is an $(\in,\in\vee q)$–IFID of X. Hence, $(\widetilde{R},tA)$ is an $(\in\wedge q,\in)$-IFSID of X. □

Theorem 3. *An IFSS $(\widetilde{R},tA)$ is an $(\in\wedge q,\in)$-IFSID of $X\Leftrightarrow(\widetilde{R},tA)$ is an IFSID of X with thresholds $(1/2,t1)$.*

Proof. ($\Rightarrow$) Suppose $(\widetilde{R},tA)$ of X is an $(\in\wedge q,\in)$–IFSID of X. We prove that $(\widetilde{R},tA)$ is an IFSID of X with thresholds $(1/2,t1)$. It is enough to show that

(1) $\varphi_{\widetilde{R}[\delta]}(0)\vee 1/2\geq\varphi_{\widetilde{R}[\delta]}(x)$ and $\psi_{\widetilde{R}[\delta]}(0)\wedge 1/2\leq\psi_{\widetilde{R}[\delta]}(x)$
(2) $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2\geq\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$ and $\psi_{\widetilde{R}[\delta]}(x)\wedge 1/2\leq\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)$ for all $x,y\in X$ and $\delta\in A$

(1) Let $x\in X$ and $t=\varphi_{\widetilde{R}[\delta]}(x)$. If $\varphi_{\widetilde{R}[\delta]}(0)\vee 1/2<t$, then $\varphi_{\widetilde{R}[\delta]}(x)=t>1/2$, which indicates that $[(x,t)\in\wedge q\widetilde{R}[\delta]]=1$. Now, $[(0,t)\in\widetilde{R}[\delta]]\geq[(x,t)\in\wedge q\widetilde{R}[\delta]]=1\Rightarrow[(0,t)\in\widetilde{R}[\delta]]=1\Rightarrow\varphi_{\widetilde{R}[\delta]}(0)\geq t$. This is a contradiction to $\varphi_{\widetilde{R}[\delta]}(0)<t$. Hence, $\varphi_{\widetilde{R}[\delta]}(0)\vee 1/2\geq\varphi_{\widetilde{R}[\delta]}(x)$. Let $t=1-s=\psi_{\widetilde{R}[\delta]}(x)$; then, $[(x,s)\in\widetilde{R}[\delta]]\geq 1/2$. If $\psi_{\widetilde{R}[\delta]}(0)\wedge 1/2>t$ and $t<1/2<s$, then $\psi_{\widetilde{R}[\delta]}(x)=1-s=t<s$. This indicates $[(x,s)\in q\widetilde{R}[\delta]]\geq 1/2$. Thus, $[(x,s)\in\widetilde{R}[\delta]]\geq 1/2$ and $[(x,s)q\widetilde{R}[\delta]]\geq 1/2\Rightarrow[(x,s)\in\wedge q\widetilde{R}[\delta]]\geq 1/2$. Now, $[(0,s)\in\widetilde{R}[\delta]]\geq[(x,s)\in\wedge q\widetilde{R}[\delta]]\geq 1/2\Rightarrow s=1-\psi_{\widetilde{R}[\delta]}(0)$. Therefore, $\psi_{\widetilde{R}[\delta]}(0)\leq 1-s=t$. This contradicts the assumption $\psi_{\widetilde{R}[\delta]}(0)>t$. Hence, $\psi_{\widetilde{R}[\delta]}(0)\wedge 1/2\leq t=\psi_{\widetilde{R}[\delta]}(x)$.

(2) Let $x,y\in X$ and $t=\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$. Now, if $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2<t=\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$, we have, $\varphi_{\widetilde{R}[\delta]}(x*y)\geq t>1/2$ and $\varphi_{\widetilde{R}[\delta]}(y)\geq t>1/2\Rightarrow[(x*y,t)\in\widetilde{R}[\delta]]=1,[(x*y,t)q\widetilde{R}[\delta]]=1$ and $[(y,t)\in\widetilde{R}[\delta]]=1,[(y,t)q\widetilde{R}[\delta]]=1\Rightarrow[(x*y,t)\in\wedge q\widetilde{R}[\delta]]=1$ and $[(y,t)\in\wedge q\widetilde{R}[\delta]]=1\Rightarrow[(x*y,t)\in\wedge q\widetilde{R}[\delta]]\wedge[(y,t)\in\wedge q\widetilde{R}[\delta]]=1$. Therefore, $[(x,t)\in\widetilde{R}[\delta]]\geq[(x*y,t)\in\wedge q\widetilde{R}[\delta]]\wedge[(y,t)\in\wedge q\widetilde{R}[\delta]]\Rightarrow[(x,t)\in\widetilde{R}[\delta]]=1\Rightarrow\varphi_{\widetilde{R}[\delta]}(x)\geq t$. This contradicts the assumption $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2<t=\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$. Hence, $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2\geq t=\varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$.

Let $\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)=t=1-s$; then, $1-s\geq\psi_{\widetilde{R}[\delta]}(x*y)$ and $1-s\geq\psi_{\widetilde{R}[\delta]}(y)$, but if $\psi_{\widetilde{R}[\delta]}(x)\wedge 1/2\geq t=1-s=\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)$, $s\leq 1-\psi_{\widetilde{R}[\delta]}(x*y)$, $s\leq 1-\psi_{\widetilde{R}[\delta]}(y)$ and $\psi_{\widetilde{R}[\delta]}(x)>t$, and then $s>1/2>t$. Thus, $\psi_{\widetilde{R}[\delta]}(x*y)\leq t<s$ and $\psi_{\widetilde{R}[\delta]}(y)\leq t<s\Rightarrow[(x*y,s)\in\widetilde{R}[\delta]]\geq 1/2$ and $[(x*y,s)q\widetilde{R}[\delta]]\geq 1/2$, $[(y,s)\in\widetilde{R}[\delta]]\geq 1/2$ and $[(y,s)q\widetilde{R}[\delta]]\geq 1/2$. Therefore, $[(x,s)\in\widetilde{R}[\delta]]\geq[(x*y,s)\in\wedge q\widetilde{R}[\delta]]\wedge[(y,s)\in\wedge q\widetilde{R}[\delta]]\geq 1/2\Rightarrow[(x,s)\in\widetilde{R}[\delta]]\geq 1/2$, which indicate that $s\leq 1-\psi_{\widetilde{R}[\delta]}(x)$. This contradicts the assumption $\psi_{\widetilde{R}[\delta]}(x)>t$. Hence, $\psi_{\widetilde{R}[\delta]}(x)\wedge 1/2\leq t=\psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)$.

($\Leftarrow$) Suppose $(\widetilde{R},tA)$ is an IFSID of X with thresholds $(1/2,t1)$. Let $x,y\in X,s,t\in(0,1]$ and $a=\varphi_{\widetilde{R}[\delta]}(x)$.

Case 1: let $a = 1$. Then, $\varphi_{\widetilde{R}[\delta]}(x) \geq s, \varphi_{\widetilde{R}[\delta]}(x) + s > 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(x) \geq 1/2$. Thus, $\varphi_{\widetilde{R}[\delta]}(0) \geq \varphi_{\widetilde{R}[\delta]}(x) \geq s \Rightarrow [(0, s) \in \widetilde{R}[\delta]] = 1$.

Case 2: let $a = 1/2$. Then, $s \leq 1 - \psi_{\widetilde{R}[\delta]}(x)$ and $\psi_{\widetilde{R}[\delta]}(x) < s \Rightarrow 1 - \psi_{\widetilde{R}[\delta]}(x) > s > \psi_{\widetilde{R}[\delta]}(x)$. Thus, $\psi_{\widetilde{R}[\delta]}(x) < 1/2$. Therefore, $\psi_{\widetilde{R}[\delta]}(0) \wedge 1/2 \leq \psi_{\widetilde{R}[\delta]}(x) \Rightarrow \psi_{\widetilde{R}[\delta]}(0) \leq \psi_{\widetilde{R}[\delta]}(x) \Rightarrow 1 - \psi_{\widetilde{R}[\delta]}(0) \geq 1 - \psi_{\widetilde{R}[\delta]}(x) \geq s$. Thus, $[(0, s) \in \widetilde{R}[\delta]] \geq 1/2$. Hence, $[(0, s) \in \widetilde{R}[\delta]] \geq [(x, s) \in \wedge q\widetilde{R}[\delta]]$.

Next, we prove that $[(x, s \wedge t) \in \widetilde{R}[\delta]] \geq [(x * y, s) \in \wedge q\widetilde{R}[\delta]] \wedge [(y, t) \in \wedge q\widetilde{R}[\delta]]$. Let $b = \varphi_{\widetilde{R}[\delta]}(x * y) \wedge \varphi_{\widetilde{R}[\delta]}(y)$.

Case 1: let $b = 1$. Then, $\varphi_{\widetilde{R}[\delta]}(x * y) \geq s, \varphi_{\widetilde{R}[\delta]}(x * y) + s > 1$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq t, \varphi_{\widetilde{R}[\delta]}(y) + t > 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(x * y) \geq 1/2$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq 1/2$. Thus, $\varphi_{\widetilde{R}[\delta]}(x) \vee 1/2 \geq \varphi_{\widetilde{R}[\delta]}(x * y) \wedge \varphi_{\widetilde{R}[\delta]}(y) \geq s \wedge t$; we get $[(x, s \wedge t) \in \widetilde{R}[\delta]] = 1$.

Case 2: let $b = 1/2$. Then, $s \leq 1 - \psi_{\widetilde{R}[\delta]}(x * y)$, $\psi_{\widetilde{R}[\delta]}(x * y) < s$ and $t \leq 1 - \psi_{\widetilde{R}[\delta]}(y), \psi_{\widetilde{R}[\delta]}(y) < t \Rightarrow 1 - \psi_{\widetilde{R}[\delta]}(x * y) \geq s > \psi_{\widetilde{R}[\delta]}(x * y)$ and $1 - \psi_{\widetilde{R}[\delta]}(y) \geq t > \psi_{\widetilde{R}[\delta]}(y) \Rightarrow \psi_{\widetilde{R}[\delta]}(x * y) < 1/2$ and $\psi_{\widetilde{R}[\delta]}(y) < 1/2$. Thus, $\psi_{\widetilde{R}[\delta]}(x) \wedge 1/2 \leq \psi_{\widetilde{R}[\delta]}(x * y) \vee \psi_{\widetilde{R}[\delta]}(y) \Rightarrow \psi_{\widetilde{R}[\delta]}(x) \leq \psi_{\widetilde{R}[\delta]}(x * y) \vee \psi_{\widetilde{R}[\delta]}(y)$. Therefore, $1 - \psi_{\widetilde{R}[\delta]}(x) \geq (1 - \psi_{\widetilde{R}[\delta]}(x * y)) \wedge (1 - \psi_{\widetilde{R}[\delta]}(y)) \geq s \wedge t$. Thus, $[(x, s \wedge t) \in \widetilde{R}[\delta]] \geq 1/2$. Hence, $[(x, s \wedge t) \in \widetilde{R}[\delta]] \geq [(x * y, s \in \wedge q\widetilde{R}[\delta]] \wedge [(y, t) \in \wedge q\widetilde{R}[\delta]]$. Therefore, $\widetilde{R}[\delta]$ is an $(\in \wedge q, \in)$–IFID of X. Hence, $(\widetilde{R}, tA)$ is an $(\in \wedge q, \in)$–IFSID of X. □

Theorem 4. *An IFSS $(\widetilde{R}, tA)$ of X is an $(\in, \in)$-IFSID of $X \Leftrightarrow (\widetilde{R}, tA)_p$ is a FSID of X for any $p \in [0, 1]$.*

Proof. $(\Rightarrow)$ Suppose $(\widetilde{R}, tA)$ is an $(\in, \in)$-IFSID of X. Let $x, y \in X, \delta \in A$, and for $p \in [0, 1]$, $[(0, p) \in \widetilde{R}[\delta]] \geq [(x, p) \in \widetilde{R}[\delta]]$ and $[(x, p) \in \widetilde{R}[\delta]] \geq [(x * y, p) \in \widetilde{R}[\delta]] \wedge [(y, p) \in \widetilde{R}[\delta]] \Rightarrow \widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x)$ and $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x * y) \wedge \widetilde{R}[\delta]_p(y)$. Therefore, $\widetilde{R}[\delta]_p$ is a FID of X. Hence, $(\widetilde{R}, tA)_p$ is a FSID of X.

$(\Leftarrow)$ Assume for any $p \in [0, 1]$, $(\widetilde{R}, tA)_p$ is a FSID of X. Let $x, y \in X, \delta \in A$, and for $s, t \in (0, 1]$, we have to show that

(1) $[(0, s) \in \widetilde{R}[\delta]] \geq [(x, s) \in \widetilde{R}[\delta]]$

(2) $[(x, s \wedge t) \in \widetilde{R}[\delta]] \geq [(x * y, s) \in \widetilde{R}[\delta]] \wedge [(y, t) \in \widetilde{R}[\delta]]$

(1) Let $c = [(x, s) \in \widetilde{R}[\delta]]$.

Case 1: for $c = 1$, $\widetilde{R}[\delta]_s(x) = 1$ and $\widetilde{R}[\delta]_s(0) \geq \widetilde{R}[\delta]_s(x) = 1$; we get $\widetilde{R}[\delta]_s(0) = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(0) \geq s$. Thus, $[(0, s) \in \widetilde{R}[\delta]] = 1$.

Case 2: for $c = 1/2$, $\widetilde{R}[\delta]_s(x) = 1/2$. So, $\widetilde{R}[\delta]_s(0) \geq \widetilde{R}[\delta]_s(x) = 1/2$, and we get $\widetilde{R}[\delta]_s(0) = 1/2 \Rightarrow s \leq 1 - \psi_{\widetilde{R}[\delta]}(0)$. Thus, $[(0, s) \in \widetilde{R}[\delta]] \geq 1/2$. Therefore, $[(0, s) \in \widetilde{R}[\delta]] \geq [(x, s) \in \widetilde{R}[\delta]]$.

Next, to prove

(2) Let $c = [(x * y, s) \in \widetilde{R}[\delta]] \wedge [(y, t) \in \widetilde{R}[\delta]]$.

Case 1: at $c = 1$, $[(x * y, s) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_s(x * y) = 1$ and $[(y, t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_t(y) = 1$; then, $[(x, s \wedge t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_{s \wedge t}(x) \geq \widetilde{R}[\delta]_{s \wedge t}(x * y) \wedge \widetilde{R}[\delta]_{s \wedge t}(y) \geq \widetilde{R}[\delta]_s(x * y) \wedge \widetilde{R}[\delta]_t(y) = [(x * y, s) \in \widetilde{R}[\delta]] \wedge [(y, t) \in \widetilde{R}[\delta]] = 1 \Rightarrow \widetilde{R}[\delta]_{s \wedge t}(x) = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(x * y) \geq s \wedge t \Rightarrow [(x, s \wedge t) \in \widetilde{R}[\delta]] = 1$.

Case 2: at $c = 1/2$, $[(x * y, s) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_s(x * y) = 1/2$ and $[(y, t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_t(y) = 1/2$. Therefore, $[(x, s \wedge t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_{s \wedge t}(x) \geq \widetilde{R}[\delta]_{s \wedge t}(x * y) \wedge \widetilde{R}[\delta]_{s \wedge t}(y) \geq \widetilde{R}[\delta]_s(x * y) \wedge \widetilde{R}[\delta]_t(y) = [(x * y, s) \in \widetilde{R}[\delta]] \wedge [(y, t) \in \widetilde{R}[\delta]] = 1/2$, and we get $\widetilde{R}[\delta]_{s \wedge t}(x) \geq 1/2 \Rightarrow s \wedge t \leq 1 - \psi_{\widetilde{R}[\delta]}(x)$. Thus, $[(x, s \wedge t) \in \widetilde{R}[\delta]] \geq 1/2$. Hence, $[(x, s \wedge t) \in \widetilde{R}[\delta]] \geq [(x * y, s) \in \widetilde{R}[\delta]] \wedge [(y, t) \in \widetilde{R}[\delta]]$. Therefore, $\widetilde{R}[\delta]$ is an $(\in, \in)$-IFID of X. Hence, $(\widetilde{R}, tA)$ is an $(\in, \in)$-IFSID of X. □

Theorem 5. *An IFSS $(\widetilde{R}, tA)$ of X is an $(\in, \in \vee q)$-IFSID of $X \Leftrightarrow$ for any $p \in (0, 1/2]$, $(\widetilde{R}, tA)_p$ is a FSID of X.*

Proof. $(\Rightarrow)$ Suppose $(\widetilde{R}, tA)$ is an $(\in, \in \vee q)$–IFSID of X. Then, for every $x, y \in X, \delta \in A$ and for $p \in (0, 1/2]$, we have to show that

(1) $[(0, p) \in \vee q\widetilde{R}[\delta]] \geq [(x, p) \in \widetilde{R}[\delta]] \Rightarrow \widetilde{R}[\delta]_p(0) \vee \widetilde{R}[\delta]_{[p]}(0) \geq \widetilde{R}[\delta]_p(x)$

(2) $[(x, p) \in \vee q\widetilde{R}[\delta]] \geq [(x * y, p) \in \widetilde{R}[\delta]] \wedge [(y, p) \in \widetilde{R}[\delta]] \Rightarrow \widetilde{R}[\delta]_p(x) \vee \widetilde{R}[\delta]_{[p]}(x) \geq \widetilde{R}[\delta]_p(x * y) \wedge \widetilde{R}[\delta]_p(y)$

As $0 < p \leq 1/2, p \leq 1/2 \leq 1 - p$. Then, $\widetilde{R}[\delta]_{[p]}(0) = \widetilde{R}[\delta]_{\underline{1-p}}(0) \leq \widetilde{R}[\delta]_{\underline{p}}(0) \leq \widetilde{R}[\delta]_p(0)$ and $\widetilde{R}[\delta]_{[p]}(x) = \widetilde{R}[\delta]_{\underline{1-p}}(x) \leq \widetilde{R}[\delta]_{\underline{p}}(x) \leq \widetilde{R}[\delta]_p(x)$. Therefore, $\widetilde{R}[\delta]_p(x) \leq \widetilde{R}[\delta]_p(0) \vee \widetilde{R}[\delta]_{[p]}(0) \leq \widetilde{R}[\delta]_p(0) \vee \widetilde{R}[\delta]_p(0) = \widetilde{R}[\delta]_p(0)$ and $\widetilde{R}[\delta]_p(x * y) \wedge \widetilde{R}[\delta]_p(y) \leq \widetilde{R}[\delta]_p(x) \vee \widetilde{R}[\delta]_{[p]}(x) \leq \widetilde{R}[\delta]_p(x) \vee \widetilde{R}[\delta]_p(x) = \widetilde{R}[\delta]_p(x)$. Thus, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x)$ and $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x * y) \wedge \widetilde{R}[\delta]_p(y)$. Therefore, for any $p \in (0, 1/2]$, $\widetilde{R}[\delta]_p$ is a FID of X. Hence, $(\widetilde{R}, tA)$ is a FSID of X.

$(\Leftarrow)$ Let $(\widetilde{R}, tA)_p$ be a FSID of X, for any $p \in [0, 1/2]$. Let $x, y \in X, \delta \in A$ and $s, t \in (0, 1]$.

(1) If $s \leq 1/2$, then let $a = \varphi_{\widetilde{R}[\delta]}(x)$.

Case 1: for $a = 1$, $\widetilde{R}[\delta]_s(x) = 1, \widetilde{R}[\delta]_s(0) \geq \widetilde{R}[\delta]_s(x) = 1 \Rightarrow \widetilde{R}[\delta]_s(0) = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(0) \geq s \Rightarrow [(0, s) \in \widetilde{R}[\delta]] = 1$. Hence, $[(0, s) \in \vee q\widetilde{R}[\delta]] = [(0, s) \in \widetilde{R}[\delta]] \vee [(0, s) q\widetilde{R}[\delta]] = 1$.

Case 2: let $a = 1/2$. Then, $\widetilde{R}[\delta]_s(x) = 1/2, \widetilde{R}[\delta]_s(0) \geq \widetilde{R}[\delta]_s(x) = 1/2 \Rightarrow s \leq 1 - \varphi_{\widetilde{R}[\delta]}(0)$. Thus, $[(0, s) \in \widetilde{R}[\delta]] \geq 1/2$, from which $[(0, s) \in \vee q\widetilde{R}[\delta]] = [(0, s) \in \widetilde{R}[\delta]] \vee [(0, s)\, q\widetilde{R}[\delta]] = 1/2$. Hence, $[(0, s) \in \vee q\widetilde{R}[\delta]] \geq [(x, s) \in \widetilde{R}[\delta]]$. If $s > 1/2$, then let $p \in [0, 1]$ be such that $1 - s < p < 1/2 < s$. Then, $\widetilde{R}[\delta]_{[s]}(0) = \widetilde{R}[\delta]_{\underline{1-s}}$

$(0) \geq \widetilde{R}[\delta]_s(0)$ and $\widetilde{R}[\delta]_{\underline{[s]}}(0) = \widetilde{R}[\delta]_{\underline{1-s}}(0) \geq \widetilde{R}[\delta]_p(0)$. Therefore, $[(0,s) \in\vee q\widetilde{R}[\delta]] = [(0,s) \in \widetilde{R}[\delta]]\vee[(0,s)q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_s(0)\vee\widetilde{R}[\delta]_{\underline{[s]}}(0) = \widetilde{R}[\delta]_{\underline{[s]}}(0) \geq \widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_s(x) = [(x,s) \in \widetilde{R}[\delta]]$. Hence, $[(0,s) \in\vee q\widetilde{R}[\delta]] \geq [(x,s) \in \widetilde{R}[\delta]]$.

(2) $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] \geq [(x*y,s) \in \widetilde{R}[\delta]]\wedge[(y,t) \in \widetilde{R}[\delta]]$. If $s\wedge t \leq 1/2$, then let $a = [(x*y,s) \in \widetilde{R}[\delta]]\wedge[(y,t) \in \widetilde{R}[\delta]]$.

Case 1: for $a = 1$, $[(x*y,s) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_s(x*y) = 1$ and $[(y,t) \in \widetilde{R}[\delta]] = \widetilde{R}[\delta]_t(y) = 1$, and we have $\widetilde{R}[\delta]_{s\wedge t}(x) \geq \widetilde{R}[\delta]_{s\wedge t}(x*y)\wedge\widetilde{R}[\delta]_{s\wedge t}(y) \geq \widetilde{R}[\delta]_s(x*y)\wedge\widetilde{R}[\delta]_t(y) \Rightarrow \widetilde{R}[\delta]_{s\wedge t}(x) = 1 \Rightarrow [(x, s\wedge t) \in \widetilde{R}[\delta]] = 1$. Thus, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] = [(x, s\wedge t) \in \widetilde{R}[\delta]]\vee[(x, s\wedge t)q\widetilde{R}[\delta]] = 1$.

Case 2: at $a = 1/2$, $\widetilde{R}[\delta]_s(x*y) \geq 1/2$ and $\widetilde{R}[\delta]_t(y) \geq 1/2$; so, $\widetilde{R}[\delta]_{s\wedge t}(x) \geq \widetilde{R}[\delta]_{s\wedge t}(x*y)\wedge\widetilde{R}[\delta]_{s\wedge t}(y) \geq \widetilde{R}[\delta]_s(x*y)\wedge\widetilde{R}[\delta]_t(y) \geq 1/2 \Rightarrow \widetilde{R}[\delta]_{s\wedge t}(x) \geq 1/2 \Rightarrow [(x, s\wedge t) \in \widetilde{R}[\delta]] \geq 1/2$. Therefore, $[(x, s\wedge t) \in \vee q\widetilde{R}[\delta]] = [(x, s\wedge t) \in \widetilde{R}[\delta]]\vee[(x, s\wedge t)q\widetilde{R}[\delta]] \geq 1/2$. Hence, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] \geq [(x*y,s) \in \widetilde{R}[\delta]]\wedge[(y,t) \in \widetilde{R}[\delta]]$. If $s\wedge t > 1/2$, then let $p \in [0,1]$ be such that $1 - s\wedge t < p < 1/2 < s\wedge t$. Now, $\widetilde{R}[\delta]_{\underline{s\wedge t}}(x) = \widetilde{R}[\delta]_{\underline{1-s\wedge t}}(x) \geq \widetilde{R}[\delta]_{s\wedge t}(x)$ and $\widetilde{R}[\delta]_{\underline{[s\wedge t]}}(x) = \widetilde{R}[\delta]_{\underline{1-s\wedge t}}(x) \geq \widetilde{R}[\delta]_p(x)$. Thus, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] = [(x, s\wedge t) \in \widetilde{R}[\delta]]\vee[(x, s\wedge t)q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_{s\wedge t}(x)\vee\widetilde{R}[\delta]_{\underline{[s\wedge t]}}(x) = \widetilde{R}[\delta]_{\underline{[s\wedge t]}}(x) \geq \widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x*y)\wedge\widetilde{R}[\delta]_p(y) \geq \widetilde{R}[\delta]_s(x*y)\wedge\widetilde{R}[\delta]_t(y) = [(x*y,s) \in \widetilde{R}[\delta]]\wedge[(y,t) \in \widetilde{R}[\delta]]$. Thus, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] = [(x*y,s) \in \widetilde{R}[\delta]]\wedge[(y,t) \in \widetilde{R}[\delta]]$. Therefore, $\widetilde{R}[\delta]$ is an $(\in, \in\vee q)$-IFID of X. Hence, $(\widetilde{R}, tA)$ is an $(\in, \in\vee q)$-IFSID of X. □

Theorem 6. *An IFSS $(\widetilde{R}, tA)$ of X is an $(\in\wedge q, \in)$-IFSID of $X \Leftrightarrow$ for any $p \in (1/2, 1]$, $(\widetilde{R}, tA)_p$ is a FSID of X.*

Proof. $(\Rightarrow)$ Suppose $(\widetilde{R}, tA)$ is an $(\in\wedge q, \in)$-IFSID of X. For every $x, y \in X, \delta \in A$ and for $p \in (1/2, 1]$, $\widetilde{R}[\delta]_{\underline{[p]}}(x) \geq \widetilde{R}[\delta]_p(x)$. Then, $\widetilde{R}[\delta]_p(0) = [(0,p) \in \widetilde{R}[\delta]] \geq [(x,p) \in\wedge q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_p(x)\wedge\widetilde{R}[\delta]_{\underline{[p]}}(x) = \widetilde{R}[\delta]_p(x)$ and $\widetilde{R}[\delta]_p(x) = [(x,p) \in \widetilde{R}[\delta]] \geq [(x*y,p) \in\wedge q\widetilde{R}[\delta]]\wedge[(y,p) \in\wedge q\widetilde{R}[\delta]] = \widetilde{R}[\delta]_p(x*y)\wedge\widetilde{R}[\delta]_{\underline{[p]}}(x*y)\wedge\widetilde{R}[\delta]_p(y)\wedge\widetilde{R}[\delta]_{\underline{[p]}}(y) = \widetilde{R}[\delta]_p(x*y)\wedge\widetilde{R}[\delta]_p(y)$. Therefore, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x)$ and $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x*y)\wedge\widetilde{R}[\delta]_p(y)$. Thus, $\widetilde{R}[\delta]_p$ is a FID of X. Hence, $(\widetilde{R}, tA)_p$ is a FSID of X.

$(\Leftarrow)$ We claim that, for any $p \in (1/2, 1]$, $(\widetilde{R}, tA)_p$ is a FSID of X. Let $x, y \in X, \delta \in A$ and for $s, t \in (0, 1]$. Let $c = [(x,s) \in\wedge q\widetilde{R}[\delta]]$.

Case 1: let $c = 1$. Then, $\widetilde{R}[\delta]_s(x) \geq s$ and $\varphi_{\widetilde{R}[\delta]}(x) > 1 - s$. Let $p = \varphi_{\widetilde{R}[\delta]}(x)$. For $\varphi_{\widetilde{R}[\delta]}(x) > 1/2$, we have $p > 1/2$ and $\widetilde{R}[\delta]_p(x) = 1$. Then, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x) = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(0) = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(0) \geq p = \varphi_{\widetilde{R}[\delta]}(x) = s$. Hence, $[(0,s) \in \widetilde{R}[\delta]] = 1$.

Case 2: let $c = 1/2$. Then, $1 - \psi_{\widetilde{R}[\delta]}(x) \geq s$ and $s > \psi_{\widetilde{R}[\delta]}(x)$. Thus, $1 - \psi_{\widetilde{R}[\delta]}(x) > 1/2$ and $\psi_{\widetilde{R}[\delta]}(x) < 1/2$. Let $p = 1 - \psi_{\widetilde{R}[\delta]}(x)$. Then, $p > 1/2$. Therefore, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x) \geq 1/2 \Rightarrow 1 - \psi_{\widetilde{R}[\delta]}(0) \geq p = 1 - \psi_{\widetilde{R}[\delta]}(x) = s$. Hence, $[(0,s) \in \widetilde{R}[\delta]] \geq 1/2 = [(x,s) \in\wedge q\widetilde{R}[\delta]]$.

Next, we prove $[(x, s\wedge t) \in \widetilde{R}[\delta]] \geq [(x*y,s) \in\wedge q\widetilde{R}[\delta]]\wedge[(y,t) \in\wedge q\widetilde{R}[\delta]]$. Let $c = [(x*y,s) \in\wedge q\widetilde{R}[\delta]]\wedge[(y,t) \in\wedge q\widetilde{R}[\delta]]$.

Case 1: let $c = 1$. Then, $\varphi_{\widetilde{R}[\delta]}(x*y) \geq s, \varphi_{\widetilde{R}[\delta]}(x*y) > 1 - s$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq t, \varphi_{\widetilde{R}[\delta]}(y) > 1 - t$. Therefore, $\varphi_{\widetilde{R}[\delta]}(x*y) > 1/2, \varphi_{\widetilde{R}[\delta]}(y) > 1/2$. Let $p = \varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$; then, $p > 1/2$, and we get $\varphi_{\widetilde{R}[\delta]}(x*y) \geq p$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq p \Rightarrow \widetilde{R}[\delta]_p(x*y) = 1$ and $\widetilde{R}[\delta]_p(y) = 1$. Thus, $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x*y)\wedge\widetilde{R}[\delta]_p(y) = 1 \Rightarrow \widetilde{R}[\delta]_p(x) = 1$, and so, $\varphi_{\widetilde{R}[\delta]}(x) \geq p = \varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y) = s\wedge t$. Hence, $[(x, s\wedge t) \in \widetilde{R}[\delta]] = 1$.

Case 2: let $c = 1/2$. Then, $1 - \psi_{\widetilde{R}[\delta]}(x*y) \geq s, s > \psi_{\widetilde{R}[\delta]}(x*y)$ and $1 - \psi_{\widetilde{R}[\delta]}(y) \geq t, t > \psi_{\widetilde{R}[\delta]}(y) \Rightarrow \psi_{\widetilde{R}[\delta]}(x*y) < 1/2$ and $\psi_{\widetilde{R}[\delta]}(y) < 1/2$. Then, $1 - \psi_{\widetilde{R}[\delta]}(x*y) > 1/2$ and $1 - \psi_{\widetilde{R}[\delta]}(y) > 1/2$. Let $p = (1 - \psi_{\widetilde{R}[\delta]}(x*y))\wedge(1 - \psi_{\widetilde{R}[\delta]}(y))$. Then, $p > 1/2$. Therefore, $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x*y)\wedge\widetilde{R}[\delta]_p(y) = 1/2$ as $1 - \psi_{\widetilde{R}[\delta]}(x*y) \geq p$ and $1 - \psi_{\widetilde{R}[\delta]}(y) \geq p$. Thus, $1 - \psi_{\widetilde{R}[\delta]}(x) \geq p = (1 - \psi_{\widetilde{R}[\delta]}(x*y))\wedge(1 - \psi_{\widetilde{R}[\delta]}(y)) \geq s\wedge t$. Hence, $[(x, s\wedge t) \in \widetilde{R}[\delta]] \geq 1/2 = [(x*y, s) \in\wedge q\widetilde{R}[\delta]]\wedge[(y,t) \in\wedge q\widetilde{R}[\delta]]$. Thus, $\widetilde{R}[\delta]$ is an $(\in\wedge q, \in)$-IFID of X. Hence, $(\widetilde{R}, tA)$ is an $(\in\wedge q, \in)$-IFSID of X. □

Theorem 7. *An IFSS $(\widetilde{R}, tA)$ of X is an $(\in\wedge q, \in\vee q)$-IFSID of $X \Leftrightarrow$ for any $x, y \in X$ and $\delta \in A$:*

(1) $\varphi_{\widetilde{R}[\delta]}(0) \geq \varphi_{\widetilde{R}[\delta]}(x)\wedge 1/2$ *or* $\varphi_{\widetilde{R}[\delta]}(0)\vee 1/2 \geq \varphi_{\widetilde{R}[\delta]}(x)$

(2) $\varphi_{\widetilde{R}[\delta]}(x) \geq \varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge 1/2$ *or* $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2 \geq \varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$

(3) $\psi_{\widetilde{R}[\delta]}(0) \leq \psi_{\widetilde{R}[\delta]}(x)\vee 1/2$ *or* $\psi_{\widetilde{R}[\delta]}(0)\wedge 1/2 \leq \psi_{\widetilde{R}[\delta]}(x)$

(4) $\psi_{\widetilde{R}[\delta]}(x) \leq \psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)\vee 1/2$ *or* $\psi_{\widetilde{R}[\delta]}(x)\wedge 1/2 \leq \psi_{\widetilde{R}[\delta]}(x*y)\vee\psi_{\widetilde{R}[\delta]}(y)$

Proof. $(\Rightarrow)$ We prove only (2) and (4), and (1) and (3) can be similarly proved.

(2) If $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2 < t = \varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)$, then $\varphi_{\widetilde{R}[\delta]}(x*y) \geq t > 1/2, \varphi_{\widetilde{R}[\delta]}(y) \geq t > 1/2$. Then, $[(x*y, 1/2) \in\wedge q\widetilde{R}[\delta]] = 1$ and $[(y, 1/2) \in\wedge q\widetilde{R}[\delta]] = 1$. Thus, $[(x, 1/2\wedge 1/2) \in\vee q\widetilde{R}[\delta]] \geq [(x*y, 1/2) \in\wedge q\widetilde{R}[\delta]]\wedge[(y, 1/2) \in\wedge q\widetilde{R}[\delta]] = 1 \Rightarrow [(x, 1/2) \in\wedge q\widetilde{R}[\delta]] = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(x) \geq 1/2$ or $\varphi_{\widetilde{R}[\delta]}(x) + 1/2 > 1$. Hence, $\varphi_{\widetilde{R}[\delta]}(x) \geq 1/2 \geq \varphi_{\widetilde{R}[\delta]}(x*y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge 1/2$.

(4) If $\psi_{\widetilde{R}[\delta]}(x)\wedge 1/2 > t = 1 - s = \psi_{\widetilde{R}[\delta]}(x * y)\vee\psi_{\widetilde{R}[\delta]}(y)$, then $s \leq 1 - \psi_{\widetilde{R}[\delta]}(x * y), s \leq 1 - \psi_{\widetilde{R}[\delta]}(y)$, and $s > 1/2$ as $[(x * y, 1/2) \in\wedge q\widetilde{R}[\delta]] \geq 1/2$ and $[(y, 1/2) \in\wedge q\widetilde{R}[\delta]] \geq 1/2$. Thus, $[(x, 1/2\wedge 1/2) \in\vee q\widetilde{R}[\delta]] \geq [(x * y, 1/2) \in\wedge q\widetilde{R}[\delta]]\wedge[(y, 1/2) \in\wedge q\widetilde{R}[\delta]] \geq 1/2 \Rightarrow [(x, 1/2) \in\vee q\widetilde{R}[\delta]] \geq 1/2 \Rightarrow 1/2 \leq 1 - \psi_{\widetilde{R}[\delta]}(x)$ or $\psi_{\widetilde{R}[\delta]}(x) < 1/2$. Hence, $\psi_{\widetilde{R}[\delta]}(x) \leq 1/2 \leq \psi_{\widetilde{R}[\delta]}(x * y)\vee \psi_{\widetilde{R}[\delta]}(y)\vee 1/2$.

($\Leftarrow$) For every $x, y \in X, \delta \in A$ and $s, t \in (0, 1]$, let $c = [(x * y, s) \in\wedge q\widetilde{R}[\delta]]\wedge[(y, t) \in\wedge q\widetilde{R}[\delta]]$.

Case 1: let $c = 1$. Then, $\varphi_{\widetilde{R}[\delta]}(x * y) \geq s, \varphi_{\widetilde{R}[\delta]}(x * y) > 1 - s$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq t, \varphi_{\widetilde{R}[\delta]}(y) > 1 - t \Rightarrow \varphi_{\widetilde{R}[\delta]}(x * y)\wedge \varphi_{\widetilde{R}[\delta]}(y) > 1/2$. Suppose $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] \leq 1/2$; then, $\varphi_{\widetilde{R}[\delta]}(x) < s\wedge t$ and $\varphi_{\widetilde{R}[\delta]}(x) < 1 - s\wedge t$. Then, $\varphi_{\widetilde{R}[\delta]}(x) < 1/2 < \varphi_{\widetilde{R}[\delta]}(x * y)\wedge\varphi_{\widetilde{R}[\delta]}(y), \varphi_{\widetilde{R}[\delta]}(x) < \varphi_{\widetilde{R}[\delta]}(x * y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge 1/2$ and $\varphi_{\widetilde{R}[\delta]}(x)\vee 1/2 < \varphi_{\widetilde{R}[\delta]}(x * y)\wedge \varphi_{\widetilde{R}[\delta]}(y)$. This is a contradiction to (2). Hence, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] = 1$.

Case 2: let $c = 1/2$. Then, $1 - \psi_{\widetilde{R}[\delta]}(x * y) \geq s > \psi_{\widetilde{R}[\delta]}(x * y), 1 - \psi_{\widetilde{R}[\delta]}(y) \geq t > \psi_{\widetilde{R}[\delta]}(y)$. Thus, $\psi_{\widetilde{R}[\delta]}(x * y)\vee \psi_{\widetilde{R}[\delta]}(y) < 1/2$. If $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] = 0$, then $\psi_{\widetilde{R}[\delta]}(x) \geq s\wedge t > 1 - \psi_{\widetilde{R}[\delta]}(x)$. Thus, $\psi_{\widetilde{R}[\delta]}(x) > 1/2$, and so, $\psi_{\widetilde{R}[\delta]}(x)\wedge 1/2 = 1/2 > \psi_{\widetilde{R}[\delta]}(x * y)\vee\psi_{\widetilde{R}[\delta]}(y)$ and $\psi_{\widetilde{R}[\delta]}(x) > \psi_{\widetilde{R}[\delta]}(x * y)\vee\psi_{\widetilde{R}[\delta]}(y)\vee 1/2$ which is a contradiction to (4). Hence, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] \geq 1/2$. Therefore, $[(x, s\wedge t) \in\vee q\widetilde{R}[\delta]] \geq [(x * y, s) \in\wedge q\widetilde{R}[\delta]]\wedge[(y, t) \in\wedge q\widetilde{R}[\delta]]$. This shows that $\widetilde{R}[\delta]$ is an $(\in\wedge q, \in\vee q)$-IFID of X. Hence, $(\widetilde{R}, tA)$ is an $(\in\wedge q, \in\vee q)$-IFSID of X. □

Theorem 8. *An IFSS $(\widetilde{R}, tA)$ of X is an IFSID of X with thresholds $(s, t) \Leftrightarrow$ for every $p \in (s, t]$, $(\widetilde{R}, tA)_p$ is a FSID of X.*

Proof. ($\Rightarrow$) Suppose $(\widetilde{R}, tA)$ is an IFSID with thresholds (s, t) of X. Let $x, y \in X$ and $p \in (s, t]$. Let $c = \widetilde{R}[\delta]_p(x)$.

Case 1: let $c = 1$. Then, $\varphi_{\widetilde{R}[\delta]}(x) \geq p > s \Rightarrow \varphi_{\widetilde{R}[\delta]}(0)\vee s \geq \varphi_{\widetilde{R}[\delta]}(x)\wedge t \geq p\wedge t = p$. Thus, $\varphi_{\widetilde{R}[\delta]}(0) \geq p \Rightarrow \widetilde{R}[\delta]_p(0) = 1$.

Case 2: let $c = 1/2$. Then, $1 - \psi_{\widetilde{R}[\delta]}(x) \geq p$. Thus, $\psi_{\widetilde{R}[\delta]}(x) \leq 1 - p < 1 - s$. As $t \geq p$, now, $\psi_{\widetilde{R}[\delta]}(0)\wedge(1 - s) \leq \psi_{\widetilde{R}[\delta]}(x)\vee(1 - t) \leq (1 - p)\wedge(1 - t) = 1 - p$. Therefore, $1 - \psi_{\widetilde{R}[\delta]}(0) \geq p \Rightarrow \widetilde{R}[\delta]_p(0) \geq 1/2 = \widetilde{R}[\delta]_p(x)$. Hence, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x)$.

Next, we have to prove $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x * y)\wedge\widetilde{R}[\delta]_p(y)$. Let $c = \widetilde{R}[\delta]_p(x * y)\wedge\widetilde{R}[\delta]_p(y)$.

Case 1: let $c = 1$. Then, $\widetilde{R}[\delta]_p(x * y) = 1$ and $\widetilde{R}[\delta]_p(y) = 1 \Rightarrow \varphi_{\widetilde{R}[\delta]}(x * y) \geq p > s$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq p > s$. Now, $\varphi_{\widetilde{R}[\delta]}(x)\vee s \geq \varphi_{\widetilde{R}[\delta]}(x * y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge t \geq (p\vee p)\wedge t = p \Rightarrow \varphi_{\widetilde{R}[\delta]}(x) \geq p$. Thus, $\widetilde{R}[\delta]_p(x) = 1$.

Case 2: let $c = 1/2$. Then, $\widetilde{R}[\delta]_p(x * y) = 1/2$ and $\widetilde{R}[\delta]_p(y) = 1/2 \Rightarrow 1 - \psi_{\widetilde{R}[\delta]}(x * y) \geq p$ or $1 - \psi_{\widetilde{R}[\delta]}(y) \geq p$. Thus, $\psi_{\widetilde{R}[\delta]}(x * y)\wedge\psi_{\widetilde{R}[\delta]}(y) \leq 1 - p < 1 - s$, from which we get $\psi_{\widetilde{R}[\delta]}(x)\wedge(1 - s) \leq (\psi_{\widetilde{R}[\delta]}(x * y)\wedge\psi_{\widetilde{R}[\delta]}(y))\vee(1 - t) \leq (1 - p)\vee(1 - t) = 1 - p$. Since $t \geq p$ and $1 - s > 1 - p$, therefore, $1 - \psi_{\widetilde{R}[\delta]}(x) \geq p$, and so, $\psi_{\widetilde{R}[\delta]}(x) \geq 1/2 = \psi_{\widetilde{R}[\delta]}(x * y)\wedge \psi_{\widetilde{R}[\delta]}(y)$. Thus, $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x * y)\wedge\widetilde{R}[\delta]_p(y)$. Therefore, $\widetilde{R}[\delta]_p$ is a FID of X. Hence, $(\widetilde{R}, tA)_p$ is a FSID of X.

($\Leftarrow$) We assume for every $p \in (s, t]$, $(\widetilde{R}, tA)_p$ is a FSID of X. We show $\varphi_{\widetilde{R}[\delta]}(0)\vee s \geq \varphi_{\widetilde{R}[\delta]}(x)\wedge t$.

If $\varphi_{\widetilde{R}[\delta]}(0)\vee s < p = \varphi_{\widetilde{R}[\delta]}(x)\wedge t$, then $p \in (s, t]$ and $\varphi_{\widetilde{R}[\delta]}(x) \geq p$. Thus, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x) = 1$, and we have $\varphi_{\widetilde{R}[\delta]}(0) = 1$, and so, $\varphi_{\widetilde{R}[\delta]}(0) \geq p$. This is a contradiction to $\varphi_{\widetilde{R}[\delta]}(0) < p$. Hence, $\varphi_{\widetilde{R}[\delta]}(0)\vee s \geq \varphi_{\widetilde{R}[\delta]}(x)\wedge t$.

We have to show $\psi_{\widetilde{R}[\delta]}(0)\wedge(1 - s) \leq \psi_{\widetilde{R}[\delta]}(x)\vee(1 - t)$. If $\psi_{\widetilde{R}[\delta]}(0)\wedge(1 - s) > p = \psi_{\widetilde{R}[\delta]}(x)\vee(1 - t)$, then $(1 - \psi_{\widetilde{R}[\delta]}(0))\vee s < c = 1 - p = (1 - \psi_{\widetilde{R}[\delta]}(x))\wedge t$, and so, $c \in (s, t]$ and $1 - \psi_{\widetilde{R}[\delta]}(x) \geq c$. Thus, $\widetilde{R}[\delta]_p(0) \geq \widetilde{R}[\delta]_p(x) \geq 1/2 \Rightarrow \widetilde{R}[\delta]_p(0) \geq 1/2$ and $1 - \psi_{\widetilde{R}[\delta]}(0) \geq c = 1 - p$. Therefore, $\psi_{\widetilde{R}[\delta]}(0) \leq p$, which is a contradiction to $\psi_{\widetilde{R}[\delta]}(0) > p$. Hence, $\psi_{\widetilde{R}[\delta]}(0)\wedge(1 - s) \leq \psi_{\widetilde{R}[\delta]}(x)\vee(1 - t)$.

Next, we have to show $\varphi_{\widetilde{R}[\delta]}(x)\vee s \geq \varphi_{\widetilde{R}[\delta]}(x * y)\wedge \varphi_{\widetilde{R}[\delta]}(y)\wedge t$. If $\varphi_{\widetilde{R}[\delta]}(x)\vee s < p = \varphi_{\widetilde{R}[\delta]}(x * y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge t$, then $p \in (s, t], \varphi_{\widetilde{R}[\delta]}(x * y) \geq p$ and $\varphi_{\widetilde{R}[\delta]}(y) \geq p$. Thus, $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x * y)\wedge\widetilde{R}[\delta]_p(x) = 1 \Rightarrow \widetilde{R}[\delta]_p(x) = 1$, and so, $\varphi_{\widetilde{R}[\delta]}(x) \geq p$. This is a contradiction to $\varphi_{\widetilde{R}[\delta]}(x) < p$. Hence, $\varphi_{\widetilde{R}[\delta]}(x)\vee s \geq \varphi_{\widetilde{R}[\delta]}(x * y)\wedge\varphi_{\widetilde{R}[\delta]}(y)\wedge t$.

We have to show $\psi_{\widetilde{R}[\delta]}(x)\wedge(1 - s) \leq \psi_{\widetilde{R}[\delta]}(x * y)\vee\psi_{\widetilde{R}[\delta]}(y)\vee(1 - t)$. If $\psi_{\widetilde{R}[\delta]}(x)\wedge(1 - s) > p = \psi_{\widetilde{R}[\delta]}(x * y)\vee\psi_{\widetilde{R}[\delta]}(y)\vee(1 - t)$, then $(1 - \psi_{\widetilde{R}[\delta]}(x))\vee s < c = 1 - p = (1 - \psi_{\widetilde{R}}[\delta](x * y))\wedge(1 - \psi_{\widetilde{R}[\delta]}(y))\wedge t$, and so, $c \in (s, t]$ and $1 - \psi_{\widetilde{R}[\delta]}(x * y) \geq c$ and $1 - \psi_{\widetilde{R}[\delta]}(y) \geq c$. Thus, $\widetilde{R}[\delta]_p(x) \geq \widetilde{R}[\delta]_p(x * y)\wedge\widetilde{R}[\delta]_p(y) \geq 1/2 \Rightarrow \widetilde{R}[\delta]_p(x) \geq 1/2$, and so, $1 - \psi_{\widetilde{R}[\delta]}(x) \geq c = 1 - p$. Therefore, $\psi_{\widetilde{R}[\delta]}(x) \leq p$, which is a contradiction to $\psi_{\widetilde{R}[\delta]}(x) > p$. Hence, $\psi_{\widetilde{R}[\delta]}(x)\wedge(1 - s) \leq \psi_{\widetilde{R}[\delta]}(x * y)\vee\psi_{\widetilde{R}[\delta]}(y)\vee(1 - t)$. Therefore, $\widetilde{R}[\delta]$ is an IFID of X with thresholds (s, t). Hence, $(\widetilde{R}, tA)$ is an IFSID of X with thresholds (s, t). □

5. Conclusion

The main goal of the present paper is to introduce the notion of (α, β)-intuitionistic fuzzy soft ideal of BCK/BCI-algebras, where α and β are the membership values between an intuitionistic fuzzy soft point and intuitionistic fuzzy set. Moreover, intuitionistic fuzzy soft ideals with thresholds are introduced, and their related properties are investigated.

We hope that this work will give a deep impact on the upcoming research in this field and other soft algebraic studies to open up new horizons of interest and innovations. In future directions, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras, B-algebras, MV-algebras, d-algebras, and Q-algebras.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript.

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