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Research Article

Generalization of (Q,L)-Fuzzy Soft Subhemirings of a Hemiring

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This paper investigates the properties and results of (Q,L)-fuzzy soft subhemirings ((Q,L)-FSSHR) of a hemiring R. The motivation behind this study is to utilize the concept of L-fuzzy soft set of a hemiring and to derive a few specific outcomes on (Q, L)-FSSHR. The concepts of strongest Q-fuzzy soft set relation, Q-isomorphism, pseudo-Q-fuzzy soft coset, and some of their related properties are implemented while analyzing the results. Finally, the properties are verified with a numerical example from the 2000 AMS subject classification: 05C38, 05A15, and 15A18.

1. Introduction

The pioneering work on fuzzy sets was presented by Zadeh in [1]. Following this, intuitionistic fuzzy sets (IFS) were introduced by Attansov [2] and Agarwal et al. [3] who studied the generalized IFS with applications in decision making. Goguen [4] generalized fuzzy sets to L-fuzzy sets. Molodtsov introduced soft sets and Maji et al. [5] did a combined study on fuzzy sets and soft sets and introduced fuzzy soft sets. After that, Maji et al. [6] started a combined work on IFS and soft sets called intuitionistic fuzzy soft sets (IFSS). Hooda et al. [7] introduced the intuitionistic fuzzy soft set theory and its applications in medical diagnosis.

Anjan et al. [8] introduced fuzzy soft multiset and presented some results on this. The concept of an approximation space associated with each parameter in a soft set is discussed in [9]. The intuitionistic neutrosophic soft set was studied in [10] and reformulated by Feng [11] and introduced into soft semiring by means of level soft sets and an adjustable approach to the fuzzy soft set. The notion of semiring was introduced by Vandiver and Anjum et al. [12] to characterize the hemirings by falling fuzzy k-ideals. In [13], vague sets were introduced and analyzed the various operations. Under a vulnerability climate, the neutrosophic soft sets [16] have been effectively applied, and numerical models have been effectively applied in dynamic issues. By using these definitions, the applications of the soft set hypothesis have been concentrated progressively.

In this paper, by introducing the concepts of hemiring and subhemiring for fuzzy soft sets along with L-fuzzy soft sets, the logarithmic structure of (Q, L)-fuzzy soft subhemiring concept is developed and investigated for their major properties. Fuzzy soft sets were also applied as an operational tool of (Q, L)-fuzzy soft relation homomorphic prepicture and synthesis activity and not many of its connected properties are broken down [16]. Molodtsov [17] presented the soft set hypothesis which draws in many creators since it has a wide scope of utilizations in fields of dynamic, gauging, and information examination. Presently, many researchers attempt to hybridize the soft set with various numerical models such as in [18], Shabir et al. [19] studied the characterization of hemirings by the properties of their k-ideals. Muhammed [21] discussed fuzzy ideals in nearing with respect to t-norm and investigated quotients near rings. Dudek [22] introduced the notion of intuitionistic fuzzy left k-ideals of semirings and min-max-plus semiring connections with left k-ideals of the corresponding semirings. Shabir et al. [23] studied on k-bi-ideals in hemirings. The ambiguous soft set [24] and the reluctant fuzzy soft set [25] are presented and then further augmentations of soft sets such as the span esteemed fuzzy soft set [26], the multifuzzy soft set [27], and the trapezoidal fuzzy soft set [28] were carried out. Xu et al. [29] presented the vague delicate sets and their properties.

This universe is stacked with second thoughts, imprecision, and vagueness. In actuality, most of the thoughts we bargain contain muddled data in contrast to accurate. Overseeing the second thought, vulnerability is a vital issue in various regions, for instance, financial aspects, designing, regular science, medicinal science, and social sciences. Countless researchers have ended up with the same conclusion, exhibiting a lack of definition (Table 1).

2. Preliminaries

In this section, we provide some basic definitions those are related to this study.

Definition 1 (see [8]). A pair is identified as a soft set ⇔G is a function K in to these to fall subset of the set U.

Example 1. Suppose U is the set of five laptops under consideration. Here, let $U = \{l_1, l_2, l_3, l_4, l_5\}$ and $K = \{p_1$

$$\text{(good looking)}, p_2\,\text{(quality)}, p_3\,\text{(storage space)},$$

$$p_4\,\text{(modern technology)}, p_5\,\text{(price)}\}$$

be the set of parameters.

$$(K, G) = \{(p_1, (l_2, l_4)), (p_2, (l_1, l_3)), (p_3, (l_4, l_2)), (p_4, (l_3, l_1)), (p_5, (l_1, l_5))\}. \tag{1}$$

Definition 2 (see [14]). Let $X, Q \neq \phi$. A Q-fuzzy $\subseteq H$ of X, then $H: X \times Q \longrightarrow [0, 1]$.

Definition 3 (see [7]). Let $\mathfrak{R}$ be a hemiring. A L–FS $\subseteq (S_1, J_1)$ of $\mathfrak{R}$ is called a L-fuzzy soft subhemiring (LFSSHR) of $\mathfrak{R}$ if the following axioms hold:

(1) $j_{(S_1,J_1)}(u_{(S_1,J_1)} + v_{(S_1,J_1)}) \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)} \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}))\}$,

(2) $j_{(S_1,J_1)}(u_{(S_1,J_1)} v_{(S_1,J_1)}) \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}) \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)})\}$, $\forall u_{(S_1,J_1)}$ and $v_{(S_1,J_1)} \in \mathfrak{R}$.

Example 2. Let $R = A = Z_6 = \{0, 1, 2, 3, 4, 5\}$. Then, consider $F: R \longrightarrow \mathfrak{R}(R)$ given by $F(x) = \{y \in R, x.y = 0\}$. Then, $F(0) = R, F(1) = \{0\}, F(2) = \{0, 3\}, F(3) = \{0, 2, 4\}$, $F(4) = \{0, 3\}$, and $F(5) = \{0\}$. All these sets are subhemirings of R. Therefore, (S_1, J_1) is a soft subhemiring over R.

Definition 4 (see [10]). Let $\mathfrak{R}$ be a hemiring. A (Q, L)–FS $\subseteq H$ of $\mathfrak{R}$ is called (Q, L)-fuzzy soft subhemiring (Q-FSSHR) if the following conditions hold:

(1) $j_{(S_1,J_1)}(u_{(S_1,J_1)} + v_{(S_1,J_1)}), q \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}), q \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}, q)\}$,

(2) $j_{(S_1,J_1)}(u_{(S_1,\ J_1)} v_{(S_1,J_1)}), q \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}), q \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}, q)\}$, for all $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in $\mathfrak{R}$ and $q \in Q$.

Definition 5 (see [19]). If $(\mathfrak{R}, +,.)$ and $(\mathfrak{R}^1, +, .)$ are any two hemirings and Q is a nonempty set, then $\psi: R \times Q \longrightarrow R^1 \times Q$ is called a (Q, L)-known as a Homomorphism if $\psi(u + v, r) = \psi(u, r) + \psi(v, r)$ and $\psi(uv, r) = \psi(u, r)\psi(v, r)$ for all $u, v \in \mathfrak{R}$ and r in Q.

Definition 6 (see [17]). If $(\mathfrak{R}, +,.)$ and $(\mathfrak{R}^1, +, .)$ are any two hemirings and Q is a nonempty set, then $\psi: R \times Q \longrightarrow R^1 \times Q$ is known as an anti-(Q, L)-homomorphism if $\psi(u + v, r) = \psi(v, r) + \psi(u, r)$ and $\psi(uv, r) = \psi(v, r)\psi(u, r)$, $\forall u, v \in \mathfrak{R}$, and r in Q.

Definition 7 (see [9]). Let (S_1, J_1) be a (Q, L)-fuzzy soft subset. For μ in $[0, 1]$, the sets $U_{j(s_1,j_1),\mu} = \{u_{(S_1,J_1)} \in X\colon j_{(S_1,J_1)}(u_{(S_1,J_1)}), q \geq \mu\}$ and is known as (Q, L)-fuzzy soft level μ-cut.

Definition 8 (see [16]). Let (S_1, J_1) be a (Q, L)-fuzzy soft subhemiring of X. For μ in $[0, 1]$, the level subset of (S_1, J_1) is the set $j_{(S_1,J_1)}\mu = \{u_{(S_1,J_1)} \in X\colon j_{(S_1,J_1)}(u_{(S_1,J_1)}), q \geq \mu\}$. This is known as a (Q, L)-fuzzy soft level subset.

Definition 9 (see [18]). Let (S_1, J_1) be a (Q, L)-FSSHR of a hemiring $(\mathfrak{R}, +,.)$. The level subhemiring $j_{(S_1,J_1)}\mu$, for μ in $[0, 1]$ with the end goal that $\mu \leq j_{(S_1,J_1)}(0), \mu \leq j_{(S_1,J_1)}(1)$, is known as a (Q, L)-fuzzy soft level soft subhemiring.

Table 1: How the person chooses the laptop.

Laptop	Good-looking	Quality	Storage space	Modern technology	Price
l_1	0	1	0	1	1
l_2	1	0	1	0	0
l_3	0	1	0	1	0
l_4	1	0	1	0	0
l_5	0	0	0	0	1

3. Some Properties of (Q, L)-Fuzzy Soft Subhemirings of a Hemiring

In this section, we provide main results using properties of (Q, L)-FSSHR of a hemiring.

Theorem 1. *If $(S_1, J_1)$is a (Q, L)-FSSHR of a hemiring $\mathfrak{R}$, then $H = \{u_{(S_1,J_1)}/u_{(S_1,J_1)} \in \mathfrak{R}\colon j_{(S_1,J_1)}(u_{(S_1,J_1)}), q = 0\}$ is either empty or is a subhemiring of $\mathfrak{R}$.*

Proof. If no elements satisfy these conditions, then H is empty.

Let $(u_{(S_1,J_1)}, v_{(S_1,J_1)}$ in H, then

$$j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}\right), q \geq \left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right) = \{0\wedge 0\} = 0\right\}. \tag{2}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)} + v_{(S_1,J_1)}), q = 0$. For every $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in R and q in Q. We get $u_{(S_1,J_1)} + v_{(S_1,J_1)}$ in H.

$$j\left(u_{(S_1,J_1)}v_{(S_1,J_1)}\right), q \geq \left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right) = \{0\wedge 0\} = 0\right\}. \tag{3}$$

So, $j_{(S_1,J_1)}(u_{(S_1,J_1)}v_{(S_1,J_1)}), q = 0$, for all $u_{(S_1,J_1)}$, and $v_{(S_1,J_1)}$ in R and q in Q. We get $u_{(S_1,J_1)}, v_{(S_1,J_1)}$ in H. Thus, H is a subhemiring of $\mathfrak{R}$. □

Theorem 2. *Let (S_1, J_1) be a (Q, L)-FSSHR of a hemiring $\mathfrak{R}$. Then,*

(1) If $j_{(S_1,J_1)}(u_{(S_1,J_1)} + v_{(S_1,J_1)}), q = 1$, then either $j_{(S_1,J_1)}(u_{(S_1,J_1)}, q) = 1$ or $j_{(S_1,J_1)}(v_{(S_1,J_1)}), q = 1$, for $u_{(S_1,J_1)}$and $v_{(S_1,J_1)}$ in R and q in Q.

(2) If $j_{(S_1,J_1)}(u_{(S_1,J_1)}v_{(S_1,J_1)}), q = 1$, then either $j_{(S_1,J_1)}(u_{(S_1,J_1)}), q = 1$ or $j_{(S_1,J_1)}(v_{(S_1,J_1)}, q) = 1$, for every $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ of $\mathfrak{R}$ and $q \in Q$.

Proof. Let $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)} \in \mathfrak{R}$. By the definition,

$$j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}\right), q \geq \left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right)\right\}. \tag{4}$$

We have

$$1 \geq \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right)\right\}. \tag{5}$$

Thus, whichever $j_{(S_1,J_1)}(u_{(S_1,J_1)}, q) = 1$ or $j_{(S_1,J_1)}(v_{(S_1,J_1)}, q) = 1$, for every $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in R and q in Q,

$$j_{(S_1,J_1)}\left(u_{(S_1,J_1)} v_{(S_1,J_1)}\right), q \geq \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right)\right\}. \tag{6}$$

We have

$$1 \geq \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right)\right\}. \tag{7}$$

$j_{(S_1,J_1)}(u_{(S_1,J_1)}, q) = 1$ or $j_{(S_1,J_1)}(v_{(S_1,J_1)}, q) = 1$, for every $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ of $\mathfrak{R}$ and $q \in Q$. □

Theorem 3. *Let* (S_1, J_1) *and* ($\mathfrak{R}$, *D*) *be two (Q, L)-FSSHR of a hemiring* ($\mathfrak{R}$, +*D*). *Then, their intersection* $(S_1, J_1) \cap (R, D)$ *is a (Q, L)-FSSHR of* $\mathfrak{R}$.

Proof. Let u and v be in the right place to $\mathfrak{R}$. Then,

$$(S_1, J_1) = \left\{ \langle \left(u_{(S_1,J_1)}, q\right), j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right)\rangle / u_{(S_1,J_1)} \text{of } \mathfrak{R}, q \in Q\right\}, \tag{8}$$

and

$$(R, D) = \left\{ \langle \left(u_{(R,D)}, q\right), j_{(R,D)}\left(u_{(R,D)}, q\right)\rangle / u_{(R,D)} \in N, q \in Q\right\}. \tag{9}$$

Let $(S, T) = (S_1, J_1) \cap (R, D)$ and $(S, T) = \left\{ \langle (u_{(S,T)}, q), j_{(S,T)}(u_{(S,T)}, q)\rangle / u_{(S,T)} \text{ in } \mathfrak{R}\, q \text{ in } Q\right\}$, where $j_{(S,T)}(u_{(S,T)}, q) = \left\{ j_{(S_1,J_1)}(u_{(S_1,J_1)}, q) \wedge j_{(R,D)}(u_{(R,D)}, q)\right\}$. Now,

$$\begin{aligned}
j_{(S,T)}\left(u_{(S,T)} + v_{(S,T)}, q\right) &= \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}, q\right) \wedge j_{(R,D)}\left(u_{(R,D)} + v_{(R,D)}, q\right)\right\} \\
&\geq \left\{ \begin{array}{c} \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right) \wedge \right\} \\ \left\{ j_{(R,D)}\left(v_{(R,D)}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)}, q\right)\right\} \end{array} \right] \\
&\geq \left\{ \begin{array}{c} \left\{ \left(j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(R,D)} u_{(R,D)}, q\right)\right\} \wedge \\ \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)}, q\right)\right\} \end{array} \right] \\
&\geq \left\{ j_{(S,T)}\left(v_{(S,T)}, q\right) \wedge j_{(S,T)}\left(v_{(S,T)}, q\right)\right\},
\end{aligned} \tag{10}$$

for every $u_{(S,T)}$ and $v_{(S,T)}$ in $\mathfrak{R}$ and q in Q. Again,

$$\begin{aligned}
j_{(S,T)}\left(u_{(S,T)} + v_{(S,T)}, q\right) &= \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}, q\right) \wedge j_{(R,D)}\left(u_{(R,D)} + v_{(R,D)}, q\right)\right\} \\
&\geq \left\{ \begin{array}{c} \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right) \wedge \right\}, \\ \left\{ j_{(R,D)}\left(v_{(R,D)}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)}, q\right)\right\} \end{array} \right] \\
&\geq \left\{ \begin{array}{c} \left\{ \left(j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(R,D)} u_{(R,D)}, q\right)\right\} \wedge, \\ \left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)}, q\right)\right\} \end{array} \right] \\
&\geq \left\{ j_{(S,T)}\left(v_{(S,T)}, q\right) \wedge j_{(S,T)}\left(v_{(S,T)}, q\right)\right\}.
\end{aligned} \tag{11}$$

Thus, $j_{(S,T)}(u_{(S,T)}v_{(S,T)}, q) \geq \{j_{(S,T)}(v_{(S,T)}, q) \wedge j_{(S,T)}(v_{(S,T)}, q)\}$, for each $u_{(S,T)}$ and $v_{(S,T)}$ in R and q in Q. □

Theorem 4. *The* $\cap$ *of a* (Q, L)*-FSSHR of a hemiring* $(\mathfrak{R}, +, .)$ *is a* (Q, L)*-FSSHR of* $\mathfrak{R}$.

Proof. Consider $\{j_{(S_1,J_1)}i\}_{i\in I}$ as a family of (Q, L)-FSSHR of a hemiring $\mathfrak{R}$ and $j_{(S_1,J_1)} = \prod_{i\in I} j_{(S_1,J_1)}i$. Here, $u, v \in R$, we have the following two cases: □

Case 1.

$$\begin{aligned} j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}, q\right) &= \left(j_{(S_1,J_1)}\right)i\left(\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}\right), q\right) \\ &\geq \inf_{i\in I}\left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}i\left(v_{(S_1,J_1)}, q\right)\right\} \\ \geq \inf_{i\in I}\left\{\left(j_{(S_1,J_1)}i\left(u_{(S_1,J_1)}, q\right)\right)\right\} \wedge \inf_{i\in I}\left\{j_{(S_1,J_1)}i\left(v_{(S_1,J_1)}, q\right)\right\} &= \left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}i\left(v_{(S_1,J_1)}, q\right)\right\}. \end{aligned} \tag{12}$$

Therefore,

$$j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}, q\right) \geq \left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right)\right\}, \tag{13}$$

for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ of $\mathfrak{R}$ and $q \in Q$.

Case 2.

$$\begin{aligned} j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}, q\right) &= \left(j_{(S_1,J_1)}\right)i\left(\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}\right), q\right) \geq \inf_{i\in I}\left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}i\left(v_{(S_1,J_1)}, q\right)\right\} \\ &\geq \inf_{i\in I}\left\{\left(j_{(S_1,J_1)}i\left(u_{(S_1,J_1)}, q\right)\right)\right\} \wedge \inf_{i\in I}\left\{j_{(S_1,J_1)}i\left(v_{(S_1,J_1)}, q\right)\right\} = \left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)}, q\right) \wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)}, q\right)\right\}. \end{aligned} \tag{14}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)} v_{(S_1,J_1)}, q) \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}, q) \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}, q)\}$, for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ of $\mathfrak{R}$ and $q \in Q$.

Theorem 5. *If* (S_1, J_1) *and* (R, D) *are any two* (Q, L)*-FSSHR of a hemiring* $(\mathfrak{R}, +, .)$*, then* $(S_1, J_1) \cup (R, D)$ *is a* (Q, L)*-FSSHR of* $\mathfrak{R}$.

Proof. Consider $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ belonging to $\mathfrak{R}$ and $q \in Q$,

$$\begin{aligned} j_{(S_1,J_1)} &= \left\{\frac{\langle\left(u_{(R,D)}, q\right), j_{(R,D)}\left(u_{(R,D)}, q\right)\rangle}{u_{(R,D)}} \text{ in } \mathfrak{R}, q \text{ in } Q\right\}\left\{\frac{\langle\left(u_{(S_1,J_1)}, q\right), j_{(S_1,J_1)}\left(u_{(S_1,J_1)}, q\right)\rangle}{u_{(S_1,J_1)}} \text{ in } \mathfrak{R} \text{ and } q \text{ in } Q\right\}, \\ j_{(R,D)} &= \left\{\frac{\langle\left(u_{(R,D)}, q\right), j_{(R,D)}\left(u_{(R,D)}, q\right)\rangle}{u_{(R,D)}} \text{ of } R, q \in Q\right\}. \end{aligned} \tag{15}$$

Let $(S, T) = (S_1, J_1) \cup (R, D)$ and $(S, T) = \{\langle(u_{(S,T)}, q), j_{(S,T)}(u_{(S,T)}, q)\rangle / u_{(S,T)} \text{ in } \mathfrak{R}, q \text{ in } Q\}$, where $j_{(S,T)}(u_{(S,T)}, q) = \{j_{(S_1,J_1)}(u_{(S_1,J_1)}, q) \vee j_{(R,D)}(u_{(R,D)}, q)\}$. Now,

$$\begin{aligned} j_{(S_1,J_1)}\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}, q\right) &= \left(j_{(S_1,J_1)}\right)i\left(\left(u_{(S_1,J_1)} + v_{(S_1,J_1)}\right), q\right) \geq \left\{\left(j_{(R,D)}\right)i\left(u_{(R,D)}, q\right) \wedge j_{(R,D)}i\left(v_{(R,D)}, q\right)\right\} \\ &\geq \left\{\left(j_{(S_1,J_1)}i\left(u_{(S_1,J_1)}, q\right)\right)\right\} \wedge \left\{j_{(R,D)}i\left(v_{(R,D)}, q\right)\right\} = \left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)}, q\right)\right\}. \end{aligned} \tag{16}$$

Thus, $j_{(S,T)}(u_{(S,T)}+v_{(S,T)},q)\geq j\{(S,T)(u_{(S,T)},q)\wedge j_{(S,T)}(v_{(S,T)},q)\}$, for each $u_{(S,T)}$ and $v_{(S,T)}$ in R and q in Q.

Now,

$$j_{(S_1,J_1)}\left(u_{(S_1,J_1)}+v_{(S_1,J_1)},q\right)=\left(j_{(S_1,J_1)}\right)\left(\left(u_{(S_1,J_1)}+v_{(S_1,J_1)}\right),q\right)\geq\left\{\left(j_{(R,D)}\right)i\left(u_{(R,D)},q\right)\wedge j_{(R,D)}i\left(v_{(R,D)},q\right)\right\}$$
$$\geq\left\{\left(j_{(S_1,J_1)}i\left(u_{(S_1,J_1)},q\right)\right)\right\}\wedge\left\{j_{(R,D)}i\left(v_{(R,D)},q\right)\right\}=\left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)},q\right)\wedge j_{(R,D)}\left(v_{(R,D)},q\right)\right\}. \tag{17}$$

So, $j_{(S,T)}(u_{(S,T)}+v_{(S,T)},q)\geq j\{(S,T)(u_{(S,T)},q)\wedge j_{(S,T)}(v_{(S,T)},q)\}$, for each $u_{(S,T)}$ and $v_{(S,T)}$ in R and q in Q. □

Theorem 6. *The ∪ of a (Q,L)-FSSHR of a hemiring $\mathfrak{R}$ is a (Q,L)-FSSHR of $\mathfrak{R}$.*

Proof. Given as a chance consider $\left\{j_{(S_1,J_1)i}\right\}$ to be a family of a (Q, L)-FSSHR of a hemiring $\mathfrak{R}$ and $j_{(S_1,J_1)}=\cup_{i\in I}j_{(S_1,J_1)i}$. Then, $u_{(S,T)}$ and $v_{(S,T)}$ belong to $\mathfrak{R}$ and q in Q,

$$j_{(S_1,J_1)}\left(u_{(S_1,J_1)}+v_{(S_1,J_1)},q\right)=\sup_{i\in I}\left(j_{(S_1,J_1)}\right)i\left(\left(u_{(S_1,J_1)}+v_{(S_1,J_1)}\right),q\right)\sup_{i\in I}\geq\left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)},q\right)\wedge\sup_{i\in I}j_{(S_1,J_1)}i\left(v_{(R,D)},q\right)\right\}$$
$$\geq\sup_{i\in I}\left\{\left(j_{(S_1,J_1)}i\left(u_{(S_1,J_1)},q\right)\right)\right\}\wedge\sup_{i\in I}\left\{j_{((S_1,J_1))}i\left(v_{(S_1,J_1)},q\right)\right\}=\left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)},q\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)},q\right)\right\}. \tag{18}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)}v_{(S_1,J_1)},q)\geq\left\{j_{(S_1,\quad J_1)}(u_{(S_1,J_1)},q)\wedge j_{(S_1,J_1)}(v_{(S_1,J_1)},q)\right\}$, for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in $\mathfrak{R}$ and q in Q.

$$\begin{aligned}j_{(S_1,J_1)}\left(u_{(S_1,J_1)}+v_{(S_1,J_1)},q\right)&=\sup_{i\in I}\left(j_{(S_1,J_1)i}\right)\left(\left(u_{(S_1,J_1)}+v_{(S_1,J_1)}\right),q\right)\sup_{i\in I}\geq\left\{\left(j_{(S_1,J_1)i}\right)\left(u_{(S_1,J_1)},q\right)\wedge\sup_{i\in I}j_{(S_1,J_1)}i\left(v_{(R,D)},q\right)\right\}\\&\geq\sup_{i\in I}\left\{\left(j_{(S_1,J_1)}i\left(u_{(S_1,J_1)},q\right)\right)\right\}\wedge\sup_{i\in I}\left\{j_{((S_1,J_1))}i\left(v_{(S_1,J_1)},q\right)\right\}=\left\{\left(j_{(S_1,J_1)}\right)i\left(u_{(S_1,J_1)},q\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)},q\right)\right\}.\end{aligned} \tag{19}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)}v_{(S_1,J_1)},q)\geq\left\{j_{(S_1,J_1)}(u_{(S_1,J_1)},q)\wedge j_{(S_1,J_1)}\ (v_{(S_1,J_1)},q)\right\}$ for each $u_{(S_1,J_1)}$ and $u_{(S_1,J_1)}$ of $\mathfrak{R}$ and $q\in Q$.

Similarly, the union of a family of a (Q,L)-FSSHR of a hemiring $\mathfrak{R}$ is a (Q,L)-FSSHR of $\mathfrak{R}$. □

Theorem 7. *If (S_1,J_1) and $(\mathfrak{R},\ D)$ be two (Q,L)-FSSHR of the hemirings $\mathfrak{R}$ and S, correspondingly, then the product $(S_1,J_1)\times(R,D)$ is a (Q,L)-FSSHR of $\mathfrak{R}\times S$.*

Proof. Consider (S_1,J_1) and (R,D) to be two (Q,L)-FSSHR of the hemirings $\mathfrak{R}$ and S correspondingly. Let $u_{(S_1,J_1)1}$ and $u_{(S_1,J_1)2}$ be in $\mathfrak{R}$, and $v_{(S_1,J_1)1}$ and $v_{(R,D)2}$ be in S. Then, $(u_{(S_1,J_1)\times(R,D)1},v_{(S_1,J_1)\times(R,D)1})$ and $(u_{(S_1,J_1)\times(R,D)2},v_{(S_1,J_1)\times(R,D)2})$ are in $\mathfrak{R}\times S$. Now,

$$\begin{aligned}&j_{(S_1,J_1)\times(R,D)}\left(u_{(S_1,J_1)\times(R,D)1},v_{(S_1,J_1)\times(R,D)1}\right)+\left(\left(u_{(S_1,J_1)\times(R,D)2},v_{(R,D)\times(R,D)2}\right),q\right)\\&=j_{(S_1,J_1)\times(R,D)}\left(\left(u_{(S_1,J_1)1}+u_{(S_1,J_1)2},v_{(R,D)1}+v_{(R,D)2}\right),q\right)\\&=\left\{j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)1}+u_{(S_1,J_1)2}\right),q\right)\wedge j_{(R,D)1}\left(\left(v_{(R,D)1}+v_{(R,D)2}\right),q\right)\right\}\\&\geq\left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},q\right)\wedge j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},q\right)\right\}\wedge\left\{j_{(R,D)}\left(v_{(R,D)1},q\right)\wedge j_{(R,D)}\left(v_{(R,D)2},q\right)\right\}\right\}.\end{aligned} \tag{20}$$

Thus,

$$\begin{aligned}
& j_{(S_1,J_1)\times(R,D)}\left(u_{(S_1,J_1)\times(R,D)1}, v_{(S_1,J_1)\times(R,D)1}\right) + \left(\left(u_{(S_1,J_1)\times(R,D)2}, v_{(R,D)\times(R,D)2}\right), q\right) \\
& = j_{(S_1,J_1)\times(R,D)}\left(\left(u_{(S_1,J_1)1} + u_{(S_1,J_1)2}, v_{(R,D)1} + v_{(R,D)2}\right), q\right) \\
& = \left\{ j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)1} + u_{(S_1,J_1)2}\right), q\right) \wedge j_{(R,D)1}\left(\left(v_{(R,D)1} + v_{(R,D)2}\right), q\right)\right\} \\
& \geq \left\{\left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)1}, q\right) \wedge j_{(S_1,J_1)}\left(u_{(S_1,J_1)2}, q\right)\right\} \wedge \left\{ j_{(R,D)}\left(v_{(R,D)1}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)2}, q\right)\right\}\right\},
\end{aligned} \tag{21}$$

for each $u_{(S_1,J_1)\times(R,D)1}$ and $u_{(S_1,J_1)\times(R,D)2}$ in R and $v_{(S_1,J_1)\times(R,D)1}$ and $v_{(S_1,J_1)\times(R,D)2}$ in S and q in Q. Again,

$$\begin{aligned}
& j_{(S_1,J_1)\times(R,D)}\left(u_{(S_1,J_1)\times(R,D)1}, v_{(S_1,J_1)\times(R,D)1}\right) + \left(\left(u_{(S_1,J_1)\times(R,D)2}, v_{(R,D)\times(R,D)2}\right), q\right) \\
& = j_{(S_1,J_1)\times(R,D)}\left(\left(u_{(S_1,J_1)1} + u_{(S_1,J_1)2}, v_{(R,D)1} + v_{(R,D)2}\right), q\right) \\
& = \left\{ j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)1} + u_{(S_1,J_1)2}\right), q\right) \wedge j_{(R,D)1}\left(\left(v_{(R,D)1} + v_{(R,D)2}\right), q\right)\right\} \\
& \geq \left\{\left\{ j_{(S_1,J_1)}\left(u_{(S_1,J_1)1}, q\right) \wedge j_{(S_1,J_1)}\left(u_{(S_1,J_1)2}, q\right)\right\} \wedge \left\{ j_{(R,D)}\left(v_{(R,D)1}, q\right) \wedge j_{(R,D)}\left(v_{(R,D)2}, q\right)\right\}\right\}.
\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
& j_{(S_1,J_1)\times(R,D)}\left(\left(u_{(S_1,J_1)\times(R,D)1}, v_{(S_1,J_1)\times(R,D)1}\right)\left(u_{(S_1,J_1)\times(R,D)2}, v_{(S_1,J_1)\times(R,D)2}\right), q\right) \\
& \geq \left\{ j_{(S_1,J_1)\times(R,D)1}\left(\left(u_{(S_1,J_1)\times(R,D)1}, v_{(S_1,J_1)\times(R,D)}, q\right) \wedge j_{(S_1,J_1)\times(R,D)}\left(u_{(S_1,J_1)\times(R,D)2}, v_{(S_1,J_1)\times(R,D)2}, q\right)\right)\right\},
\end{aligned} \tag{23}$$

for every $u_{(S_1,J_1)\times(R,D)1}$ and $u_{(S_1,J_1)\times(R,D)2}$ in $\mathfrak{R}$ and $v_{(S_1,J_1)\times(R,D)1}$ and $v_{(S_1,J_1)\times(R,D)2}$ in S and q in Q.

Similarly, the product $(S_1, J_1) \times (R, D)$ is a (Q, L)-FSSHR of $\mathfrak{R} \times S$. □ □

4. Properties of (Q, L)-Fuzzy Soft Subhemiring of a Hemiring

Some additional properties of (Q, L)-FSSHR of a hemiring are discussed as follows:

Theorem 8. *Let (S_1, J_1) be a FS subset of a hemiring $\mathfrak{R}$ and (L_1, O_1) be the strongest Q-fuzzy soft relation of $\mathfrak{R}$. Then, (S_1, J_1) is a (Q, L)-FSSHR subhemiring of $\mathfrak{R} \Leftrightarrow (L_1, O_1)$ is a (Q, L)-FSSHR of $\mathfrak{R} \times \mathfrak{R}$.*

Proof. Assume that (S_1, J_1) is a (Q, L)-FSSHR of $\mathfrak{R}$. Here, $u = ((u_{(S_1,J_1)1}, u_{(S_1,J_1)2})$ and $v = (v_{(S_1,J_1)1}, v_{(S_1,J_1)2})$ are in $R \times R$. Now,

$$
\begin{aligned}
& j_{(L_1,O_1)}\left(u_{(L_1,O_1)}+v_{(L_1,O_1)},r\right) \\
& = j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)+\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right) \\
& = j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)+\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right) \\
& = \left\{j_{(S_1,J_1)}\left(\left(\left(u_{(S_1,J_1)1}+v_{(S_1,J_1)1}\right),r\right)\wedge j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)2}+v_{(S_1,J_1)2}\right),r\right)\right)\right\} \\
& \geq \left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right)\right\}\right\} \\
& \geq \left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right)\right\}\right\} \\
& = \left\{j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right),r\right)\wedge j_{(L_1,O_1)}\left(\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2}\right),r\right)\right\} \\
& = \left\{j_{(L_1,O_1)}\left(u_{(L_1,O_1)},r\right)\wedge j_{(L_1,O_1)}\left(v_{(L_1,O_1)},r\right)\right\}.
\end{aligned}
\tag{24}
$$

Therefore, $j_{(L_1,O_1)}(u_{(L_1,O_1)}+v_{(L_1,O_1)},r)\geq\{j_{(L_1,O_1)}(u_{(L_1,O_1)},r)\wedge j_{(L_1,O_1)}(v_{(L_1,O_1)},r)\}$ for each $u_{(L_1,O_1)}$ and $v_{(L_1,O_1)}$ in $R\times R$ and r in Q. Again,

$$
\begin{aligned}
& j_{(L_1,O_1)}\left(u_{(L_1,O_1)}+v_{(L_1,O_1)},r\right) \\
& = j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)+\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right) \\
& = j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)+\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right) \\
& = \left\{j_{(S_1,J_1)}\left(\left(\left(u_{(S_1,J_1)1}+v_{(S_1,J_1)1}\right),r\right)\wedge j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)2}+v_{(S_1,J_1)2}\right),r\right)\right)\right\} \\
& \geq \left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right)\right\}\right\} \\
& \geq \left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right)\right\}\right\} \\
& = \left\{j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right),r\right)\wedge j_{(L_1,O_1)}\left(\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2}\right),r\right)\right\} \\
& = \left\{j_{(L_1,O_1)}\left(u_{(L_1,O_1)},r\right)\wedge j_{(L_1,O_1)}\left(v_{(L_1,O_1)},r\right)\right\}.
\end{aligned}
\tag{25}
$$

Thus, $j_{(S_1,J_1)}(u_{(L_1,O_1)}v_{(L_1,O_1)},r)\geq\{j_{(L_1,O_1)}(u_{(L_1,O_1)},r)\wedge j_{(S_1,J_1)}(v_{(L_1,O_1)},r)\}$ for all $u_{(L_1,O_1)}$ and $v_{(L_1,O_1)}$ in $R\times R$ and r in Q. Hence, (L_1,O_1) is a Q-fuzzy subhemiring of $R\times R$.

Furthermore, we consider that (L_1,O_1) is a Q-fuzzy subhemiring of $R\times R$, then $u=(u_{(L_1,O_1)1},u_{(L_1,O_1)2})$ and $v=((v_{(L_1,O_1)1},v_{(L_1,O_1)2})$ are in $R\times R$.

$$\begin{aligned}
&j_{(L_1,O_1)}\left(u_{(L_1,O_1)}+v_{(L_1,O_1)},r\right)\\
&\quad=j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)+\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right)\\
&\quad=j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)+\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right)\\
&\quad=\left\{j_{(S_1,J_1)}\left(\left(\left(u_{(S_1,J_1)1}+v_{(S_1,J_1)1}\right),r\right)\wedge j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)2}+v_{(S_1,J_1)2}\right),r\right)\right)\right\}\\
&\quad\geq\left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right)\right\}\right\}\\
&\quad\geq\left\{\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right)\right\}\right\}\\
&\quad=\left\{j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right),r\right)\wedge j_{(L_1,O_1)}\left(\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2}\right),r\right)\right\}\\
&\quad=\left\{j_{(L_1,O_1)}\left(u_{(L_1,O_1)},r\right)\wedge j_{(L_1,O_1)}\left(v_{(L_1,O_1)},r\right)\right\}.
\end{aligned}\tag{26}$$

If

$$j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)1}+v_{(S_1,J_1)1}\right),r\right)\geq j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)2}+v_{(S_1,J_1)2}\right),r\right),\ j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\geq j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right),\tag{27}$$

and

$$j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\geq j_{(S_1,J_1)}\left(v_{(S_1,J_1)2},r\right),\tag{28}$$

we get $(S_1,J_1)((u_{(S_1,J_1)1}+v_{(S_1,J_1)1}),r)\geq\{j_{(S_1,J_1)}(u_{(S_1,J_1)1},r)\wedge j_{(S_1,J_1)}(v_{(S_1,J_1)1},r)\}$ for every $u_{(S_1,J_1)1}$ and $v_{(S_1,J_1)1}$ in $\mathfrak{R}$ and r in Q. Again,

$$\begin{aligned}
&\left\{j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)1}v_{(S_1,J_1)1}\right),r\right)\wedge j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)1}v_{(S_1,J_1)1}\right),r\right)\right\}\\
&\quad=j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1}v_{(L_1,O_1)1},u_{(L_1,O_1)2}v_{(L_1,O_1)2}\right),r\right)\\
&\quad=j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right)\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2}\right),r\right)\\
&\quad=j_{(L_1,O_1)}\left(u_{(L_1,O_1)}v_{(L_1,O_1)},r\right)\\
&\quad\geq\left\{j_{(L_1,O_1)}\left(u_{(L_1,O_1)},r\right)\wedge j_{(L_1,O_1)}\left(v_{(L_1,O_1)},r\right)\right\}\\
&\quad=\left\{j_{(L_1,O_1)}\left(\left(u_{(L_1,O_1)1},u_{(L_1,O_1)2}\right),r\right)\wedge j_{(S_1,J_1)}\left(v_{(L_1,O_1)1},v_{(L_1,O_1)2},r\right)\right\}\\
&\quad=\left\{\left\{j_{(S_1,J_1)1}\left(u_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(u_{(S_1,J_1)1},r\right)\right\}\wedge\left\{j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\wedge j_{(S_1,J_1)}\left(v_{(S_1,J_1)1},r\right)\right\}\right\}.
\end{aligned}\tag{29}$$

If

$$\begin{aligned}
j_{(S_1,J_1)}\left(u_{(S_1,J_1)1}v_{(S_1,J_1)},r\right)&\geq j_{(S_1,J_1)2}\left(u_{(S_1,J_1)}v_{(S_1,J_1)},r\right),\ j_{(S_1,J_1)2}\left(u_{(S_1,J_1)},r\right)\\
&\geq j_{(S_1,J_1)}\left(u_{(S_1,J_1)2},r\right),
\end{aligned}\tag{30}$$

and

$$j_{(S_1,J_1)}\left(v_{(S_1,J_1)1}, r\right) \geq j_{(S_1,J_1)}\left(v_{(S_1,J_1)2}, r\right), \tag{31}$$

we get $j_{(S_1,J_1)}(u_{(S_1,J_1)1}v_{(S_1,J_1)1}, r) \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)1}, r) \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)1}, r)\}$ for every $u_{(S_1,J_1)1}$ and $v_{(S_1,J_1)1}$ in R and $r \in Q$. We have (S_1, J_1) which is a (Q, L)-fuzzv soft subhemiring of $\mathfrak{R}$. □

In the following theorem, we use the composition operation of functions.

Suppose $U_{(S_1,J_1)}, V_{(S_2,J_2)}, Z_{(S_3,J_3)} \in R$.

We can define a composition mapping $F\colon U_{(S_1,J_1)} \longrightarrow V_{(S_2,J_2)}, G\colon V_{(S_2,J_2)} \longrightarrow Z_{(S_3,J_3)}, F.G = H\colon U_{(S_1,J_1)} \longrightarrow Z_{(S_3,J_3)}$

Also, using some properties of (Q, L)-FSSHR of a hemiring with the and composition operation of functions with the some more theorem are derived.

Theorem 9. *Let (S_1, J_1) be a (Q, L)-FSSHR of a hemiring R' and g is Q-isomorphism from a hemiring N onto R'. Then, $(S_1, J_1)\circ\psi$ is a (Q, L)-FSSHR of $\mathfrak{R}$.*

Proof. Let $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in $\mathfrak{R}$ and (S_1, J_1) be a Q-FSSHR of R'. Then,

$$\begin{aligned}\left(bj_{(S_1,J_1)^p}\right)(u+v,r) &= p(b)j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)}+v_{(S_1,J_1)}\right),r\right) \geq p(b)\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)},r\right), j_{(S_1,J_1)}\left(v_{(S_1,J_1)},r\right)\right\}\\ &= \left\{p(b)j_{(S_1,J_1)}\left(u_{(S_1,J_1)},r\right)\wedge p(b)j_{(S_1,J_1)}\left(v_{(S_1,J_1)},r\right)\right\}\\ &= \left\{\left(bj_{(S_1,J_1)^p}\right)\left(u_{(S_1,J_1)},r\right)\wedge\left(bj_{(S_1,J_1)^p}\right)\left(v_{(S_1,J_1)},r\right)\right\},\end{aligned} \tag{32}$$

for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in $\mathfrak{R}$ and

$$\begin{aligned}\left(bj_{(S_1,J_1)^p}\right)(u+v,r) &= p(b)j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)}+v_{(S_1,J_1)}\right),r\right) \geq p(b)\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)},r\right), j_{(S_1,J_1)}\left(v_{(S_1,J_1)},r\right)\right\}\\ &= \left\{p(b)j_{(S_1,J_1)}\left(u_{(S_1,J_1)},r\right)\wedge p(b)j_{(S_1,J_1)}\left(v_{(S_1,J_1)},r\right)\right\}\\ &= \left\{\left(bj_{(S_1,J_1)^p}\right)\left(u_{(S_1,J_1)},r\right)\wedge\left(bj_{(S_1,J_1)^p}\right)\left(v_{(S_1,J_1)},r\right)\right\},\end{aligned} \tag{33}$$

for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)} \neq 0$ in $\mathfrak{R}$. □

Theorem 10. *If (S_1, J_1) is a (Q, L)-FSSHR of $\mathfrak{R}$, then the pseudo-Q-fuzzy soft coset $(bj_{(S_1,J_1)^p})$ is a (Q, L)-FSSHR of $\mathfrak{R}$, for each $a \in \mathfrak{R}$.*

Proof. Let (S_1, J_1) be a (Q, L)-FSSHR of $\mathfrak{R}$, for each $u, v \in N$. Then,

$$\begin{aligned}\left(bj_{(S_1,J_1)^p}\right)(u+v,r) &= p(b)j_{(S_1,J_1)}\left(\left(u_{(S_1,J_1)}+v_{(S_1,J_1)}\right),r\right) \geq p(b)\left\{j_{(S_1,J_1)}\left(u_{(S_1,J_1)},r\right), j_{(S_1,J_1)}\left(v_{(S_1,J_1)},r\right)\right\}\\ &= \left\{p(b)j_{(S_1,J_1)}\left(u_{(S_1,J_1)},r\right)\wedge p(b)j_{(S_1,J_1)}\left(v_{(S_1,J_1)},r\right)\right\}\\ &= \left\{\left(bj_{(S_1,J_1)^p}\right)\left(u_{(S_1,J_1)},r\right)\wedge\left(bj_{(S_1,J_1)^p}\right)\left(v_{(S_1,J_1)},r\right)\right\}.\end{aligned} \tag{34}$$

Thus, $(b(j_{(S_1,J_1)^p}))(u+v,r) \geq \{(b(j_{(S_1,J_1)})p)(u_{(S_1,J_1)}, r) \wedge (b(j_{(S_1,J_1)}p))\ (v_{(S_1,J_1)}, r)\}$ for every $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in $\mathfrak{R}$ and r in Q and for every $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)} \neq 0$ in $\mathfrak{R}$.

$$\begin{aligned} b\Big(j_{(S_1,J_1)^p}\Big)\Big(u_{(S_1,J_1)}v_{(S_1,J_1)},r\Big) &= p(b)\Big(j_{(S_1,J_1)}\Big(u_{(S_1,J_1)}v_{(S_1,J_1)},r\Big)\Big) \geq p(b)\Big\{\Big(j_{(S_1,J_1)}\Big(u_{(S_1,J_1)},r\Big)\Big)\wedge j_{(S_1,J_1)}\Big(v_{(S_1,J_1)},r\Big)\Big\} \\ &= \Big\{p(b)\Big(j_{(S_1,J_1)}\Big(u_{(S_1,J_1)},r\Big)\Big)\wedge p(b)\Big(j_{(S_1,J_1)}\Big(v_{(S_1,J_1)},r\Big)\Big)\Big\} \\ &= \Big\{b\Big(j_{(S_1,J_1)^p}\Big)\Big(u_{(S_1,J_1)},r\Big)\wedge(b)\Big(\Big(j_{(S_1,J_1)^p}\Big)\Big(v_{(S_1,J_1)},r\Big)\Big)\Big\}. \end{aligned} \tag{35}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)} + v_{(S_1,J_1)}, r) \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}, r)\wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}, r)\}$ for all $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)} \neq 0$ in $\mathfrak{R}$. In this manner, $(b(j_{(S_1,J_1)^p}))$ is a (Q, L)-FSSHR subhemiring of a hemiring $\mathfrak{R}$. □

Theorem 11. *If (S_1, J_1) is a (Q, L)-FSSHR of a hemiring $\mathfrak{R}$, then $H = \{\langle (u_{(S_1,J_1)}, r), j(u_{(S_1,J_1)}, r)\rangle\colon 0 < j(u_{(S_1,J_1)}, r) \leq 1\}$ is either empty or a (Q, L)-fuzzy soft subhemiring of $\mathfrak{R}$.*

Proof. In the event that no segment holds the axioms, by then H is unfilled. If $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ satisfy this condition, then

$$\begin{aligned} &j_{(S_1,J_1)}\Big(u_{(S_1,J_1)}v_{(S_1,J_1)},r\Big), \\ &\geq \Big\{j_{(S_1,J_1)}\Big(u_{(S_1,J_1)},r\Big)\wedge j\Big(v_{(S_1,J_1)},r\Big)\Big\}, \\ &\geq \{0\wedge 0\}. \end{aligned} \tag{36}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)} + v_{(S_1,J_1)}, r) \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}, r)\wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}, r)\}$ for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)}$ in R and in Q.

$$\begin{aligned} &j_{(S_1,J_1)}\Big(u_{(S_1,J_1)}v_{(S_1,J_1)},r\Big), \\ &\geq \Big\{j_{(S_1,J_1)}\Big(u_{(S_1,J_1)},r\Big)\wedge j\Big(v_{(S_1,J_1)},r\Big)\Big\} \\ &\geq \{0\wedge 0\}. \end{aligned} \tag{37}$$

Thus, $j_{(S_1,J_1)}(u_{(S_1,J_1)}v_{(S_1,J_1)}), r \geq \{j_{(S_1,J_1)}(u_{(S_1,J_1)}, r), \wedge j_{(S_1,J_1)}(v_{(S_1,J_1)}, r)\}$ for each $u_{(S_1,J_1)}$ and $v_{(S_1,J_1)} \neq e$ in $\mathfrak{R}$. In this way, H is either empty or a (Q, L)-fuzzy soft subhemiring of $\mathfrak{R}$. □

5. Conclusion

The main idea of this research work is to briefly explain and establish the results, the properties, and some theorems on the morphism of (Q, L)-FSSHR of a hemiring. This work can be extended in future to ideals of (Q, L)-fuzzy soft subhermiring and inter-valued (Q, L)-FSSHR of a hemiring. We trust that this work will have a profound effect on the forthcoming exploration in this field and other soft algebraic investigations to open up new horizons of premium and advancement.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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