Research Article

Numerical Solutions for Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equations

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Analyzing the stability of many control systems required solving a couple of crisp Sylvester matrix equations (CSMEs) simultaneously. However, there are some situations in which the crisp Sylvester matrix equations are not well equipped to deal with the uncertainty problem during the stability analysis of control systems. This paper constructs analytical and numerical methods for solving a couple of trapezoidal fully fuzzy Sylvester matrix equations (CTrFFSMEs) to overcome the drawbacks of the existing crisp methods. In developing these new methods, fuzzy arithmetic multiplication is applied on the CTrFFSME to transform it into an equivalent system of four CSMEs. Then, the fuzzy solution is obtained analytically by the fuzzy matrix vectorization method and numerically by gradient and least square methods. The analytical method can obtain the exact solution; however, it is limited to small-sized systems while the numerical methods can approximate the solution for large dimensional systems up to 100×100 with a very small error bound for any initial value. In addition, the proposed methods are applied to other fuzzy systems such as Sylvester and Lyapunov matrix equations. The proposed methods are illustrated by solving numerical examples with different size systems.

1. Introduction

The Sylvester matrix equation (SME) has massive applications in control theory [1, 2], system theory [3], optima control [4], linear descriptor systems [5], sensitivity analysis [6], perturbation theory [7], system design [8], theory of orbits [9], design and analysis of linear control systems [10], reduction of large-scale dynamical systems [11], restoration of noisy images [12, 13], medical imaging data acquisition and model reduction [14], and stochastic control, image processing, and filtering [13]. CSME must be solved simultaneously in many applications, such as analyzing the stability of control systems [15]. Researchers for many years have proposed many analytical and numerical methods for solving CSME with crisp numbers.

Although analytical solutions, which can be computed using Vec-operator and Kronecker product, are important,

the computational efforts rapidly increase with the dimensions of the matrices to be solved. For example, it required getting the inverse of $mn \times mn$ matrix for a system of size $m \times n$ which leads to computation complexity. Therefore, this method is limited to systems with small coefficients only. In addition, for some applications such as stability analysis, it is often not necessary to compute analytical solutions; approximate solutions or bounds of solutions are sufficient. Also, if the parameters in system matrices are uncertain, it is not possible to obtain analytical solutions for robust stability results [16, 17]. Alternative ways exist which transform the matrix equations into forms for which solutions may be readily computed, such as the Jordan canonical form [18] and Hessenberg–Schur form [19]. However, these methods are computationally expensive for large systems. In the field of matrix algebra and system identification, iterative algorithms for large systems have

received much attention [20]. Starke and Niethammer [21] presented an iterative method for solutions of the SME by using the SOR technique while Jonsson and Kägström proposed recursive block algorithms for solving the coupled Sylvester matrix equations [22]. Kägström derived an approximate solution of the coupled Sylvester equation [23].

Many authors studied the least square solutions of CSME [24–31] while authors in [32] discussed the solvability conditions and general solutions for mixed Sylvester equations. Recently, a relaxed gradient-based algorithm for solving generalized CSME was introduced by [33] in addition to the conjugate gradient least square algorithm [34] and gradient-based approach [35] and the BCR algorithm proposed by [36–38]. However, in many applications, some of the system parameters are represented by fuzzy numbers rather than crisp numbers due to uncertainty problems such as conflicting requirements during the system process and the distraction of any elements and noise. When all parameters of the CSME are in the fuzzy form, then it is called the coupled fully fuzzy Sylvester matrix equation (CFFSME).

Definition 1. The couple fully fuzzy matrix equation can be written as

$$\begin{cases} \widetilde{A}\widetilde{X} + \widetilde{Y}\widetilde{B} = \widetilde{E}, \\ \widetilde{C}\widetilde{X} + \widetilde{Y}\widetilde{D} = \widetilde{F}, \end{cases} \tag{1}$$

where $\widetilde{A} = (\widetilde{a}_{ij})_{m\times m}$, $\widetilde{X} = (\widetilde{x}_{ij})_{m\times n}$, $\widetilde{Y} = (\widetilde{y}_{ij})_{m\times n}$, $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{C} = (\widetilde{c}_{ij})_{m\times m}$, $\widetilde{D} = (\widetilde{d}_{ij})_{n\times n}$, $\widetilde{E} = (\widetilde{e}_{ij})_{m\times n}$, and $\widetilde{F} = (\widetilde{f}_{ij})_{m\times n}$.

Equation (1) is of interest in many different applications. However, until now, there are fewer studies for the solution of this equation. In the fuzzy literature, most of the solution methods are proposed for its special cases, such as fully fuzzy Sylvester matrix equations (FFSMEs), fully fuzzy matrix equations (FFMEs), fully fuzzy linear systems (FFLSs), and fuzzy linear system (FLS).

The first approach of solving FLS was accomplished by [39], which proposed a general model for solving a FLS by transferring FLS to a linear system. Allahviranloo et al. [40] proposed a method to obtain symmetric solutions of the FLS based on a 1-cut expansion. They extended the same method in [41] to obtain symmetric solutions of the FFLS. Sufficient conditions needed for getting positive solutions of the FFLS were discussed by Malkawi and his colleagues [42, 43]. Otadi and Mosleh extended the FFLS to FFME in [44]. Several analytical methods have been proposed for solving the triangular fully fuzzy Sylvester matrix equation (TFFSME) [45–47]. However, these methods are restricted only for positive triangular fuzzy numbers and require a long multiplication process and consequently long computational timing. Consequently, researchers limit the sizes of the TFFSME to $n = 2$ or 3. Recently, authors in [48] considered solution of the trapezoidal fully fuzzy Sylvester matrix equation (TrFFSME) by transforming the TrFFSME to a system of crisp linear matrix equations where the positive and negative fuzzy solutions are obtained by applying Kronecker product and Vec-operator method. However,

these algorithms are not suitable for TrFFSME with large sizes.

In addition, a few studies have been conducted for solving a pair of fuzzy matrix equations. Sadeghi, Abbasbandy, and Abbasnejad [49] proposed a method for solving a pair of fuzzy matrix equations in the form as follows:

$$\begin{cases} A\widetilde{X} + \widetilde{X}B = \widetilde{C}, \\ D\widetilde{X}E = \widetilde{F}. \end{cases} \tag{2}$$

Moreover, Daud, Ahmad, and Malkawi [50–53] proposed analytical methods for solving FFSME and a pair of fully fuzzy matrix equations (PFFME) in the form as follows:

$$\begin{cases} \widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{B} = \widetilde{C}, \\ \widetilde{D}\widetilde{X}\widetilde{E} = \widetilde{F}. \end{cases} \tag{3}$$

In that study, a direct method was proposed to solve the PFFME by applying the Kronecker product and Vec-operator. However, both methods required a long multiplication process and were consequently limited to small-sized systems. In general, the existing methods proposed for solving PFFME, TFFSME, and TrFFSME are based on Kronecker product and Vec-operator and therefore limited to small systems (2×2) or (3×3). Only a few researchers considered fuzzy systems with sizes 10×10 [43]. Fuzzy systems with sizes greater than 10×10 are not investigated till now. In addition, the CFFSME is not investigated in the fuzzy literature.

To deal with this shortcoming, in this paper, three different methods are proposed for solving CFFSME with trapezoidal fuzzy numbers (CTrFFSME) and its special cases. The fuzzy solution to the CTrFFSME is obtained analytically by the fuzzy matrix vectorization method and numerically by gradient and least square methods. The fuzzy matrix vectorization method can obtain the exact solution; however, it is restricted to small systems. Therefore, it is important to develop mathematical models and numerical procedures that solve the CFFSME and special cases with big sizes while the numerical methods can obtain the solution for large dimensional systems up to 100×100 with a very small error bound compared with the existing numerical approaches, which were applied up to 10×10 fuzzy systems [54–63]. Moreover, the proposed methods can also be applied to other fuzzy systems such as Sylvester and Lyapunov matrix equations with triangular fuzzy numbers (TFNs) and trapezoidal (TrFNs) fuzzy numbers.

To illustrate the effectiveness of the proposed methods for solving the CTrFFSME in equation (1), we consider various sizes of fuzzy systems, namely, small 2×2 and large 100×100. In addition, we compare the performance of the proposed methods by calculating the number of iterations (k), convergence factor (α), error $\delta^l(k)$, error bound (ε), convergence rate, CPU time, real-time, and memory usage. In addition to the graphical representation of the relative error $\delta^l(k)$ when the number of iterations (k) increases.

This paper is organized as follows. Section 2 introduces preliminary arithmetic operations of trapezoidal fuzzy numbers. In Section 3, three proposed methods for solving

CTrFFSME are developed along with a presentation of its algorithms. In Section 4, numerical examples are presented to illustrate the proposed methods. Section 5 is dedicated to the conclusion.

2. Preliminaries

The following are the basic definitions and results related to TrFNs in fuzzy theory [64–66] and matrix theory [67–69].

Definition 2. Let X be a universal set. Then, the fuzzy subset $\widetilde{A}$ of X is defined by its membership function $\mu_{\widetilde{A}}: X \longrightarrow [0, 1]$ which assigns to each element $x \in X$ a real number $\mu_{\widetilde{A}}(x)$ in the interval $[0, 1]$, where the function value of $\mu_{\widetilde{A}}(x)$ represents the grade of membership of x in $\widetilde{A}$. A fuzzy set $\widetilde{A}$ is written as $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)), x \in X, \mu_{\widetilde{A}}(x) \in [0, 1]\}$.

Definition 3. A fuzzy set $\widetilde{A}$, defined on the universal set of real number R, is said to be a fuzzy number if its membership function has the following characteristics:

(i) $\widetilde{A}$ is convex, i.e.,

$$\mu_{\widetilde{A}}(\lambda x_1 + (1 - \lambda))x_2 \geq \min\left(\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(x)\right) \tag{4}$$
$$\forall x_1, x_2 \in R, \forall \lambda \in [0, 1].$$

(ii) $\widetilde{A}$ is normal, i.e.,

$$\exists x_0 \in R \text{ such that } \mu_{\widetilde{A}}(x_0) = 1.$$

(iii) $\mu_{\widetilde{A}}$ is piecewise continuous.

Definition 4. A fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is a TrFN in the general form if its membership function is as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x < a_1, \\ \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ \\ 1, & a_2 \leq x \leq a_3, \\ \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ \\ 0, & x > a_4. \end{cases} \tag{5}$$

In Figure 1, the TrFN in general form is presented.

Definition 5. The sign of the TrFN $\widetilde{A} = (a_1, a_2, a_3, a_4)$ can be classified as follows:

$\widetilde{A}$ is positive (negative) iff $a_1 \geq 0, (a_4 \leq 0)$
$\widetilde{A}$ is zero iff $(a_1, a_2, a_3 \text{ and } a_4 = 0)$
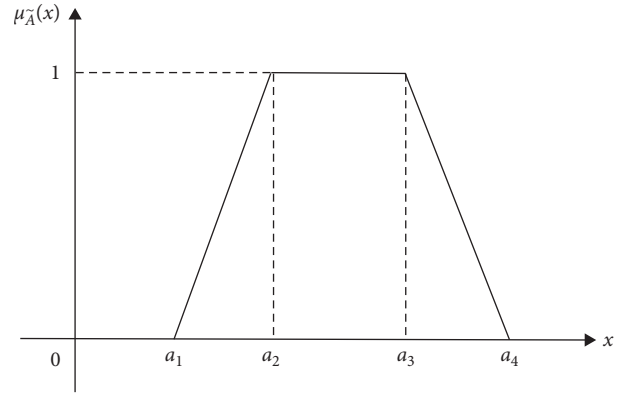$\widetilde{A}$ is near zero iff $a_1 \leq 0 \leq a_4$

FIGURE 1: Representation of TrFN in a general form.

Definition 6. Operations of TrFNs.

The arithmetic operations of TrFNs are presented as follows: let $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ be two TrFNs, then

(i) Addition:

$$\begin{aligned} \widetilde{A} + \widetilde{B} &= (a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4). \end{aligned} \tag{6}$$

(ii) Subtraction:

$$\begin{aligned} \widetilde{A} - \widetilde{B} &= (a_1, a_2, a_3, a_4) - (b_1, b_2, b_3, b_4) \\ &= (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1). \end{aligned} \tag{7}$$

(iii) Symmetric image:

$$-\widetilde{A} = (-a_4, -a_3, -a_2, -a_1). \tag{8}$$

(iv) Scalar multiplication: let $\lambda \in \mathbb{R}$, then

$$\lambda \otimes (a_1, a_2, a_3, a_4) = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4), & \lambda \geq 0, \\ (\lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1), & \lambda < 0. \end{cases} \tag{9}$$

(v) Multiplication: the multiplication between fuzzy numbers is neither commutative nor associative. Thus, TrFNs multiplication operations can be classified as follows:

Case I. If $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ be two arbitrary TrFNs, then

$$\widetilde{A}\widetilde{B} = (a, h, m, d), \tag{10}$$

where

$$\begin{aligned} a &= \min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4), \\ h &= \min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3), \\ m &= \max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3), \\ d &= \max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4). \end{aligned} \tag{11}$$

Case II. If $\widetilde{A}, \widetilde{B} > 0$, then

$$\widetilde{A}\widetilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4). \tag{12}$$

Case III. If $\widetilde{A}, \widetilde{B} < 0$, then

$$\widetilde{A}\widetilde{B} = (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1). \tag{13}$$

Case IV. If $\widetilde{A} > 0$ and $\widetilde{B} < 0$, then

$$\widetilde{A}\widetilde{B} = (a_4 b_1, a_3 b_2, a_2 b_3, a_1 b_4). \tag{14}$$

Case V. If $\widetilde{A} < 0$ and $\widetilde{B} > 0$, then

$$\widetilde{A}\widetilde{B} = (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1). \tag{15}$$

(vi) Equality: the fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ are equal iff

$$\begin{aligned} a_1 &= b_1, \\ a_2 &= b_2, \\ a_3 &= b_3, \\ a_4 &= b_4. \end{aligned} \tag{16}$$

Definition 7. A matrix $\widetilde{A} = (\widetilde{a}_{ij})_{m \times n}$ is called a trapezoidal fuzzy matrix if each element of $\widetilde{A}$ is a TrFN.

Definition 8. A fuzzy matrix $\widetilde{A}$ will be as follows:

(i) Positive (negative) and denoted by $\widetilde{A} > 0$, ($\widetilde{A} < 0$) if each element of $\widetilde{A}$ is positive (negative) TrFN

(ii) Nonnegative (nonpositive) and denoted by $\widetilde{A} \geq 0$, ($\widetilde{A} \leq 0$) if each element of $\widetilde{A}$ is nonnegative (non-positive) TrFNs

(iii) Arbitrary if at least one element of $\widetilde{A}$ is near zero TrFNs

In Remark 1, the positive trapezoidal fuzzy matrix is written as four separated crisp matrices.

Remark 1. The positive trapezoidal fuzzy matrices $\widetilde{A} = (\widetilde{a}_{ij})_{m \times m}$ can be written as four separated crisp matrices as follows:

$$\widetilde{A} = (\widetilde{a}_{ij})_{m \times m} = (A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}), \tag{17}$$

where $A^{(1)}, A^{(2)}, A^{(3)},$ and $A^{(4)}$ are four crisp matrices sized $m \times m$.

In Remark 2, the multiplication of positive trapezoidal fuzzy matrices is introduced.

Remark 2. The product of the two positive trapezoidal fuzzy matrices $\widetilde{A} = (\widetilde{a}_{ij})_{m \times m} = (A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)})$, $\forall 1 \leq i, j \leq m$, and $\widetilde{X} = (\widetilde{x}_{pq})_{m \times n} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$, $\forall 1 \leq p \leq m, 1 \leq q \leq n$, can be represented as follows:

$$\widetilde{A}\widetilde{X} = \sum_{k=1}^{n} \widetilde{a}_{ik} \widetilde{x}_{kq}, \tag{18}$$

where $\widetilde{a}_{ik} \widetilde{x}_{kq}$ represent the multiplication of the fuzzy k − th number of i − row of matrix $\widetilde{A}$ with q − column of matrix $\widetilde{X}$.

In addition, this product is equivalent to the product of the following crisp matrices:

$$\widetilde{A}\widetilde{X} = (A^{(1)} X^{(1)}, A^{(2)} X^{(2)}, A^{(3)} X^{(3)}, A^{(4)} X^{(4)}). \tag{19}$$

Definition 9. The Vec-operator generates a column vector from a matrix A by stacking the column vectors of $A =$

$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ as $\text{Vec}(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix}$. In addition, if

$A = \text{Vec}^{-1} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix}$, then $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

Theorem 1 (see [70]). *If the crisp linear matrix equation $AX = E$ has a unique solution X, then the gradient iterative solution $X(k)$ given by $X(k) = X(k-1) + \alpha \cdot (A)^T (E - AX(k-1))$ converges to X or $\lim_{k \to \infty} (X(k)) = X$ for any initial value $X(0)$.*

Theorem 2 (see [71]). *If the crisp linear matrix equation $AX = E$ has a unique solution X, then the gradient iterative solution $X(k)$ given by $X(k) = X(k-1) + \alpha \cdot ((A)^T \cdot A)^{-1} (A)^T (E - AX(k-1))$ converges to X or $\lim_{k \to \infty} (X(k)) = X$ for any initial value $X(0)$.*

In Section 3, the solution to the CTrFFSME in equation (1) is discussed.

3. The Solution of Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equation

In this section, the solution to the positive CTrFFSME is considered. To get the solution, the positive CTrFFSME is converted to an equivalent system of CSME, and then the solution to this system of CSME is obtained by three different methods. In Section 3.1, the positive CTrFFSME in equation (1) is converted to an equivalent system CSME based on the arithmetic multiplication operation in Definition 6.

3.1. Systems of CSME.

In this section, the positive CTrFFSME in equation (1) is converted to four systems of CSME. The next theorem shows that the CTrFFSME can be written as four systems of CSME.

Theorem 3. *Fundamental theorem of the coupled trapezoidal fully fuzzy Sylvester matrix equation.*

In the CTrFFSME in equation (1), if $\widetilde{A} = (\widetilde{a}_{ij})_{m \times m} = (A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}) > 0$ and $\widetilde{C} = (\widetilde{c}_{ij})_{m \times m} = (C^{(1)}, C^{(2)}, C^{(3)}, C^{(4)}) > 0$, $\forall 1 \leq i, j \leq m$, $\widetilde{B} = (\widetilde{b}_{ij})_{n \times n} = (B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}) > 0$ and $\widetilde{D} = (\widetilde{d}_{ij})_{n \times n} = (D^{(1)}, D^{(2)}, D^{(3)}, D^{(4)}) > 0$, $\forall 1 \leq i, j \leq n$, and $\widetilde{X} = (\widetilde{x}_{ij})_{m \times n} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}) > 0$, $\widetilde{Y} = (\widetilde{y}_{ij})_{m \times n} = (Y^{(1)}, Y^{(2)}, Y^{(3)}, Y^{(4)}) > 0$, $\widetilde{E} = (\widetilde{e}_{ij})_{m \times n} = (E^{(1)}, E^{(2)}, E^{(3)}, E^{(4)}) > 0$,

and $\widetilde{F} = (\widetilde{f}_{ij})_{m \times n} =$ $(F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)}) > 0,$ $\forall 1 \le i \le m, 1 \le j \le n,$ *then the positive CTrFFSME is equivalent to the following systems of CSME:*

$$
\begin{cases}
\sum_{k=1}^{n} \widetilde{a}_{ik} \widetilde{x}_{kj} + \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{b}_{kj} = \widetilde{e}_{ij}, & \forall 1 \le i \le m, 1 \le j \le n, \\
\sum_{k=1}^{n} \widetilde{c}_{ik} \widetilde{x}_{kj} + \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{d}_{kj} = \widetilde{f}_{ij}, & \forall 1 \le i \le m, \ 1 \le j \le n.
\end{cases}
$$

$$(20)$$

Proof. Since $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, $\widetilde{E}$, $\widetilde{F}$, $\widetilde{X}$, *and* $\widetilde{Y}$ in equation (1) are positive trapezoidal fully fuzzy matrices, respectively, equation (12) in Definition 6 can be used to find $\widetilde{A}\widetilde{X}$, $\widetilde{Y}\widetilde{B}$, $\widetilde{C}\widetilde{X}$, *and* $\widetilde{Y}\widetilde{D}$ in equation (1) as follows:

$$
\widetilde{A}\widetilde{X} = \sum_{k=1}^{n} \widetilde{a}_{ik} \widetilde{x}_{kj} = \left(A^{(1)}X^{(1)}, A^{(2)}X^{(2)}, A^{(3)}X^{(3)}, A^{(4)}X^{(4)} \right),
$$

$$
\widetilde{Y}\widetilde{B} = \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{b}_{kj} = \left(Y^{(1)}B^{(1)}, Y^{(2)}B^{(2)}, Y^{(3)}B^{(3)}, Y^{(4)}B^{(4)} \right),
$$

$$
\widetilde{C}\widetilde{X} = \sum_{k=1}^{n} \widetilde{c}_{ik} \widetilde{x}_{kj} = \left(C^{(1)}X^{(1)}, C^{(2)}X^{(2)}, C^{(3)}X^{(3)}, C^{(4)}X^{(4)} \right),
$$

$$
\widetilde{Y}\widetilde{D} = \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{d}_{kj} = \left(Y^{(1)}D^{(1)}, Y^{(2)}D^{(2)}, Y^{(3)}D^{(3)}, Y^{(4)}D^{(4)} \right),
$$

$$(21)$$

s.t all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Combining $\widetilde{A}\widetilde{X}$ and $\widetilde{Y}\widetilde{B}$, $\widetilde{C}\widetilde{X}$ and $\widetilde{Y}\widetilde{D}$, we get

$$
\begin{cases}
\sum_{k=1}^{n} \widetilde{a}_{ik} \widetilde{x}_{kj} + \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{b}_{kj}, & \forall 1 \le i \le m, 1 \le j \le n, \\
\sum_{k=1}^{n} \widetilde{d}_{ik} \widetilde{x}_{kj} + \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{e}_{kj}, & \forall 1 \le i \le m, 1 \le j \le n.
\end{cases}
$$

$$(22)$$

By Definition 1, the positive CTrFFSME is equivalent to the following systems of CSME:

$$
\begin{cases}
\sum_{k=1}^{n} \widetilde{a}_{ik} \widetilde{x}_{kj} + \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{b}_{kj} = \widetilde{e}_{ij}, & \forall 1 \le i \le m, 1 \le j \le n, \\
\sum_{k=1}^{n} \widetilde{c}_{ik} \widetilde{x}_{kj} + \sum_{k=1}^{n} \widetilde{y}_{ik} \widetilde{d}_{kj} = \widetilde{f}_{ij}, & \forall 1 \le i \le m, 1 \le j \le n.
\end{cases}
$$

$$(23) \qquad \square$$

Remark 3. The equivalent systems of CSME in equation (20) to the CTrFFSME in equation (1) can also be written as follows:

$$
\begin{cases}
\begin{cases}
A^{(1)}X^{(1)} + Y^{(1)}B^{(1)} = E^{(1)}, \\
C^{(1)}X^{(1)} + Y^{(1)}D^{(1)} = F^{(1)},
\end{cases} \\
\begin{cases}
A^{(2)}X^{(2)} + Y^{(2)}B^{(2)} = E^{(2)}, \\
C^{(2)}X^{(2)} + Y^{(2)}D^{(2)} = F^{(2)},
\end{cases} \\
\begin{cases}
A^{(3)}X^{(3)} + Y^{(3)}B^{(3)} = E^{(3)}, \\
C^{(3)}X^{(3)} + Y^{(3)}D^{(3)} = F^{(3)},
\end{cases} \\
\begin{cases}
A^{(4)}X^{(4)} + Y^{(4)}B^{(4)} = E^{(4)}, \\
C^{(4)}X^{(4)} + Y^{(4)}D^{(4)} = F^{(4)}.
\end{cases}
\end{cases}
$$

$$(24)$$

To solve the CTrFFSME in equation (1), we consider the corresponding systems of CSME in equation (24).

Remark 4. The nature of the solutions of the CTrFFSME in equation (1) depends upon the nature of the solutions of the system of CSME in equation (24), which may be unique, trivial, or infinitely many solutions [72]; that is, the CTrFFSME may yield no solution, unique solution, or infinitely many solutions.

Since the systems of CSME obtained in equation (24) are similar, in Remark 5, the systems of CSME are represented in a more general form.

Remark 5. Based on equation (24), the CTrFFSME in equation (1) can be written as follows: for $1 \le l \le 4$, we have

$$
\begin{cases}
A^{(l)}X^{(l)} + Y^{(l)}B^{(l)} = E^{(l)}, \\
C^{(l)}X^{(l)} + Y^{(l)}D^{(l)} = F^{(l)}.
\end{cases}
$$

$$(25)$$

Based on Theorem 3, the CTrFFSME in equation (1) is transferred to an equivalent linear system of CSME in crisp form, which can be solved analytically and numerically by many classical methods in linear algebra like the matrix inversion method or Gaussian method. The main advantage of the analytical methods is that the exact fuzzy solution to the CTrFFSME in equation (1) can be obtained.

However, since most of the analytical methods are based on Vec-operator and Kronecker products, the system's size in equation (1) is limited to small sizes ($n < 10$). For CTrFFSME with large dimensions ($n \ge 10$), iterative algorithms to find an approximated solution are more practical [20]. Therefore, in the following section, three different methods are proposed for solving the CTrFFSME in equation (1). The first method aims to find the exact fuzzy solution by extending the concept of matrix vectorization and the Kronecker product. In addition, two iterative methods

are applied to approximate the fuzzy solution of the CTrFFSME with large dimensions. In Definition 10, the positive trapezoidal fuzzy solution is defined.

Definition 10. Trapezoidal positive fuzzy solution matrix in a general form.

Let $\widetilde{X} = (\widetilde{x}_{ij})_{m \times n} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$ be a trapezoidal fuzzy matrix. If $\widetilde{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$ is an exact solution of equation (20) such that $0 < X^{(1)} \leq X^{(2)} \leq X^{(3)} \leq X^{(4)}$, $\forall 1 \leq i, j \leq m, n$, then $\widetilde{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$ is called a positive fuzzy solution of equation (1).

3.2. Proposed Analytical Method for Solving CTrFFSME. In the following method, the analytical solution for the CTrFFSME in equation (1) is obtained by extending the method of the fuzzy matrix vectorization method (FMVM) proposed by [48].

3.2.1. Fuzzy Matrix Vectorization Method (FMVM) for Solving CTrFFSME. In this method, the CTrFFSME in equation (1) is solved analytically using Vec-Operator and Kronecker product. The following steps summarize the methods:

Step 1. Decompose $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, \widetilde{F}, \widetilde{X}$, and $\widetilde{Y}$ into $A^{(l)}$, $B^{(l)}$, $C^{(l)}$, $D^{(l)}$, $E^{(l)}$, $F^{(l)}$, $X^{(l)}$, and $Y^{(l)}$ where $l = 1, 2, 3, 4$, respectively, and convert the CTrFFSME in equation (1) to the system of linear matrix equations in equation (24) using Theorem 3.

Step 2. Applying Vec-operator and Kronecker product on equation (24) gives the following:

$$\begin{cases} \begin{pmatrix} I_n \otimes A^{(1)} & \left(B^{(1)}\right)^T \otimes I_m \\ I_n \otimes C^{(1)} & \left(D^{(1)}\right)^T \otimes I_m \end{pmatrix} \begin{pmatrix} \mathrm{vec}\left(X^{(1)}\right) \\ \mathrm{vec}\left(Y^{(1)}\right) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}\left(E^{(1)}\right) \\ \mathrm{vec}\left(F^{(1)}\right) \end{pmatrix}, \\ \begin{pmatrix} I_n \otimes A^{(2)} & \left(B^{(2)}\right)^T \otimes I_m \\ I_n \otimes C^{(2)} & \left(D^{(2)}\right)^T \otimes I_m \end{pmatrix} \begin{pmatrix} \mathrm{vec}\left(X^{(2)}\right) \\ \mathrm{vec}\left(Y^{(2)}\right) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}\left(E^{(2)}\right) \\ \mathrm{vec}\left(F^{(2)}\right) \end{pmatrix}, \\ \begin{pmatrix} I_n \otimes A^{(3)} & \left(B^{(3)}\right)^T \otimes I_m \\ I_n \otimes C^{(3)} & \left(D^{(3)}\right)^T \otimes I_m \end{pmatrix} \begin{pmatrix} \mathrm{vec}\left(X^{(3)}\right) \\ \mathrm{vec}\left(Y^{(3)}\right) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}\left(E^{(3)}\right) \\ \mathrm{vec}\left(F^{(3)}\right) \end{pmatrix}, \\ \begin{pmatrix} I_n \otimes A^{(4)} & \left(B^{(4)}\right)^T \otimes I_m \\ I_n \otimes C^{(4)} & \left(D^{(4)}\right)^T \otimes I_m \end{pmatrix} \begin{pmatrix} \mathrm{vec}\left(X^{(4)}\right) \\ \mathrm{vec}\left(Y^{(4)}\right) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}\left(E^{(4)}\right) \\ \mathrm{vec}\left(F^{(4)}\right) \end{pmatrix}. \end{cases} \tag{26}$$

Step 3. By equation (25), the systems of coupled linear matrix equations in equation (27) can be combined and written as follows:

$$PQ = U, \tag{27}$$

where

$$P = \begin{pmatrix} \begin{pmatrix} I_n \otimes A^{(1)} & \left(B^{(1)}\right)^T \otimes I_m \\ I_n \otimes C^{(1)} & \left(D^{(1)}\right)^T \otimes I_m \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} I_n \otimes A^{(2)} & \left(B^{(2)}\right)^T \otimes I_m \\ I_n \otimes C^{(2)} & \left(D^{(2)}\right)^T \otimes I_m \end{pmatrix} & 0 & \vdots \\ \vdots & 0 & \begin{pmatrix} I_n \otimes A^{(3)} & \left(B^{(3)}\right)^T \otimes I_m \\ I_n \otimes C^{(3)} & \left(D^{(3)}\right)^T \otimes I_m \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} I_n \otimes A^{(4)} & \left(B^{(4)}\right)^T \otimes I_m \\ I_n \otimes C^{(4)} & \left(D^{(4)}\right)^T \otimes I_m \end{pmatrix} \end{pmatrix},$$

$$Q = \begin{pmatrix} \begin{pmatrix} \mathrm{vec}\left(X^{(1)}\right) \\ \mathrm{vec}\left(Y^{(1)}\right) \end{pmatrix}, \\ \begin{pmatrix} \mathrm{vec}\left(X^{(2)}\right) \\ \mathrm{vec}\left(Y^{(2)}\right) \end{pmatrix}, \\ \begin{pmatrix} \mathrm{vec}\left(X^{(3)}\right) \\ \mathrm{vec}\left(Y^{(3)}\right) \end{pmatrix}, \\ \begin{pmatrix} \mathrm{vec}\left(X^{(4)}\right) \\ \mathrm{vec}\left(Y^{(4)}\right) \end{pmatrix}, \end{pmatrix}$$

$$U = \begin{pmatrix} \begin{pmatrix} \mathrm{vec}\left(E^{(1)}\right) \\ \mathrm{vec}\left(F^{(1)}\right) \end{pmatrix}, \\ \begin{pmatrix} \mathrm{vec}\left(E^{(2)}\right) \\ \mathrm{vec}\left(F^{(2)}\right) \end{pmatrix}, \\ \begin{pmatrix} \mathrm{vec}\left(E^{(3)}\right) \\ \mathrm{vec}\left(F^{(3)}\right) \end{pmatrix}, \\ \begin{pmatrix} \mathrm{vec}\left(E^{(4)}\right) \\ \mathrm{vec}\left(F^{(4)}\right) \end{pmatrix}, \end{pmatrix}$$

(28)

where the crisp matrix P is $8\,mn \times 8\,mn$, and the crisp vectors Q and U are $8\,mn \times 1$.

Step 4. Multiplying the system of equation in equation (27) by matrix multiplicative inverse gives

$$Q = P^{-1}U. \tag{29}$$

Step 5: By Definitions 9 and 10, the obtained solution in equation (29) can be written as follows:

$$\begin{cases} \widetilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)} & \cdots & x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)} \end{pmatrix}, \\ \widetilde{Y} = \begin{pmatrix} \left(y_{11}^{(1)}, y_{11}^{(2)}, y_{11}^{(3)}, y_{11}^{(4)}\right) & \cdots & \left(y_{1n}^{(1)}, y_{1n}^{(2)}, y_{1n}^{(3)}, y_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ y_{m1}^{(1)}, y_{m1}^{(2)}, y_{m1}^{(3)}, y_{m1}^{(4)} & \cdots & y_{mn}^{(1)}, y_{mn}^{(2)}, y_{mn}^{(3)}, y_{mn}^{(4)} \end{pmatrix}. \end{cases}$$

(30)

In the following remark, the system of equations in equation (26) is written in a general form.

Remark 6. Based on equations (25) and (27), the CTrFFSME in equation (1) is equivalent to the following system: for $1 \leq l \leq 4$, we have

$$
\begin{pmatrix} I_n \otimes A^{(l)} & \left(B^{(l)} \right)^T \otimes I_m \\ I_n \otimes C^{(l)} & \left(D^{(l)} \right)^T \otimes I_m \end{pmatrix} \begin{pmatrix} \mathrm{vec}\left(X^{(l)} \right) \\ \mathrm{vec}\left(Y^{(l)} \right) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}\left(E^{(l)} \right) \\ \mathrm{vec}\left(F^{(l)} \right) \end{pmatrix}.
$$

$$(31)$$

The sufficient conditions to have a unique fuzzy solution to the CTrFFSME are discussed in Corollary 1.

Corollary 1. *For $1 \leq l \leq 4$, the CTrFFSME in equation (1) has a unique solution if and only if the matrix $\begin{pmatrix} I_n \otimes A^{(l)} & \left(B^{(l)} \right)^T \otimes I_m \\ I_n \otimes C^{(l)} & \left(D^{(l)} \right)^T \otimes I_m \end{pmatrix}$ is nonsingular. Then, this solution is obtained by*

$$
\begin{pmatrix} \mathrm{vec}\left(X^{(l)} \right) \\ \mathrm{vec}\left(Y^{(l)} \right) \end{pmatrix} = \begin{pmatrix} I_n \otimes A^{(l)} & \left(B^{(l)} \right)^T \otimes I_m \\ I_n \otimes C^{(l)} & \left(D^{(l)} \right)^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{vec}\left(E^{(l)} \right) \\ \mathrm{vec}\left(F^{(l)} \right) \end{pmatrix},
$$

$$(32)$$

and can be written as

$$
\begin{cases} \widetilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)} \right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)} & \cdots & x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)} \end{pmatrix}, \\ \widetilde{Y} = \begin{pmatrix} \left(y_{11}^{(1)}, y_{11}^{(2)}, y_{11}^{(3)}, y_{11}^{(4)} \right) & \cdots & \left(y_{1n}^{(1)}, y_{1n}^{(2)}, y_{1n}^{(3)}, y_{1n}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ y_{m1}^{(1)}, y_{m1}^{(2)}, y_{m1}^{(3)}, y_{m1}^{(4)} & \cdots & y_{mn}^{(1)}, y_{mn}^{(2)}, y_{mn}^{(3)}, y_{mn}^{(4)} \end{pmatrix}. \end{cases}
$$

$$(33)$$

3.2.2. Feasibility of the Positive Fuzzy Solution to the CTrFFSME. The obtained positive fuzzy solution in equation (30) to the CTrFFSME in equation (1) is feasible (strong fuzzy solution) if the following conditions are satisfied: for $1 \leq l \leq 4$,

   (i) $X^{(l)} > 0, \forall \{1 \leq i, j \leq m, n\}$
   (ii) $Y^{(l)} > 0, \forall \{1 \leq i, j \leq m, n\}$
   (iii) $X^{(1)} \leq X^{(2)} \leq X^{(3)} \leq X^{(4)}, \forall \{1 \leq i, j \leq m, n\}$
   (iv) $Y^{(1)} \leq Y^{(2)} \leq Y^{(3)} \leq Y^{(4)}, \forall \{1 \leq i, j \leq m, n\}$

Remark 7. If the solution fails to satisfy the feasibility conditions, it is infeasible (weak fuzzy solution).

The algorithm of the FMVM for solving the CTrFFMSE in equation (1) is given in the following five steps.

$$
\begin{cases} \widetilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)} \right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)} & \cdots & x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)} \end{pmatrix}, \\ \widetilde{Y} = \begin{pmatrix} \left(y_{11}^{(1)}, y_{11}^{(2)}, y_{11}^{(3)}, y_{11}^{(4)} \right) & \cdots & \left(y_{1n}^{(1)}, y_{1n}^{(2)}, y_{1n}^{(3)}, y_{1n}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ y_{m1}^{(1)}, y_{m1}^{(2)}, y_{m1}^{(3)}, y_{m1}^{(4)} & \cdots & y_{mn}^{(1)}, y_{mn}^{(2)}, y_{mn}^{(3)}, y_{mn}^{(4)} \end{pmatrix}. \end{cases}
$$

$$(34)$$

In Section 3.3, the positive solution to the CTrFFSME is approximated numerically by extending the GI method in Theorem 1 and LSI method in Theorem 2.

3.3. Proposed Numerical Methods for Solving CTrFFSME. In this section, two numerical methods are developed: the fuzzy gradient iterative (FGI) method and the fuzzy least square iterative (FLSI) method. The details of the FGI methods are discussed as follows.

3.3.1. Fuzzy Gradient Iterative (FGI) Method for CTrFFSME. In this method, the CTrFFSME in equation (1) is solved numerically by extending the GI method for solving the crisp linear matrix equation $AX = B$ in Theorem 1 to the CTrFFSME in equation (1). The following steps summarize the methods:

> *Step 1.* Decompose $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, \widetilde{F}, \widetilde{X}$, and $\widetilde{Y}$ into $A^{(l)}$, $B^{(l)}$, $C^{(l)}$, $D^{(l)}$, $E^{(l)}$, $F^{(l)}$, $X^{(l)}$, and $Y^{(l)}$ where $l = 1, 2, 3, 4$, respectively, and convert the CTrFFSME in equation (1) to the system of linear matrix equations in equation (24) using Theorem 3.
>
> *Step 2.* Using the hierarchical identification principle and Remark 5, the system of CSME in equation (25) can be decomposed into two subsystems: for $1 \leq l \leq 4$,

$$
\begin{aligned} \boldsymbol{\xi}_1^{(l)} &= \begin{pmatrix} E^{(l)} - Y^{(l)} B^{(l)} \\ F^{(l)} - Y^{(l)} D^{(l)} \end{pmatrix}, \\ \boldsymbol{\xi}_2^{(l)} &= \begin{pmatrix} E^{(l)} - A^{(l)} X^{(l)} & F^{(l)} - C^{(l)} X^{(l)} \end{pmatrix}, \end{aligned}
$$

$$(35)$$

where the iterative positive solution to the system of CSME in equation (25) is the average of the iterative solution for the subsystems. Let $\gamma_l = \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}$ and $\beta_l = \begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}$.

From equations (25) and (35), the following can be obtained: for $1 \leq l \leq 4$,

$$
\boldsymbol{\xi}_2^{(l)} = \gamma_l X^{(l)},
$$

$$(36a)$$

$$
\boldsymbol{\xi}_1^{(l)} = Y^{(l)} \beta_l.
$$

$$(36b)$$

Step 1. Convert the CTrFFSME in equation (1) to four systems of linear matrix equations using Theorem 3
Step 2. Apply Vec-operator and Kronecker product on the systems obtained in Step 1
Step 3. Convert the obtained systems in Step 2 to a single system $PQ = U$
Step 4. Multiply both sides of the system obtained in Step 3 by P^{-1}
Step 5. Solve the system of matrix equations in Step 4 and write the positive fuzzy solution as follows:

ALGORITHM 1: Fuzzy matrix vectorization and Kronecker product algorithm for solving CTrFFSME

Step 3. The iterative positive solution to the system of equations in equations (36a) and (36b) can be obtained by the GI method in Theorem 1 as follows: for $1 \leq l \leq 4$, we have

$$
\begin{cases}
\hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \left(\xi_1^{(l)} - \gamma_l \hat{x}^{(l)}(k-1) \right), \\
\hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(\xi_2^{(l)} - \hat{y}_l(k-1)\beta_l \right)\beta_l^T.
\end{cases}
\tag{37}
$$

Step 4. Substitute equations (35) into (37) as follows:

$$
\begin{cases}
\hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \left(\begin{pmatrix} E^{(l)} - \hat{y}_l(k-1)B^{(l)} \\ F^{(l)} - \hat{y}_l(k-1)D^{(l)} \end{pmatrix} - \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \hat{x}^{(l)}(k-1) \right), \\
\hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(\left(E^{(l)} - A^{(l)}\hat{x}_l(k-1) \quad F^{(l)} - C^{(l)}\hat{x}_l(k-1) \right) - \hat{y}_l(k-1)\left(B^{(l)} \quad D^{(l)} \right) \right)(\beta_l)^T.
\end{cases}
\tag{38}
$$

The obtained algorithm in equation (38) can be written as follows: for $1 \leq l \leq 4$, we have

$$
\begin{cases}
\hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)} \\ F^{(l)} - C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)} \end{pmatrix}, \\
\hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)} \quad F^{(l)} - C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)} \right)(\beta_l)^T.
\end{cases}
\tag{39}
$$

Let $r_l(k-1) = E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)}$ and $s_l(k-1) = F^{(l)} - C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)}$, then for $1 \leq l \leq 4$, the approximated solution in equation (39) can be written in reduced form as follows:

$$
\begin{cases}
\hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\
\hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(r_l(k-1) \quad s_l(k-1) \right)(\beta_l)^T,
\end{cases}
\tag{40}
$$

where the convergence factor (step size) is given by

$$
0 < \alpha_l < \frac{2}{\lambda_{\max}\left[\left(A^{(l)} \right)^T A^{(l)} \right] + \lambda_{\max}\left[B^{(l)}\left(B^{(l)} \right)^T \right] + \lambda_{\max}\left[\left(C^{(l)} \right)^T C^{(l)} \right] + \lambda_{\max}\left[D^{(l)}\left(D^{(l)} \right)^T \right]}.
\tag{41a}
$$

It can also be obtained as follows:

$$0 < \alpha_l < \frac{2}{\left\|A^{(l)}\right\|^2 + \left\|B^{(l)}\right\|^2 + \left\|C^{(l)}\right\|^2 + \left\|D^{(l)}\right\|^2} = \varphi, \quad (41b)$$

where $\|A^{(l)}\|^2 = \mathrm{tr}[A^{(l)} \cdot (A^{(l)})^T]$.

At step $k-$th of the iteration, the following relative error is considered:

$$\delta^l(k) = \sqrt{\frac{\left\|\hat{x}_l(k) - \hat{x}_l(k-1)\right\|^2 + \left\|\hat{y}_l(k) - \hat{y}_l(k-1)\right\|^2}{\left\|\hat{x}_l(k)\right\|^2 + \left\|\hat{y}_l(k)\right\|^2}}. \quad (42)$$

Step 5. By Definition 10, the approximated fuzzy solutions obtained by the previous step to the CTrFFSME in equation (1) can be written as follows:

$$\begin{cases} \hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)} \right), \\ \hat{y} = \left(\hat{y}^{(1)}, \hat{y}^{(2)}, \hat{y}^{(3)}, \hat{y}^{(4)} \right). \end{cases} \quad (43)$$

It can also be written in matrix form as

$$\begin{cases} \hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)} \right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ \hat{x}_{m1}^{(1)}, \hat{x}_{m1}^{(2)}, \hat{x}_{m1}^{(3)}, \hat{x}_{m1}^{(4)} & \cdots & \hat{x}_{mn}^{(1)}, \hat{x}_{mn}^{(2)}, \hat{x}_{mn}^{(3)}, \hat{x}_{mn}^{(4)} \end{pmatrix}, \\ \hat{y} = \begin{pmatrix} \left(\hat{y}_{11}^{(1)}, \hat{y}_{11}^{(2)}, \hat{y}_{11}^{(3)}, \hat{y}_{11}^{(4)} \right) & \cdots & \left(\hat{y}_{1n}^{(1)}, \hat{y}_{1n}^{(2)}, \hat{y}_{1n}^{(3)}, \hat{y}_{1n}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ \hat{y}_{m1}^{(1)}, \hat{y}_{m1}^{(2)}, \hat{y}_{m1}^{(3)}, \hat{y}_{m1}^{(4)} & \cdots & \hat{y}_{mn}^{(1)}, \hat{y}_{mn}^{(2)}, \hat{y}_{mn}^{(3)}, \hat{y}_{mn}^{(4)} \end{pmatrix}. \end{cases}$$

$$(44)$$

In Theorem 4, we prove that the iterative solution obtained by the FGI method converges to the positive solution of the positive CTrFFSME for any initial value.

Theorem 4. *If the system of CSME in equation (25) has a unique positive solution* $(X^{(l)}, Y^{(l)})$, *then the iterative solution* $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ *in equation (43) converges to* $(X^{(l)}, Y^{(l)})$ *for any initial values* $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ *for* $1 \le l \le 4$ *(i.e., if* $k \longrightarrow \infty$, *then* $X^{(l)} = \hat{x}^{(l)}(k)$ *and* $Y^{(l)} = \hat{y}^{(l)}(k)$).

Proof. Let $\psi(k)$ be the error at each k, for $k = 1, \ldots, n$ and $1 \le l \le 4$.

$$\psi(k) = \psi_1(k) + \psi_2(k), \quad (45)$$

$$\psi_1(k) = X^{(l)} - \hat{x}^{(l)}(k), \quad (46a)$$

$$\psi_2(k) = Y^{(l)} - \hat{y}^{(l)}(k). \quad (46b)$$

From equations (24), (39), (46a), and (46b), the following is obtained:

$$\begin{cases} \psi_1(k) = \psi_1(k-1) + \alpha_l \cdot \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix} \\ \psi_2(k) = \psi_2(k-1) + \alpha_l \cdot \left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \right) \left(B^{(l)} \quad D^{(l)} \right)^T. \end{cases} \quad (47)$$

Taking $\| \cdot \|^2$ to both sides of equation (47) gives the following:

$$\begin{cases} \|\psi_1(k)\|^2 = \left\| \psi_1(k-1) + \alpha_l \cdot \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix} \right\|^2, \\ \|\psi_2(k)\|^2 = \left\| \psi_2(k-1) + \alpha_l \cdot \left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \right) \left(B^{(l)} \quad D^{(l)} \right)^T \right\|^2. \end{cases} \quad (48)$$

Remark 8. The following steps in the proof are long; therefore, the system in equation (48) needs to be separated into two equations in the following steps of the proof.

It is not hard to prove that $\|A + B\|^2 = \mathrm{tr}((A + B)^T(A + B)) = \|A\|^2 + 2\mathrm{tr}(A^T B) + \|B\|^2$. Thus, equation (48) can be written as

$$\left\|\psi_1(k)\right\|^2 = \left\|\psi_1(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\psi_1^T(k-1)\left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\right)\right]$$

$$+ \alpha_l^2 \left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\right\|^2,$$

(49a)

$$\left\|\psi_2(k)\right\|^2 = \left\|\psi_2(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\psi_2^T(k-1)\left(\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\left(B^{(l)} \quad D^{(l)}\right)^T\right)\right]$$

$$+ \alpha_l^2 \left\|\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\left(B^{(l)} \quad D^{(l)}\right)^T\right\|^2.$$

(49b)

Applying norm properties and matrix multiplication gives

$$\left\|\psi_1(k)\right\|^2 \leq \left\|\psi_1(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(A^{(l)}\psi_1(k-1)\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$

$$\left. + \left(C^{(l)}\psi_1(k-1)\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right]$$

$$+ \alpha_l^2 \left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\right\|^2,$$

(50a)

$$\left\|\psi_2(k)\right\|^2 \leq \left\|\psi_2(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(\psi_2(k-1)B^{(l)}\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$

$$\left. + \left(\psi_2(k-1)D^{(l)}\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right]$$

$$+ \alpha_l^2 \left\|\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\left(B^{(l)} \quad D^{(l)}\right)^T\right\|^2.$$

(50b)

Applying norm properties on equations (50a) and (50b) gives

$$\left\|\psi_1(k)\right\|^2 \leq \left\|\psi_1(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(A^{(l)}\psi_1(k-1)\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$

$$\left. + \left(C^{(l)}\psi_1(k-1)\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right]$$

$$+ \alpha_l^2\left(\left\|A^{(l)}\right\|^2 + \left\|C^{(l)}\right\|^2\right)\left[\left\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right\|^2 + \left\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right\|^2\right],$$

(51a)

$$\left\|\psi_2(k)\right\|^2 \leq \left\|\psi_2(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(\psi_2(k-1)B^{(l)}\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$

$$\left. + \left(\psi_2(k-1)D^{(l)}\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right]$$

$$+ \alpha_l^2\left(\left\|B^{(l)}\right\|^2 + \left\|D^{(l)}\right\|^2\right)\left[\left\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right\|^2 + \left\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right\|^2\right].$$

(51b)

By the definition of the error in equation (45) and by equations (51a) and (51b), the following is obtained:

$$
\begin{aligned}
\|\psi(k)\|^2 \leq {} & \|\psi_1(k-1)\|^2 + 2\alpha_l \mathrm{tr}\Big[\big(A^{(l)}\psi_1(k-1)\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1)\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2\Big(\big\|A^{(l)}\big\|^2 + \big\|C^{(l)}\big\|^2\Big)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big] \\
& + \|\psi_2(k-1)\|^2 + 2\alpha_l \mathrm{tr}\Big[\big(\psi_2(k-1)B^{(l)}\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(\psi_2(k-1)D^{(l)}\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2\Big(\big\|B^{(l)}\big\|^2 + \big\|D^{(l)}\big\|^2\Big)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big],
\end{aligned}
\tag{52}
$$

which can be written as

$$
\begin{aligned}
\|\psi(k)\|^2 \leq {} & \|\psi_1(k-1)\|^2 + \|\psi_2(k-1)\|^2 + 2\alpha_l \mathrm{tr}\Big[\big(A^{(l)}\psi_1(k-1)\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1)\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2\Big(\big\|A^{(l)}\big\|^2 + \big\|C^{(l)}\big\|^2\Big)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big] \\
& + 2\alpha_l \mathrm{tr}\Big[\big(\psi_2(k-1)B^{(l)}\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) + \big(\psi_2(k-1)D^{(l)}\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2\Big(\big\|B^{(l)}\big\|^2 + \big\|D^{(l)}\big\|^2\Big)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big], \\[8pt]
\|\psi(k)\|^2 \leq {} & \|\psi_1(k-1)\|^2 + \|\psi_2(k-1)\|^2 + 2\alpha_l \mathrm{tr}\Big[\big(A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2\Big(\big\|A^{(l)}\big\|^2 + \big\|C^{(l)}\big\|^2 + \big\|B^{(l)}\big\|^2 + \big\|D^{(l)}\big\|^2\Big)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big], \\[8pt]
\|\psi(k)\|^2 \leq {} & \|\psi_1(k-1)\|^2 + \|\psi_2(k-1)\|^2 - 2\alpha_l \mathrm{tr}\Big[\big(A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big)^T\big(A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big)^T\big(C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2\Big(\big\|A^{(l)}\big\|^2 + \big\|C^{(l)}\big\|^2 + \big\|B^{(l)}\big\|^2 + \big\|D^{(l)}\big\|^2\Big)\Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big], \\[8pt]
\|\psi(k)\|^2 \leq {} & \|\psi(k-1)\|^2 - 2\alpha_l\Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big] \\
& + \frac{2\alpha_l^2}{\varphi}\Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big].
\end{aligned}
\tag{53}
$$

By equation (41b), the following can be obtained:

$$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 - 2\alpha_l\left(1 - \frac{\alpha_l}{\varphi}\right)\left[\left\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\right\|^2\right.$$
$$\left. + \left\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\right\|^2\right].$$

(54)

At $k = 1$, $\|\psi(1)\|^2 \le \|\psi(0)\|^2 - 2\alpha_l$ $(1 - (\alpha_l/\varphi))[\|A^{(l)}\psi_1(0) + \psi_2(0)B^{(l)}\|^2 + \|C^{(l)}\psi_1(0) + \psi_2(0)D^{(l)}\|^2]$.
At $k = 2$, $\|\psi(2)\|^2 \le \|\psi(1)\|^2 - 2\alpha_l(1 - (\alpha_l/\varphi))$ $[\|A^{(l)}\psi_1(1) + \psi_2(1)B^{(l)}\|^2 + \|C^{(l)}\psi_1(0) + \psi_2(0)D^{(l)}\|^2]$.
At $k = 3$, $\|\psi(3)\|^2 \le \|\psi(2)\|^2 - 2\alpha_l(1 - (\alpha_l/\varphi))$ $[\|A^{(l)}\psi_1(2) + \psi_2(2)B^{(l)}\|^2 + \|C^{(l)}\psi_1(2) + \psi_2(2)D^{(l)}\|^2]$.
At $k = n-1$, $\|\psi(n-1)\|^2 \le \|\psi(n-2)\|^2 - 2\alpha_l(1 - (\alpha_l/\varphi))$ $[\|A^{(l)}\psi_1(n-2) + \psi_2(n-2)$ $B^{(l)}\|^2 + \|C^{(l)}\psi_1(n-2) + \psi_2(n-2)D^{(l)}\|^2]$.
At $k = n$, $\|\psi(n)\|^2 \le \|\psi(n-1)\|^2 - 2\alpha_l(1 - (\alpha_l/\varphi))$ $[\|A^{(l)}\psi_1(n-1) + \psi_2(n-1)B^{(l)}\|^2 + \|C^{(l)}\psi_1(n-1) + \psi_2(n-1)D^{(l)}\|^2]$.

Therefore, the following is obtained:

$$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 - 2\alpha_l\left(1 - \frac{\alpha_l}{\varphi}\right)\left[\left\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\right\|^2\right.$$
$$\left. + \left\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\right\|^2\right].$$

(55)

If the convergence factor α is chosen to satisfy equation (41b) and $k \longrightarrow \infty$, then

$$\sum_{k=1}^{\infty}\left(\left\|A^{(l)}\psi_1(k) + \psi_2(k)B^{(l)}\right\|^2 + \left\|C^{(l)}\psi_1(k) + \psi_2(k)D^{(l)}\right\|^2\right) < \infty.$$

(56)

Therefore,

$$\lim_{k \longrightarrow \infty}\left(A^{(l)}\psi_1(k) + \psi_2(k)B^{(l)}\right) = 0,$$
$$\lim_{k \longrightarrow \infty}\left(C^{(l)}\psi_1(k) + \psi_2(k)D^{(l)}\right) = 0.$$

(57)

Since $A^{(l)} > 0, B^{(l)} > 0, C^{(l)} > 0,$ and $D^{(l)} > 0$, then

$$\lim_{k \longrightarrow \infty}\psi_1(k) = 0,$$
$$\lim_{k \longrightarrow \infty}\psi_2(k) = 0.$$

(58)

By equations (46a) and (46b), the following is obtained:

$$\lim_{k \longrightarrow \infty}\left(X^{(l)} - \hat{x}^{(l)}(k)\right) = 0,$$
$$\lim_{k \longrightarrow \infty}\left(Y^{(l)} - \hat{y}^{(l)}(k)\right) = 0.$$

(59)

Consequently, if $k \longrightarrow \infty$, then $X^{(l)} = \hat{x}^{(l)}(k)$ and $Y^{(l)} = \hat{y}^{(l)}(k)$.

Therefore, if the system of CSME in equation (25) has a unique positive solution $(X^{(l)}, Y^{(l)})$, then the iterative solution $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ in equation (43) converges to $(X^{(l)}, Y^{(l)})$ for any initial values $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ for $1 \le l \le 4$ (i.e., if $k \longrightarrow \infty$, then $X^{(l)} = \hat{x}^{(l)}(k)$ and $Y^{(l)} = \hat{y}^{(l)}(k)$).

The FGI algorithm is shown as Algorithm 2. This algorithm can be used by different software for solving the CTrFFSME in equation (1).

In the following method, the LSI method in Theorem 2 is extended and applied for solving the CTrFFSME in equation (1).

3.3.2. Fuzzy Least-Square Iterative (FLSI) Method for CTrFFSME.
In this method, the least square iterative method in Theorem 2 for solving $AX = B$ is applied for the CTrFFSME in equation (1). The first two steps of this method are similar to those of the fuzzy gradient iterative method.

Step 3. The iterative positive solution to the system of equations in equations (36a) and (36b) can be obtained by the LSI method in Theorem 2 as follows: for $1 \le l \le 4$, we have

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l\left(\left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\right)^T\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\right)^{-1}\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T\begin{pmatrix} E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)} \\ F^{(l)} - C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)} \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)} \quad F^{(l)} - C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)}\right) \\ \left(B^{(l)} \quad D^{(l)}\right)^T\left(\left(B^{(l)} \quad D^{(l)}\right)\left(B^{(l)} \quad D^{(l)}\right)^T\right)^{-1}. \end{cases}$$

(60)

Step 4. Let $\gamma_l = \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}$, $\beta_l = \left(B^{(l)} \quad D^{(l)}\right)$, $r_l(k-1) = E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)}$, and $s_l(k-1) = F^{(l)}$

$- C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)}$. For $1 \le l \le 4$, the approximated fuzzy solution in equation (60) can be written as follows:

Input that $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, \widetilde{F}$ # *Split each matrix into four matrices (e.g., $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}$)*
for $l = 1, 2, 3, 4$
Choose α_l, ε, $\hat{x}_l(0) = 0$, $\hat{y}_l(0) = 0$ # 0 is the Zero matrix with the same dimension as $X^{(l)}(k)$ and $Y^{(l)}(k)$.
While $k = 0, 1, 2, \ldots, n$ do

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix} \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(r_l(k-1) \ \ s_l(k-1) \right) (\beta_l)^T \end{cases}$$

$r_l(k-1) = E^{(l)} - A^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)B^{(l)}$
$s_l(k-1) = F^{(l)} - C^{(l)}\hat{x}_l(k-1) - \hat{y}_l(k-1)D^{(l)}$

$$\gamma_l = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}, \ \beta_l = \left(b_{ij}^{(l)} \ \ d_{ij}^{(l)} \right)$$

$\delta_l(k) = \sqrt{\left(\left(\|\hat{x}_l(k) - \hat{x}_l(k-1)\|^2 + \|\hat{y}_l(k) - \hat{y}_l(k-1)\|^2 \right) / \left(\|\hat{x}_l(k)\|^2 + \|\hat{y}_l(k)\|^2 \right) \right)}$
If $\delta_l(k) < \varepsilon$ then
  print $(\hat{x}_l(k), \hat{y}_l(k))$;
  print $(''$number of iterations $= '', k)$.
else

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix} \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(r_l(k-1) \ \ s_l(k-1) \right) (\beta_l)^T \end{cases},$$

update k.
$k = k + 1$
end
  print $(\hat{x}_l(k), \hat{y}_l(k))$;
  print $(''$number of iterations $= '', k)$.
end

ALGORITHM 2: FGI algorithm for CTrFFSME

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \left((\gamma_l)^T \gamma_l \right)^{-1} (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(r_l(k-1) \ \ s_l(k-1) \right) (\beta_l)^T \left((\beta_l)^T \beta_l \right)^{-1}, \end{cases} \tag{61}$$

where the convergence factor (step size) is given by

$$0 < \alpha_l < \frac{2}{\lambda_{\max}\left[\alpha_l \left((\alpha_l)^T \alpha_l \right)^{-1} (\alpha_l)^T \right] + \lambda_{\max}\left[(\beta_l)^T \left((\beta_l)^T \beta_l \right)^{-1} \beta_l \right]} = \frac{2}{\varphi_1 + \varphi_2}. \tag{62}$$

At step k − th of the iteration, the following relative error is considered:

$$\delta^l(k) = \sqrt{\frac{\left\| \hat{x}_l(k) - \hat{x}_l(k-1) \right\|^2 + \left\| \hat{y}_l(k) - \hat{y}_l(k-1) \right\|^2}{\left\| \hat{x}_l(k) \right\|^2 + \left\| \hat{y}_l(k) \right\|^2}}. \tag{63}$$

Step 5. By Definition 10, the approximated fuzzy solutions obtained by the previous step to the CTrFFSME in equation (1) can be written as follows:

$$\begin{cases} \hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)} \right), \\ \hat{y} = \left(\hat{y}^{(1)}, \hat{y}^{(2)}, \hat{y}^{(3)}, \hat{y}^{(4)} \right). \end{cases} \tag{64}$$

It can also be written in matrix form as

$$
\begin{cases}
\hat{x} = \begin{pmatrix}
\left(\overset{\wedge(1)}{x_{11}}, \overset{\wedge(2)}{x_{11}}, \overset{\wedge(3)}{x_{11}}, \overset{\wedge(4)}{x_{11}} \right) & \cdots & \left(\overset{\wedge(1)}{x_{1n}}, \overset{\wedge(2)}{x_{1n}}, \overset{\wedge(3)}{x_{1n}}, \overset{\wedge(4)}{x_{1n}} \right) \\
\vdots & \ddots & \vdots \\
\left(\overset{\wedge(1)}{x_{m1}}, \overset{\wedge(2)}{x_{m1}}, \overset{\wedge(3)}{x_{m1}}, \overset{\wedge(4)}{x_{m1}} \right) & \cdots & \left(\overset{\wedge(1)}{x_{mn}}, \overset{\wedge(2)}{x_{mn}}, \overset{\wedge(3)}{x_{mn}}, \overset{\wedge(4)}{x_{mn}} \right)
\end{pmatrix}, \\[3em]
\hat{y} = \begin{pmatrix}
\left(\overset{\wedge(1)}{y_{11}}, \overset{\wedge(2)}{y_{11}}, \overset{\wedge(3)}{y_{11}}, \overset{\wedge(4)}{y_{11}} \right) & \cdots & \left(\overset{\wedge(1)}{y_{1n}}, \overset{\wedge(2)}{y_{1n}}, \overset{\wedge(3)}{y_{1n}}, \overset{\wedge(4)}{y_{1n}} \right) \\
\vdots & \ddots & \vdots \\
\left(\overset{\wedge(1)}{y_{m1}}, \overset{\wedge(2)}{y_{m1}}, \overset{\wedge(3)}{y_{m1}}, \overset{\wedge(4)}{y_{m1}} \right) & \cdots & \left(\overset{\wedge(1)}{y_{mn}}, \overset{\wedge(2)}{y_{mn}}, \overset{\wedge(3)}{y_{mn}}, \overset{\wedge(4)}{y_{mn}} \right)
\end{pmatrix}.
\end{cases}
\tag{65}
$$

In Theorem 5, we prove that the iterative solution obtained by the FLSI method converges to the positive solution of the positive CTrFFSME for any initial value.

Theorem 5. *If the system of CSME in equation (25) has a unique positive solution $(X^{(l)}, Y^{(l)})$, then the iterative solution $(\overset{\wedge(l)}{x}(k), \overset{\wedge(l)}{y}(k))$ in equation (60) converges to $(X^{(l)}, Y^{(l)})$ for any initial values $\overset{\wedge(l)}{x}(0), \overset{\wedge(l)}{y}(0)$ for $1 \le l \le 4$ (i.e., if $k \longrightarrow \infty$, then $X^{(l)} = \overset{\wedge(l)}{x}(k)$ and $Y^{(l)} = \overset{\wedge(l)}{y}(k)$).*

Proof. Let $\psi(k)$ be the error at each k, for $k = 1, \ldots, n$ and $1 \le l \le 4$.

$$
\psi(k) = \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \psi_1(k) + \psi_2(k) \begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix},
\tag{66}
$$

where

$$
\psi_1(k) = X^{(l)} - \overset{\wedge(l)}{x}(k),
\tag{67a}
$$

$$
\psi_2(k) = Y^{(l)} - \overset{\wedge(l)}{y}(k).
\tag{67b}
$$

From equations (24), (60), (67a), and (67b), the following is obtained:

$$
\begin{cases}
\psi_1(k) = \psi_1(k-1) + \alpha_l \cdot \left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \right)^{-1} \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}, \\
\psi_2(k) = \psi_2(k-1) + \alpha_l \cdot \left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \right) \\
\quad \left(B^{(l)} \quad D^{(l)} \right)^T \left(\left(B^{(l)} \quad D^{(l)} \right) \left(B^{(l)} \quad D^{(l)} \right)^T \right)^{-1}.
\end{cases}
\tag{68}
$$

Taking $\| \cdot \|^2$ to both sides of equation (68) gives the following:

$$
\begin{cases}
\| \psi_1(k) \|^2 = \left\| \psi_1(k-1) + \alpha_l \cdot \left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \right)^{-1} \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix} \right\|^2, \\
\| \psi_2(k) \|^2 = \Big\| \psi_2(k-1) + \alpha_l \cdot \left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \quad -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \right) \\
\quad \left(B^{(l)} \quad D^{(l)} \right)^T \left(\left(B^{(l)} \quad D^{(l)} \right) \left(B^{(l)} \quad D^{(l)} \right)^T \right)^{-1} \Big\|^2.
\end{cases}
\tag{69}
$$

The following steps in the proof are long; therefore, the system in equation (69) has to be into two equations in the following steps of the proof.

Apply the following formula to equation (69) we get

$$
\left\| A\left(X + \left((A)^T \cdot A \right)^{-1} Y \right) \right\|^2 = \mathrm{tr}\left(\left(A\left(X + \left((A)^T \cdot A \right)^{-1} Y \right) \right)^T \left(A\left(X + \left((A)^T \cdot A \right)^{-1} Y \right) \right) \right) = \| AX \|^2 + 2\mathrm{tr}\left(X^T Y \right) + \left\| \left((A)^T \cdot A \right)^{-1} Y \right\|^2
$$

$$
\left\| \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \psi_1(k) \right\|^2 = \left\| \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \psi_1(k-1) \right\|^2 + 2\alpha_l \mathrm{tr}\left[\psi_1^T(k-1) \left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix} \right) \right]
$$

$$
+ \alpha_l^2 \left\| \left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \right)^{-1} \begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T \begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix} \right\|^2,
$$

$$
\tag{70a}
$$

$$\left\|\psi_2(k)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2 = \left\|\psi_2(k-1)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2 + 2\alpha_l \mathrm{tr}\left[\psi_2^T(k-1)\right.$$
$$\cdot \left.\left(\begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} & -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}^T\right)\right]$$
$$+ \alpha_l^2 \left\|\begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} & -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\right.$$
$$\cdot \left.\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}^T\left(\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}^T\right)^{-1}\right\|^2. \tag{70b}$$

Applying norm properties and matrix multiplication gives

$$\left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\psi_1(k)\right\|^2 \leq \left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\psi_1(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(A^{(l)}\psi_1(k-1)\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$
$$+ \left.\left(C^{(l)}\psi_1(k-1)\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right] \tag{71a}$$
$$+ \alpha_l^2 \left\|\left(\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\right)^{-1}\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}^T\begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} \\ -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\right\|^2,$$

$$\left\|\psi_2(k)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2 \leq \left\|\psi_2(k-1)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(\psi_2(k-1)B^{(l)}\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$
$$+ \left.\left(\psi_2(k-1)D^{(l)}\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right] \tag{71b}$$
$$+ \alpha_l^2 \left\|\begin{pmatrix} -A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)} & -C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)} \end{pmatrix}\right.$$
$$\cdot \left.\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}^T\left(\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}^T\right)^{-1}\right\|^2.$$

Applying norm properties on equations (71a) and (71b) gives

$$\left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\psi_1(k)\right\|^2 \leq \left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\psi_1(k-1)\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(A^{(l)}\psi_1(k-1)\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$
$$+ \left.\left(C^{(l)}\psi_1(k-1)\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right] \tag{72a}$$
$$+ \alpha_l^2(\varphi_1)\left[\left\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right\|^2 + \left\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right\|^2\right],$$

$$\left\|\psi_2(k)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2 \leq \left\|\psi_2(k-1)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2 + 2\alpha_l \mathrm{tr}\left[\left(\psi_2(k-1)B^{(l)}\right)^T\left(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right)\right.$$
$$+ \left.\left(\psi_2(k-1)D^{(l)}\right)^T\left(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right)\right] \tag{72b}$$
$$+ \alpha_l^2(\varphi_2)\left[\left\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\right\|^2 + \left\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\right\|^2\right].$$

By the definition of the error in equation (66),

$$\|\psi(k)\|^2 \leq \left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix}\psi_1(k)\right\|^2 + \left\|\psi_2(k)\begin{pmatrix} B^{(l)} & D^{(l)} \end{pmatrix}\right\|^2. \tag{73}$$

From equation (73) and by equations (72a) and (72b), the following is obtained:

$$
\begin{aligned}
\|\psi(k)\|^2 \leq & \left\|\begin{pmatrix} A^{(l)} \\ C^{(l)} \end{pmatrix} \psi_1(k-1)\right\|^2 + 2\alpha_l \text{tr}\Big[\big(A^{(l)}\psi_1(k-1)\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1)\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2(\varphi_1)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big] \\
& + \|\psi_2(k-1)\big(B^{(l)}\ D^{(l)}\big)\|^2 + 2\alpha_l \text{tr}\Big[\big(\psi_2(k-1)B^{(l)}\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(\psi_2(k-1)D^{(l)}\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2(\varphi_2)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big], \\
\|\psi(k)\|^2 \leq & \|\psi(k-1)\|^2 + 2\alpha_l \text{tr}\Big[\big(A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big)^T\big(-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big)^T\big(-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2(\varphi_1 + \varphi_2)\Big[\big\|-A^{(l)}\psi_1(k-1) - \psi_2(k-1)B^{(l)}\big\|^2 + \big\|-C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big\|^2\Big], \\
\|\psi(k)\|^2 \leq & \|\psi(k-1)\|^2 - 2\alpha_l \text{tr}\Big[\big(A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big)^T\big(A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big) \\
& + \big(C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big)^T\big(C^{(l)}\psi_1(k-1) - \psi_2(k-1)D^{(l)}\big)\Big] \\
& + \alpha_l^2(\varphi_1 + \varphi_2)\Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big], \\
\|\psi(k)\|^2 \leq & \|\psi(k-1)\|^2 - 2\alpha_l \Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big] \\
& + \alpha_l^2(\varphi_1 + \varphi_2)\Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big].
\end{aligned}
\tag{74}
$$

By equation (62), the following can be obtained:

$$
\begin{aligned}
\|\psi(k)\|^2 \leq & \|\psi(k-1)\|^2 - \alpha_l\big(2 - \alpha_l(\varphi_1 + \varphi_2)\big)\Big[\big\|A^{(l)}\psi_1(k-1) + \psi_2(k-1)B^{(l)}\big\|^2 \\
& + \big\|C^{(l)}\psi_1(k-1) + \psi_2(k-1)D^{(l)}\big\|^2\Big].
\end{aligned}
\tag{75}
$$

At $k = 1$, $\|\psi(1)\|^2 \leq \|\psi(0)\|^2 - \alpha_l(2 - \alpha_l(\varphi_1 + \varphi_2))$ $[\|A^{(l)}\psi_1(0) + \psi_2(0)B^{(l)}\|^2 + \|C^{(l)}\psi_1(0) + \psi_2(0)D^{(l)}\|^2]$.

At $k = 2$, $\|\psi(2)\|^2 \leq \|\psi(1)\|^2 - \alpha_l(2 - \alpha_l(\varphi_1 + \varphi_2))$ $[\|A^{(l)}\psi_1(1) + \psi_2(1)B^{(l)}\|^2 + \|C^{(l)}\psi_1(1) + \psi_2(1)D^{(l)}\|^2]$.

At $k = 3$, $\|\psi(3)\|^2 \leq \|\psi(2)\|^2 - \alpha_l(2 - \alpha_l(\varphi_1 + \varphi_2))$ $[\|A^{(l)}\psi_1(2) + \psi_2(2)B^{(l)}\|^2 + \|C^{(l)}\psi_1(2) + \psi_2(2)D^{(l)}\|^2]$.

At $k = n - 1$, $\|\psi(n-1)\|^2 \leq \|\psi(n-2)\|^2 - \alpha_l(2 - \alpha_l(\varphi_1 + \varphi_2))$ $[\|A^{(l)}\psi_1(n-2) + \psi_2(n-2)B^{(l)}\|^2 + \|C^{(l)}\psi_1\ (n-2) + \psi_2(n-2)D^{(l)}\|^2]$.

At $k = n$, $\|\psi(n)\|^2 \leq \|\psi(n-1)\|^2 \alpha_l(2 - \alpha_l(\varphi_1 + \varphi_2))$ $[\|A^{(l)}\psi_1(n-1) + \psi_2(n-1)B^{(l)}\|^2 + \|C^{(l)}\psi_1(n-1) + \psi_2(n-1)D^{(l)}\|^2]$.

Therefore, the following is obtained:

$$
\begin{aligned}
\|\psi(k)\|^2 \leq & \|\psi(k-1)\|^2 - \alpha_l\Big(\big(2 - \alpha_l(\varphi_1 + \varphi_2)\big)\Big[\big\|a_{ij}^{(l)}\psi_1(n-1) + \psi_2(n-1)b_{ij}^{(l)}\big\|^2 \\
& + \big\|c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\big\|^2\Big].
\end{aligned}
\tag{76}
$$

If the convergence factor α is chosen to satisfy equation (41b) and $k \longrightarrow \infty$, then

$$
\sum_{k=1}^{\infty}\Big(\big\|A^{(l)}\psi_1(k) + \psi_2(k)B^{(l)}\big\|^2 + \big\|C^{(l)}\psi_1(k) + \psi_2(k)D^{(l)}\big\|^2\Big) < \infty.
\tag{77}
$$

Therefore,

$$
\begin{aligned}
\lim_{k \to \infty}\big(A^{(l)}\psi_1(k) + \psi_2(k)B^{(l)}\big) = 0, \\
\lim_{k \to \infty}\big(C^{(l)}\psi_1(k) + \psi_2(k)D^{(l)}\big) = 0.
\end{aligned}
\tag{78}
$$

Input that $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}$ # *Split each matrix into four matrices (e.g.,* $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}$)
for $l = 1, 2, 3, 4$
Choose α_l, ε, $\hat{x}_l(0) = 0$, $\hat{y}_l(0) = 0$ *# 0 is the Zero matrix with the same dimension as* $X^{(l)}(k)$ *and* $Y^{(l)}(k)$.
While $k = 0, 1, 2, \ldots, n$ **do**

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \left((\gamma_l)^T \gamma_l \right)^{-1} (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix} \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(r_l(k-1) \; s_l(k-1) \right) (\beta_l)^T \left((\beta_l)^T \beta_l \right)^{-1} \end{cases}$$

$r_l(k-1) = E^{(l)} - A^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) B^{(l)}$
$s_l(k-1) = F^{(l)} - C^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) D^{(l)}$

$$\gamma_l = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}, \; \beta_l = \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}$$

$\delta_l(k) = \sqrt{ \left(\left(\| \hat{x}_l(k) - \hat{x}_l(k-1) \|^2 + \| \hat{y}_l(k) - \hat{y}_l(k-1) \|^2 \right) / \left(\| \hat{x}_l(k) \|^2 + \| \hat{y}_l(k) \|^2 \right) \right)}$
   If $\delta_l(k) < \varepsilon$ **then**
     print $(\hat{x}_l(k), \hat{y}_l(k))$;
    print ($"$number of iterations $= ", k$).
else

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix} \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(r_l(k-1) \; s_l(k-1) \right) (\beta_l)^T \end{cases},$$

   update k.
   $k = k + 1$
end
    print $(\hat{x}_l(k), \hat{y}_l(k))$;
  print ($"$number of iterations $= ", k$).
end

ALGORITHM 3: FLSI algorithm for CTrFFSME

Since $A^{(l)} > 0, B^{(l)} > 0, C^{(l)} > 0$, and $D^{(l)} > 0$, then

$$\lim_{k \to \infty} \psi_1(k) = 0,$$
$$\lim_{k \to \infty} \psi_2(k) = 0. \tag{79}$$

By equations (46a) and (46b), the following is obtained:

$$\lim_{k \to \infty} \left(X^{(l)} - \hat{x}^{(l)}(k) \right) = 0,$$
$$\lim_{k \to \infty} \left(Y^{(l)} - \hat{y}^{(l)}(k) \right) = 0. \tag{80}$$

Consequently, if $k \to \infty$, then $X^{(l)} = \hat{x}^{(l)}(k)$ and $Y^{(l)} = \hat{y}^{(l)}(k)$.

Therefore, if the system of CSME in equation (24) has a unique positive solution $(X^{(l)}, Y^{(l)})$, then the iterative solution $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ in equation (60) converges to $(X^{(l)}, Y^{(l)})$ for any initial values $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ for $1 \le l \le 4$ (i.e., if $k \to \infty$, then $X^{(l)} = \hat{x}^{(l)}(k)$ and $Y^{(l)} = \hat{y}^{(l)}(k)$).

Algorithm 3 summarizes the FLSI, respectively. This algorithm can be implemented by any mathematical software for solving the CTrFFSME in equation (1).

In Section 3.4, the three proposed methods for solving the CTrFFSME in equation (1) are applied to different fuzzy systems. □

3.4. Applications of the Proposed Methods to Other Fuzzy Systems and Fuzzy Numbers.
The proposed methods can solve different positive fuzzy systems with triangular and trapezoidal fuzzy numbers. It can be directly applied to the following.

The fully fuzzy matrix equation is as follows:

$$\widetilde{A}\widetilde{X} = \widetilde{E}. \tag{81}$$

The fully fuzzy Sylvester matrix equation is as follows:

$$\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{D} = \widetilde{E}. \tag{82}$$

The fully fuzzy continuous-time Lyapunov matrix equation is as follows:

$$\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}^T = \widetilde{E}. \tag{83}$$

The next section illustrates the three proposed methods by solving two examples sized 2×2 and 100×100, followed by the solutions verification.

4. Numerical Examples

To illustrate the accuracy and effectiveness of the proposed methods for solving the CTrFFSME in equation (1), we consider various sizes of CTrFFSME, namely, small (2×2) and large (100×100). Analytical solutions are found by Algorithm 1 for FMVM. Then, we compare the performance of Algorithm 2 for FGI and FLSI for approximating that solution by calculating the number of iterations (k), convergence factor (α), error bound (ε), convergence rate, CPU time, real-time, and memory usage. In addition to the graphical representation of the relative error $\delta^l(k)$, when k increases, in Example 1, the proposed methods are applied to small CTrFFSME (2×2).

Example 1. Solve the following 2×2 CTrFFSME:

$$\begin{cases} \widetilde{A}\widetilde{X} + \widetilde{Y}\widetilde{B} = \widetilde{E}, \\ \widetilde{C}\widetilde{X} + \widetilde{Y}\widetilde{D} = \widetilde{F}. \end{cases} \tag{84}$$

Given

$$
\begin{aligned}
\widetilde{A} &= \begin{pmatrix} (2,3,5,7) & (1,2,4,6) \\ (1,2,3,5) & (2,4,6,9) \end{pmatrix}, \\
\widetilde{B} &= \begin{pmatrix} (2,4,5,8) & (3,6,8,10) \\ (3,5,7,9) & (1,2,4,6) \end{pmatrix}, \\
\widetilde{E} &= \begin{pmatrix} (17,48,110,252) & (17,48,114,240) \\ (21,63,130,293) & (20,66,142,289) \end{pmatrix}, \\
\widetilde{C} &= \begin{pmatrix} (2,4,5,7) & (5,7,9,11) \\ (5,6,7,8) & (2,3,4,6) \end{pmatrix}, \\
\widetilde{D} &= \begin{pmatrix} (1,3,4,6) & (3,4,6,8) \\ (2,3,5,7) & (4,5,7,9) \end{pmatrix}, \\
\widetilde{F} &= \begin{pmatrix} (22,59,121,267) & (36,85,161,305) \\ (32,66,128,258) & (41,83,159,292) \end{pmatrix}.
\end{aligned}
\tag{85}
$$

Solution. The solution to the given positive CTrFFSME is obtained by the proposed methods as follows.

4.1. Fuzzy Matrix Vectorization Method (FMVM). By decomposing the given positive CTrFFSME and applying Algorithm 1 for FMVM, the analytical solution is obtained as follows:

Step 1. Convert the given CTrFFSME to four systems of CSME using Theorem 3.

Step 2. Apply Vec-operator and Kronecker product on the systems obtained in Step 1.

Step 3. Convert the obtained systems in Step 2 to the linear system $PQ = U$

where

$$P = \begin{pmatrix}
2 & 1 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 5 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 5 & 3 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 2 & 0 & 3 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 7 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 3 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 7 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 3 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 0 & 0 & 5 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 & 0 & 5 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 8 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 & 8 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 4 & 0 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 6 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 4 & 0 & 6 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 6 & 0 & 0 & 8 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 0 & 0 & 0 & 8 & 0 & 9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 6 & 10 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 0 & 10 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 6 & 0 & 7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 & 0 & 0 & 6 & 0 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 11 & 8 & 0 & 9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 & 0 & 8 & 0 & 9 & 0 & 0 & 0
\end{pmatrix}$$

$$U = \begin{pmatrix} 17 \\ 21 \\ 17 \\ 20 \\ 22 \\ 32 \\ 36 \\ 41 \\ 48 \\ 63 \\ 48 \\ 66 \\ 59 \\ 66 \\ 85 \\ 83 \\ 110 \\ 130 \\ 114 \\ 142 \\ 121 \\ 128 \\ 161 \\ 159 \\ 252 \\ 293 \\ 240 \\ 289 \\ 267 \\ 258 \\ 305 \\ 292 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} x^{(1)}_{11} \\ x^{(1)}_{21} \\ x^{(1)}_{12} \\ x^{(1)}_{22} \\ y^{(1)}_{11} \\ y^{(1)}_{21} \\ y^{(1)}_{12} \\ y^{(1)}_{22} \\ x^{(2)}_{11} \\ x^{(2)}_{21} \\ x^{(2)}_{12} \\ x^{(2)}_{22} \\ y^{(2)}_{11} \\ y^{(2)}_{21} \\ y^{(2)}_{12} \\ y^{(2)}_{22} \\ x^{(3)}_{11} \\ x^{(3)}_{21} \\ x^{(3)}_{12} \\ x^{(3)}_{22} \\ y^{(3)}_{11} \\ y^{(3)}_{21} \\ y^{(3)}_{12} \\ y^{(3)}_{22} \\ x^{(4)}_{11} \\ x^{(4)}_{21} \\ x^{(4)}_{12} \\ x^{(4)}_{22} \\ y^{(4)}_{11} \\ y^{(4)}_{21} \\ y^{(4)}_{12} \\ y^{(4)}_{22} \end{pmatrix}.$$

$$(86)$$

Step 4. Multiplying both sides of the system obtained in Step 3 by P^{-1} and solving for Q, we get

TABLE 1: Comparison between FFMVM, FGI, and FLSI for Example 1.

	Method	Analytical solution-approximated solution	α_l	ε	k
$\hat{x}^{(1)}$	FMVM	$\begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 4.0003146781142283762 & 3.0008763877876229244 \\ 1.9996620682548063892 & 3.9991059886583548709 \end{pmatrix}$	0.01515	10^{-5}	236
	FLSI	$\begin{pmatrix} 3.9994905651474607941 & 2.9991905860792312089 \\ 1.9994905651470906697 & 3.9991905860781175687 \end{pmatrix}$	0.5	10^{-5}	213
$\hat{y}^{(1)}$	FMVM	$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 1.9988524913049122353 & 1.0013283860149024908 \\ 2.0010864923386014879 & 2.9987621291364226938 \end{pmatrix}$	0.01515	10^{-5}	236
	FLSI	$\begin{pmatrix} 2.0005343495876688624 & 1.0009007873088771599 \\ 2.0005343495884565574 & 3.0009007873076832875 \end{pmatrix}$	0.5	10^{-5}	213
$\hat{x}^{(2)}$	FMVM	$\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 4.9998640795848062688 & 4.0030110768713679898 \\ 3.0003389970085296313 & 5.9972802099500071278 \end{pmatrix}$	0.006711	10^{-5}	333
	FLSI	$\begin{pmatrix} 4.9988761522873104256 & 3.9985435355183413819 \\ 2.9990633422322140346 & 5.9990330988581697249 \end{pmatrix}$	0.5	10^{-5}	184
$\hat{y}^{(2)}$	FMVM	$\begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 2.9983853131430880787 & 3.0019716289620118953 \\ 4.0023361161663756989 & 4.9968909026675074104 \end{pmatrix}$	0.006711	10^{-5}	333
	FLSI	$\begin{pmatrix} 3.0006934360840940105 & 3.0016140300903587928 \\ 4.0008105860628660745 & 5.001423238975299027 \end{pmatrix}$	0.5	10^{-5}	184
$\hat{x}^{(3)}$	FMVM	$\begin{pmatrix} 7 & 6 \\ 5 & 8 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 6.9966176458091605751 & 5.9986527914727380584 \\ 4.9980891197185313267 & 7.9959255849905723918 \end{pmatrix}$	0.003436	10^{-5}	269
	FLSI	$\begin{pmatrix} 6.9949554419054769516 & 5.9938085953915972691 \\ 4.9961576166547481863 & 7.9958332089102526017 \end{pmatrix}$	0.5	10^{-5}	367
$\hat{y}^{(3)}$	FMVM	$\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 3.9992591622654458963 & 5.006245036314479103 \\ 6.0032068410122893031 & 7.0014106935371849297 \end{pmatrix}$	0.003436	10^{-5}	269
	FLSI	$\begin{pmatrix} 4.0034261490303281354 & 5.0060523578143844872 \\ 6.0032467233815930657 & 7.0052034740382072243 \end{pmatrix}$	0.5	10^{-5}	367
$\hat{x}^{(4)}$	FMVM	$\begin{pmatrix} 10 & 8 \\ 9 & 11 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 9.9965893617203098369 & 8.0200665293211628219 \\ 8.99879605836685188 & 10.978609176105587048 \end{pmatrix}$	0.001831	10^{-5}	551
	FLSI	$\begin{pmatrix} 9.8263328737657879555 & 7.9801395018407377488 \\ 8.9119911526917516843 & 10.991796352652112364 \end{pmatrix}$	0.5	10^{-5}	502
$\hat{y}^{(4)}$	FMVM	$\begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$	NA	0	NA
	FGI	$\begin{pmatrix} 6.9831466534571057456 & 8.0224337286592531978 \\ 9.0229413143636603062 & 9.9800757765377075115 \end{pmatrix}$	0.001831	10^{-5}	551
	FLSI	$\begin{pmatrix} 7.0083279806078923944 & 8.017488243908286638 \\ 9.0079819495371672092 & 10.015609479910344443 \end{pmatrix}$	0.5	10^{-5}	502

TABLE 2: Computational time and memory usage for FGI and FLSI for Example 1.

	Method	k	CPU time (ms)	Real-time (ms)	Memory usage (MB)
$\hat{X}^{(1)}$	FGI	236	6.09	6.15	1.05
	FLSI	213	8.15	9.89	1.32
$\hat{Y}^{(1)}$	FGI	236	6.29	6.15	1.05
	FLSI	213	7.63	7.73	1.30
$\hat{X}^{(2)}$	FGI	333	6.24	6.17	1.05
	FLSI	184	7.39	7.38	1.32
$\hat{Y}^{(2)}$	FGI	333	6.15	6.19	1.05
	FLSI	184	7.13	7.30	1.30
$\hat{X}^{(3)}$	FGI	269	6.16	6.09	1.05
	FLSI	367	7.71	7.76	1.32
$\hat{Y}^{(3)}$	FGI	269	6.16	6.06	1.05
	FLSI	367	8.00	7.99	1.30
$\hat{X}^{(4)}$	FGI	551	5.98	5.99	1.05
	FLSI	502	7.91	7.86	1.32
$\hat{Y}^{(4)}$	FGI	551	6.10	5.99	1.05
	FLSI	502	7.66	7.71	1.30

$$
Q = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 4 \\ 2 \\ 2 \\ 1 \\ 3 \\ 5 \\ 3 \\ 4 \\ 6 \\ 3 \\ 4 \\ 3 \\ 5 \\ 7 \\ 5 \\ 6 \\ 8 \\ 4 \\ 6 \\ 5 \\ 7 \\ 10 \\ 9 \\ 8 \\ 11 \\ 7 \\ 9 \\ 8 \\ 10 \end{pmatrix} \tag{87}
$$

Step 5. By Definitions 9 and 10, the obtained solution in Step 4 can be written as follows:

$$
\begin{cases} \widetilde{X} = \begin{pmatrix} (4,5,7,10) & (3,4,6,8) \\ (2,3,5,9) & (4,6,8,11) \end{pmatrix}, \\ \widetilde{Y} = \begin{pmatrix} (2,3,4,7) & (1,3,5,8) \\ (2,4,6,9) & (3,5,7,10) \end{pmatrix}. \end{cases} \tag{88}
$$

This positive fuzzy solution is approximated using Algorithms 2 and 3 as follows.

4.2. Fuzzy Gradient Iterative (FGI) Method and Fuzzy Least-Square Iterative (FLSI) Method. Algorithms 2 and 3 for FGI and FLSI are applied to compute the approximated fuzzy solutions $\hat{x}^{(l)}(k)$ and $\hat{y}^{(l)}(k)$ for the given CTrFFSME using the following initial value for $1 \leq l \leq 4$, $\hat{x}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\hat{y}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The approximated solutions $\hat{X}$ and $\hat{Y}$ are shown in Table 1 with the convergence factor (α_l), error bound (ε), and the total number of iteration (k).

Table 2 shows the computational time and memory usage for FGI and FLSI.

Figure 2 shows the change in the relative error $\delta^l(k)$ when k increases up to $k = 20$.

From Tables 1 and 2 and Figure 2, the relative error $\delta^l(k)$ is becoming smaller as k increases. This indicates that the
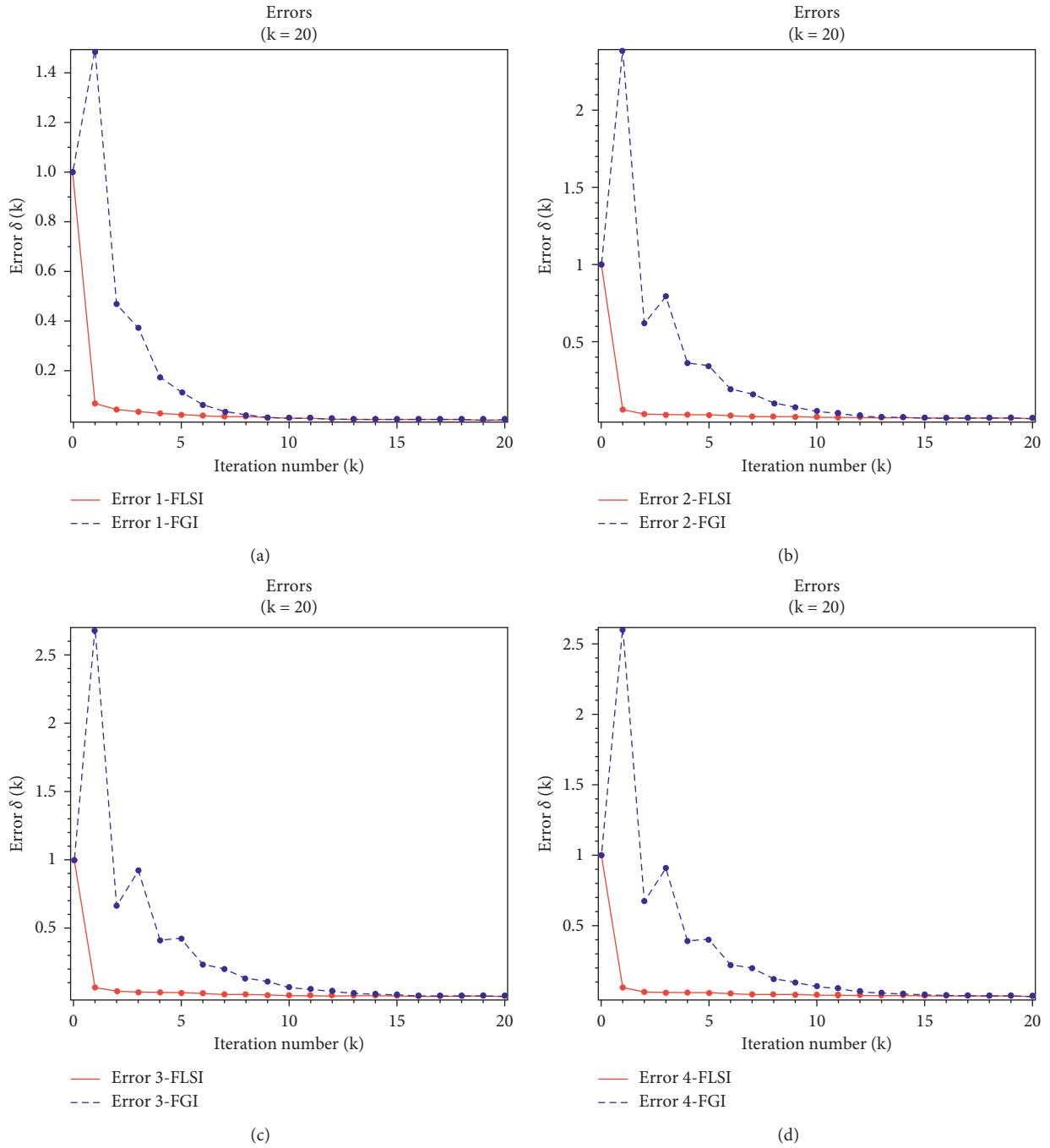
FIGURE 2: Comparison between $\delta^l(k)$ of FGI and FLSI for the first 20 iterations for Example 1.

proposed methods are effective and convergent for the given CTrFFSME. This indicates that the proposed algorithms are effective and convergent for the given positive CTrFFSME.

4.3. Verification of the Solution. To verify the obtained fuzzy solution, we first multiply $\widetilde{A}\widetilde{X}$ as follows:

$$\widetilde{A}\widetilde{X} = \begin{pmatrix} (2,3,5,7) & (1,2,4,6) \\ (1,2,3,5) & (2,4,6,9) \end{pmatrix} \begin{pmatrix} (4,5,7,10) & (3,4,6,8) \\ (2,3,5,9) & (4,6,8,11) \end{pmatrix} = \begin{pmatrix} (10,21,55,124) & (10,24,62,122) \\ (8,22,51,131) & (11,32,66,139) \end{pmatrix},$$

$$\widetilde{Y}\widetilde{B} = \begin{pmatrix} (2,3,5,7) & (1,3,5,8) \\ (2,4,6,9) & (3,5,7,10) \end{pmatrix} \begin{pmatrix} (2,4,5,8) & (3,6,8,10) \\ (3,5,7,9) & (1,2,4,6) \end{pmatrix} = \begin{pmatrix} (7,27,55,128) & (7,24,52,118) \\ (13,41,79,162) & (9,34,76,150) \end{pmatrix}.$$

$$(89)$$

We also multiply $\widetilde{C}\widetilde{X}$ as follows:

$$\widetilde{C}\widetilde{X} = \begin{pmatrix} (1,3,4,6) & (3,4,6,8) \\ (2,3,5,7) & (4,5,7,9) \end{pmatrix} \begin{pmatrix} (4,5,7,10) & (3,4,6,8) \\ (2,3,5,9) & (4,6,8,11) \end{pmatrix} = \begin{pmatrix} (18,41,80,169) & (26,58,102,177) \\ (24,39,69,134) & (23,42,74,130) \end{pmatrix},$$

$$\widetilde{Y}\widetilde{D} = \begin{pmatrix} (2,3,5,7) & (1,3,5,8) \\ (2,4,6,9) & (3,5,7,10) \end{pmatrix} \begin{pmatrix} (2,4,5,7) & (5,7,9,11) \\ (5,6,7,8) & (2,3,4,6) \end{pmatrix} = \begin{pmatrix} (4,18,41,98) & (10,27,59,128) \\ (8,27,59,124) & (18,41,85,162) \end{pmatrix}.$$

$$(90)$$

Therefore,

$$\widetilde{A}\widetilde{X} + \widetilde{Y}\widetilde{B} = \begin{pmatrix} (17,48,110,252) & (17,48,114,240) \\ (21,63,130,293) & (20,66,142,289) \end{pmatrix} = \widetilde{E},$$

$$\widetilde{C}\widetilde{X} + \widetilde{Y}\widetilde{D} = \begin{pmatrix} (22,59,121,267) & (36,85,161,305) \\ (32,66,128,258) & (41,83,159,292) \end{pmatrix} = \widetilde{F}.$$

$$(91)$$

The obtained positive fuzzy solution satisfies the given CTrFFSME, and it is feasible (strong fuzzy solution).

In Example 2, we tested the proposed method on 100×100 CTrFFSME, and we obtained the following results using Maple 2019.

Example 2. Solve the following 100×100 CTrFFSME:

$$\begin{cases} \widetilde{A}\widetilde{X} + \widetilde{Y}\widetilde{B} = \widetilde{E}, \\ \widetilde{C}\widetilde{X} + \widetilde{Y}\widetilde{D} = \widetilde{F}. \end{cases} \quad (92)$$

Given

$$A^{(1)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 1 \cdots 2),$$
$$B^{(1)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 1 \cdots 2),$$
$$C^{(1)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 1 \cdots 2),$$
$$D^{(1)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 1 \cdots 2),$$
$$E^{(1)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 7 \times 10^2 \cdots 1.5 \times 10^3),$$
$$F^{(1)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 7 \times 10^2 \cdots 1.5 \times 10^3),$$
$$A^{(2)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 3 \cdots 4),$$
$$B^{(2)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 3 \cdots 4),$$
$$C^{(2)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 3 \cdots 4),$$
$$D^{(2)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 3 \cdots 4),$$
$$E^{(2)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 4 \times 10^4 \cdots 6 \times 10^4),$$
$$F^{(2)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 4 \times 10^4 \cdots 6 \times 10^4),$$
$$A^{(3)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 5 \cdots 6),$$
$$B^{(3)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 5 \cdots 6),$$
$$C^{(3)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 5 \cdots 6),$$
$$D^{(3)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 5 \cdots 6),$$
$$E^{(3)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 1.1 \times 10^4 \cdots 1.3 \times 10^4),$$
$$F^{(3)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 1.1 \times 10^4 \cdots 1.3 \times 10^4),$$
$$A^{(4)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 7 \cdots 8),$$
$$B^{(4)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 7 \cdots 8),$$
$$C^{(4)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 7 \cdots 8),$$
$$D^{(4)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 7 \cdots 8),$$
$$E^{(4)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 2.2 \times 10^5 \cdots 3 \times 10^5),$$
$$F^{(4)} = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 2.2 \times 10^5 \cdots 3 \times 10^5).$$

$$(93)$$

TABLE 3: Comparison between FGI and FLSI for Example 2.

	Method	k	α_l	CPU time (s)	Real-time (s)	Memory usage (GB)
$\hat{x}^{(1)}$	FGI	10	2×10^{-7}	7.58	6.17	1.40
	FLSI	10	0.999	22.51	19.06	2.52
$\hat{y}^{(1)}$	FGI	10	2×10^{-7}	7.73	6.29	1.35
	FLSI	10	0.999	22.34	19.38	2.45
$\hat{x}^{(2)}$	FGI	10	2×10^{-7}	7.83	6.4	1.43
	FLSI	10	0.999	25.69	22.54	2.78
$\hat{y}^{(2)}$	FGI	10	2×10^{-7}	8.28	6.73	1.43
	FLSI	10	0.999	25.75	22.62	2.78
$\hat{x}^{(3)}$	FGI	10	2×10^{-7}	7.72	6.29	1.45
	FLSI	10	0.999	27.32	24.24	2.91
$\hat{y}^{(3)}$	FGI	10	2×10^{-7}	8.26	6.71	1.45
	FLSI	10	0.999	27.58	24.36	3.02
$\hat{x}^{(4)}$	FGI	10	2×10^{-7}	8.85	7.11	1.60
	FLSI	10	0.999	28.84	25.47	3.09
$\hat{y}^{(4)}$	FGI	10	2×10^{-7}	9.10	7.37	1.60
	FLSI	10	0.999	28.80	25.44	3.09

(a)

(b)
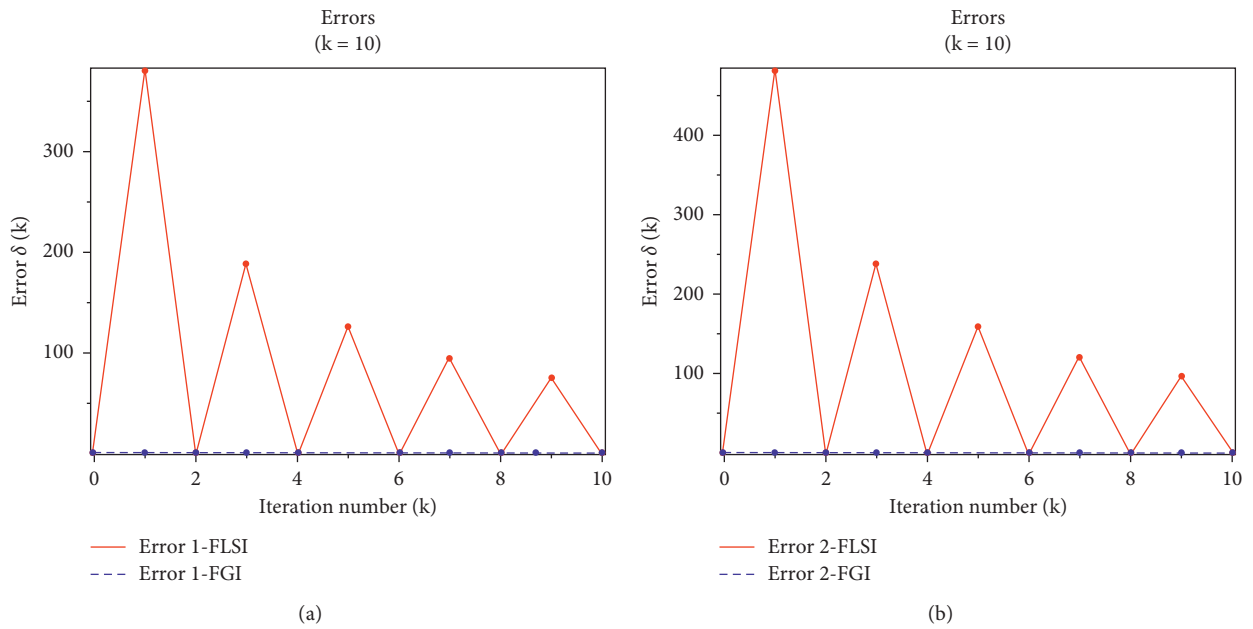
FIGURE 3: Continued.
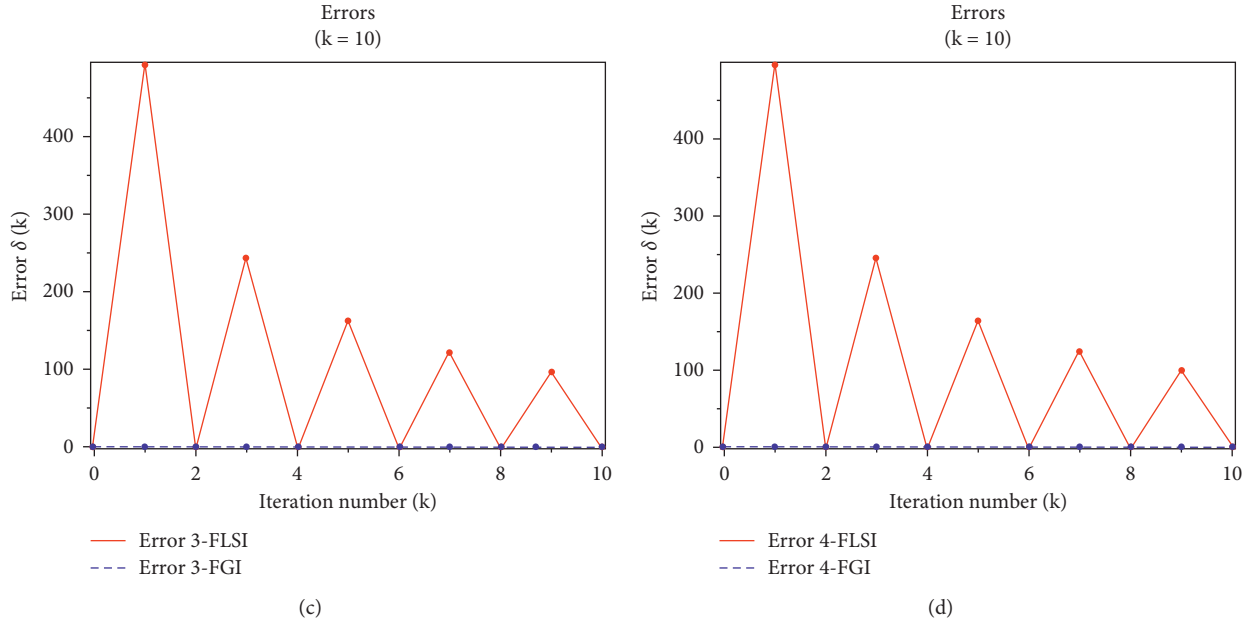
FIGURE 3: Comparison between relative $\delta^l(k)$ of FGI and FLSI for the first ten iterations for Example 2.

Solution. The solution for the given CTrFFSME is obtained by the proposed methods as follows.

4.4. Fuzzy Matrix Vectorization Method (FMVM). To apply FMVM, we need to find the inverse of the 10000×10000 matrix, which requires long computational timing and huge memory. Thus, FMVM is not a practical approach for such a large dimensional system.

4.5. Fuzzy Gradient Iterative (FGI) Method and Fuzzy Least-Square Iterative (FLSI) Method. Algorithm 2 for FGI and FLSI is applied to compute the approximated fuzzy solution $\hat{X}^{(l)}(k)$ and $\hat{Y}^{(l)}(k)$ for the given CTrFFSME with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.999$ for FLSI and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2 \times 10^{-7}$ for FGI using the following initial value:

$$
\hat{X}^{(l)}(0) = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 0),
$$
$$
\hat{Y}^{(l)}(0) = \text{LinearAlgebra:} - \text{RandomMatrix}(100, 100, \text{generator} = 0).
$$
$$(94)$$

In Table 3, the computational time and memory usage for the first ten iterations for FLSI and FGI are compared.

Figure 3 shows the change in the relative error $\delta^l(k)$ when k increases up to $k = 10$.

From Table 3 and Figure 3, the relative error $\delta^l(k)$ is becoming smaller as k increases. This indicates that the proposed methods are effective and convergent for the given CTrFFSME. This indicates that the proposed algorithms are effective and convergent for the given positive CTrFFSME.

5. Conclusion

Analyzing the stability of many control systems required solving a couple of crisp Sylvester matrix equations simultaneously. In this paper, we presented the solution to the CTrFFSME and its special cases analytically by FMVM and numerically by FGI and FLSI methods. The FMVM aims to find the exact solution. However, it is limited to

small-sized CTrFFSME, while FGI and FLSI can find an approximated fuzzy solution to large CTrFFSME. The numerical example analysis and graphical representation of the relative error indicate that the approximated solutions obtained by FGI and FLSI algorithms converge to the exact solution for any initial value and any size of the matrix system (up to 100×100). In addition, the relative error is becoming smaller as the number of iterations increases. This indicates that the proposed methods are effective and convergent for the given CTrFFSME regardless of any size of matrices. In addition, the developed method can find positive fuzzy solutions to the following fuzzy equations:

(i) Couple fully fuzzy Sylvester matrix equations with positive trapezoidal and triangular fuzzy numbers

(ii) Couple fully fuzzy Lyapunov matrix equations with positive trapezoidal and triangular fuzzy numbers

(iii) Couple fully fuzzy Sylvester matrix equations with positive trapezoidal and triangular fuzzy

(iv) Couple fully fuzzy Lyapunov matrix equations with positive trapezoidal and triangular fuzzy

(v) Couple fully fuzzy matrix equation with positive triangular and trapezoidal fuzzy numbers

The main disadvantage of this work is that it is limited to CTrFFSME with positive TrFNs only. As future research, the ideas presented in this paper will be extended to CTrFFSME with arbitrary trapezoidal fuzzy numbers.

Data Availability

No data were used to support this study.

Additional Points

This article does not contain any studies with human participants or animals performed by any of the authors.

Consent

Informed consent was obtained from all individual participants included in the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest, financial, or otherwise.

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