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Research Article

On inf-Hesitant Fuzzy Γ-Ideals of Γ-Semigroups

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The notions of an inf-hesitant fuzzy Γ-ideal and a (sup, inf)-hesitant fuzzy Γ-ideal, which are a generalization of an interval-valued fuzzy Γ-ideal, of a Γ-semigroup are introduced and some properties are investigated. Characterizations of the notions are provided in terms of sets, fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, and hesitant fuzzy sets. Furthermore, characterizations of a Γ-ideal of a Γ-semigroup are given in terms of inf-hesitant and (sup, inf)-hesitant fuzzy Γ-ideals.

1. Introduction

The notion of a fuzzy set, proposed by Zadeh [1], has provided a useful mathematical tool and method for describing the behavior of complex and ill-defined systems. The notion has huge applications in decision making, artificial intelligence, automata theory, control engineering, finite state machine, expert, graph theory, robotics, and many branches of pure and applied mathematics (cf. [2]). Nevertheless, there are limitations for using the notion to deal with vague and imprecise information when different sources of vagueness appear simultaneously. In order to overcome such limitations, Torra and Narukawa [3, 4] proposed an extension of the notion so-called a hesitant fuzzy set which is a function from a reference set to a power set of the unit interval. Hesitant fuzzy set theory has been applied to several practical problems, primarily in the area of decision making (see [5–9]) and different algebraic structures; for example, Jun and Ahn [10] introduced hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras and investigated related properties. Mosrijai et al. [11–13] studied hesitant fuzzy sets on UP-algebras. Kim et al. [14] studied the concepts and properties of a hesitant fuzzy subgroupoid (left ideal, right ideal, and ideal) of a groupoid, a hesitant fuzzy subgroup (normal subgroup and quotient subgroup) of a group, and a hesitant fuzzy subring (left ideal, right ideal, and ideal) of a ring. Jittburus and Julatha [15] proposed the concepts of a sup-hesitant fuzzy ideal of a semigroup and its sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions. They showed that the sup-hesitant fuzzy ideal is a general concept of a hesitant fuzzy ideal and an interval-valued fuzzy ideal and gave its characterizations in terms of sets, fuzzy sets, hesitant fuzzy sets, and interval-valued fuzzy sets. Julatha and Iampan [16] introduced sup-types of hesitant fuzzy sets based on ideal theory of ternary semigroups and examined their properties via a fuzzy set, an interval-valued fuzzy set, and a hesitant fuzzy set.

In 1981, Sen [17] introduced the concept and notion of the Γ-semigroup as a generalization of the plain semigroup and ternary semigroup. Many classical notions and results of (ternary) semigroups have been extended and generalized to Γ-semigroups, by many mathematicians, for instance, Siripitukdet and Iampan [18, 19], Siripitukdet and Julatha [20], Dutta and Adhikari [21, 22], Saha and Sen [23–25], Hila [26, 27], and Chinram [28, 29]. Simuen, Iampan, Chinram, Sardar, Majumder, Dutta, and Davvaz [30–35] studied theory of Γ-semigroups via fuzzy subsets. Uckun et al. [36] studied theory of Γ-semigroup via intuitionistic fuzzy subsets. Abbasi et al. [37] introduced hesitant fuzzy left (resp., right, bi-, interior, and two-sided) Γ-ideals of Γ-semigroups and characterized simple Γ-semigroups by hesitant fuzzy sets. Julatha and Iampan [38] introduced a sup-hesitant fuzzy Γ-ideal, which is a general concept of an interval-valued fuzzy Γ-ideal and a hesitant fuzzy Γ-ideal, of a Γ-semigroup and studied its properties via level sets, fuzzy sets, interval-valued fuzzy sets, and hesitant fuzzy sets.

In this paper, the notions of an inf-hesitant fuzzy Γ-ideal and a (sup, inf)-hesitant fuzzy Γ-ideal, which are a general notion of an interval-valued fuzzy Γ-ideal, of a Γ-semigroup are introduced and their properties are investigated. Equivalent conditions for a hesitant fuzzy set to be an inf-hesitant fuzzy Γ-ideal and a (sup, inf)-hesitant fuzzy Γ-ideal are provided in terms of sets, fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, and hesitant fuzzy sets. We show that every interval-valued fuzzy set on a Γ-semigroup is an interval-valued fuzzy Γ-ideal if and only if it is a (sup, inf)-hesitant fuzzy Γ-ideal. Furthermore, characterizations of a Γ-ideal of a Γ-semigroup are given in terms of inf-hesitant and (sup, inf)-hesitant fuzzy Γ-ideals.

2. Preliminaries

We will introduce some definitions and results that are important for study in this paper.

First, we recall the definition of Γ-semigroups which is defined by Sen and Saha [25]. By a *Γ-semigroup*, we mean a nonempty set G with a nonempty set Γ and a mapping $G\times\Gamma\times G\longrightarrow G$, written as $(u,\gamma,v)\mapsto u\gamma v$ satisfying the identity $(u\gamma v)\delta w = u\gamma(v\delta w)$ for all $u,v,w\in G$ and $\gamma,\delta\in\Gamma$. From now on throughout this paper, G is represented as a Γ-semigroup and X a nonempty set unless otherwise specified. For nonempty subsets U and V of G, let $U\Gamma V=\{u\gamma v | u\in U, v\in V, \gamma\in\Gamma\}$. By a *Γ-ideal* (ΓId) of G, we mean a nonempty subset V of G such that $G\Gamma V\subseteq V$ and $V\Gamma G\subseteq V$. Then, a nonempty subset V of G is a ΓId of G if and only if $u\gamma v, v\gamma u\in V$ for all $u\in G$, $v\in V$, and $\gamma\in\Gamma$.

A *fuzzy subset* (FS) [1] of X is a function from X into the unit segment of the real line $[0,1]$. A FS ϕ of G is called a *fuzzy Γ-ideal* (FΓId) of G if

$$\max\{\phi(u),\phi(v)\}\le\phi(u\gamma v),\quad \text{for all } u,v\in G \text{ and } \gamma\in\Gamma. \tag{1}$$

An *intuitionistic fuzzy set* (IFS) A [39] in X is an object having the form $A=\{(x,\phi(x),\varphi(x)) | x\in X\}$, where the functions $\phi\colon X\longrightarrow[0,1]$ and $\varphi\colon X\longrightarrow[0,1]$ define the degree of membership and the degree of nonmembership, respectively, and $0\le\phi(x)+\varphi(x)\le 1$ for all $x\in X$. An IFS $A=\{(x,\phi(x),\varphi(x)) | x\in X\}$ in X can be identified to an ordered pair (ϕ,φ) in $[0,1]^X\times[0,1]^X$. For a FS ϕ of X, we define a FS $(\phi/2)$ by $(\phi/2)(x)=(\phi(x)/2)$ for all $x\in X$. Then, $((\phi/2),(\varphi/2))$ is an IFS in X for all FSs ϕ and φ of X. An IFS (ϕ,φ) in G is called an *intuitionistic fuzzy Γ-ideal* (IFΓId) [36] of G if the following two conditions hold:

(i) (IFΓId1) $\phi(u\gamma v)\ge\max\{\phi(u),\phi(v)\}$ for all $u,v\in G$ and $\gamma\in\Gamma$

(ii) (IFΓId2) $\varphi(u\gamma v)\le\min\{\varphi(u),\varphi(v)\}$ for all $u,v\in G$ and $\gamma\in\Gamma$

By an interval number $\overline{a}$, we mean an interval $[a^-,a^+]$, where $0\le a^-\le a^+\le 1$. The set of all interval numbers is denoted by $\mathscr{D}[0,1]$. Especially, we denoted $\overline{1}:=[1,1]$ and $\overline{0}:=[0,0]$. For two elements $\overline{a}=[a^-,a^+]$ and $\overline{b}=[b^-,b^+]$ in $\mathscr{D}[0,1]$, define the operations $\preceq, =, \prec$ and rmax in case of two elements in $\mathscr{D}[0,1]$ as follows:

(i) $\overline{a}\preceq\overline{b}\Leftrightarrow a^-\le b^-$ and $a^+\le b^+$

(ii) $\overline{a}=\overline{b}\Leftrightarrow a^-=b^-$ and $a^+=b^+$

(iii) $\overline{a}\prec\overline{b}\Leftrightarrow\overline{a}\preceq\overline{b}$ and $\overline{a}\ne\overline{b}$

(iv) $r\max\{\overline{a},\overline{b}\}=[\max\{a^-,b^-\},\max\{a^+,b^+\}]$

Denote the case that $a^->b^-$ or $a^+>b^+$ by $\overline{a}\npreceq\overline{b}$. A function $\widetilde{\omega}\colon X\longrightarrow\mathscr{D}[0,1]$ is called an *interval-valued fuzzy set* (IvFS) [40] on X, where $\widetilde{\omega}(x)=[\omega^-(x),\omega^+(x)]$ for all $x\in X$ and ω^- and ω^+ are FSs of X such that $\omega^-(x)\le\omega^+(x)$ for all $x\in X$. Let IvFS(X) be the set of all IvFSs on X. An IvFS $\widetilde{\omega}$ on G is called an *interval-valued fuzzy Γ-ideal* (IvFΓId) of G if

$$r\max\{\widetilde{\omega}(u),\widetilde{\omega}(v)\}\preceq\widetilde{\omega}(u\gamma v),\quad \text{for all } u,v\in G \text{ and } \gamma\in\Gamma. \tag{2}$$

Then, $\widetilde{\omega}$ is an IvFΓId of G if and only if $\widetilde{\omega}(u)\preceq\widetilde{\omega}(u\gamma v)$ and $\widetilde{\omega}(v)\preceq\widetilde{\omega}(u\gamma v)$ for all $u,v\in G$ and $\gamma\in\Gamma$.

A *hesitant fuzzy set* (HFS) [3, 4] on X in terms of a function $\widehat{\psi}$ is that when applied to X returns a subset of $[0,1]$, that is, $\widehat{\psi}\colon X\longrightarrow\mathscr{P}([0,1])$, where $\mathscr{P}([0,1])$ denotes the set of all subsets of $[0,1]$. Let HFS(X) be the set of all HFSs on X, that is, $\text{HFS}(X)=\{\widehat{\psi} | \widehat{\psi}\colon X\longrightarrow\mathscr{P}([0,1])\}$ and let $\text{HFS}^*(X)=\{\widehat{\psi}\in\text{HFS}(X) | \widehat{\psi}(x)\ne\varnothing \text{ for all } x\in X\}$. Then, $\text{IvFS}(X)\subseteq\text{HFS}^*(X)\subseteq\text{HFS}(X)$. A HFS $\widehat{\psi}$ on G is called a *hesitant fuzzy Γ-ideal* (HFΓId) [37] of G if

$$\widehat{\psi}(u)\cup\widehat{\psi}(v)\subseteq\widehat{\psi}(u\gamma v),\quad \text{for all } u,v\in G \text{ and } \gamma\in\Gamma. \tag{3}$$

Then, $\widehat{\psi}$ is a HFΓId of G if and only if $\widehat{\psi}(v)\subseteq\widehat{\psi}(u\gamma v)\cap\widehat{\psi}(v\gamma u)$ for all $u,v\in G$ and $\gamma\in\Gamma$.

For $\widehat{\psi}\in\text{HFS}(X)$ and $\nabla\in\mathscr{P}([0,1])$, we define the element SUP∇ of [0, 1], the subset $[\widehat{\psi};\nabla]_{\text{SUP}}$ of X, the fuzzy subset $\mathscr{F}^{\widehat{\psi}}$ of X, and the hesitant fuzzy set $\mathscr{H}_{\text{SUP}}(\widehat{\psi};\nabla)$ on X [15, 16] by

(1) $\text{SUP}\nabla=\begin{cases}\sup\nabla & \text{if } \nabla\ne\varnothing\\ 0 & \text{otherwise}\end{cases}$

(2) $[\widehat{\psi};\nabla]_{\text{SUP}}=\{x\in X | \text{SUP}\widehat{\psi}(x)\ge\text{SUP}\nabla\}$

(3) $\mathscr{F}^{\widehat{\psi}}(x)=\text{SUP}\widehat{\psi}(x)$ for all $x\in X$

(4) $\mathscr{H}_{\text{SUP}}(\widehat{\psi};\nabla)(x)=\{t\in\nabla | \text{SUP}\widehat{\psi}(x)\ge t\}$ for all $x\in X$

We denote $\mathscr{H}_{\text{SUP}}(\widehat{\psi};[0,1])$ by $\widehat{\mathscr{H}^{\psi}_{\text{SUP}}}$, and then, $\widehat{\mathscr{H}^{\psi}_{\text{SUP}}}\in\text{IvFS}(X)$.

Julatha and Iampan [38] introduced a sup-hesitant fuzzy Γ-ideal, which is a generalization of the concepts of an IvFΓId and a HFΓId, of a Γ-semigroup and studied its properties in terms of FSs, IFSs, HFSs, and IvFSs in the following.

Definition 1 (see [38]). Given $\nabla\in\mathscr{P}([0,1])$, a HFS $\widehat{\psi}$ on G is called a sup-*hesitant fuzzy Γ-ideal of G related to* ∇ (briefly,

∇-sup-hesitant fuzzy Γ-ideal of G) if the set $[\hat{\psi};\nabla]_{SUP}$ is a ΓId of G. We say that $\hat{\psi}$ is a sup-*hesitant fuzzy* Γ-*ideal* (sup-HFΓId) of G if $\hat{\psi}$ is a ∇-sup-hesitant fuzzy Γ-ideal of G for all $\nabla\in\mathscr{P}([0,1])$ when $[\hat{\psi};\nabla]_{SUP}\neq\varnothing$.

Lemma 1 (see [38]). *Every IvFΓId of G is a sup-HFΓId of G.*

Lemma 2 (see [38]). *Every HFΓId of G is a sup-HFΓId of G.*

Theorem 1 (see [38]). *For $\hat{\psi}\in HFS(G)$, the following are equivalent:*

(1) $\hat{\psi}$ is a sup-HFΓId of G

(2) $\mathscr{F}^{\hat{\psi}}$ is a FΓId of G

(3) $\mathscr{H}^{\hat{\psi}}_{SUP}$ is a HFΓId of G

(4) $\mathscr{H}^{\psi}_{SUP}$ is an IvFΓId of G

(5) $\mathscr{H}^{\psi}_{SUP}$ is a sup-HFΓId of G

(6) $\mathscr{H}_{SUP}(\hat{\psi};\nabla)$ is a HFΓId of G for all $\nabla\in\mathscr{P}([0,1])$

Given $\hat{\psi}\in \mathrm{HFS}(X)$, the HFS $\hat{\psi}^*$, defined by $\hat{\psi}^*(x)=\{1-\mathrm{SUP}\hat{\psi}(x)\}$ for all $x\in X$, is called the *supremum complement* [13, 38] of $\hat{\psi}$ on X. Then, $\mathrm{SUP}\hat{\psi}^*(x)=1-\mathrm{SUP}\hat{\psi}(x)$ for all $x\in X$ and $(\mathscr{F}^{\hat{\psi}},\mathscr{F}^{\hat{\psi}^*})$ is an IFS in X.

Theorem 2 (see [38]). *A HFS $\hat{\psi}$ on G is a sup-HFΓId of G if and only if $(\mathscr{F}^{\hat{\psi}},\mathscr{F}^{\hat{\psi}^*})$ is an IFΓId of G.*

For every HFS $\hat{\psi}$ on X and every element t of $[0,1]$, the set

$$U_{SUP}(\hat{\psi};t)=\{x\in X\mid \mathrm{SUP}\hat{\psi}(x)\geq t\}, \tag{4}$$

is called a sup-*upper t-level subset* [13, 38] of $\hat{\psi}$.

Theorem 3 (see [38]). *A HFS $\hat{\psi}$ on G is a sup-HFΓId of G if and only if $U_{SUP}(\hat{\psi};t)$ is either empty or a ΓId of G for all $t\in[0,1]$.*

For a subset Y of X and two elements Δ,∇ of $\mathscr{P}([0,1])$, define the characteristic interval-valued fuzzy set (CIvFS) $\mathrm{CI}_Y\colon X\longrightarrow\mathscr{D}[0,1]$, the characteristic hesitant fuzzy set (CHFS) $\mathrm{CH}_Y\colon X\longrightarrow\mathscr{P}([0,1])$, and the hesitant fuzzy set $\chi_Y^{(\Delta,\nabla)}\colon X\longrightarrow\mathscr{P}([0,1])$ by for all $x\in X$,

$$\mathrm{CI}_Y(x)=\begin{cases}\bar{1}, & \text{if } x\in Y,\\ \bar{0}, & \text{otherwise,}\end{cases}$$
$$\mathrm{CH}_Y(x)=\begin{cases}[0,1], & \text{if } x\in Y,\\ \varnothing, & \text{otherwise,}\end{cases} \tag{5}$$
$$\chi_Y^{(\Delta,\nabla)}(x)=\begin{cases}\nabla & \text{if } x\in Y,\\ \Delta & \text{otherwise.}\end{cases}$$

Then, the HFS $\chi_Y^{(\Delta,\nabla)}$ is a general concept of the CHFS and CIvFS, that is, $\chi_Y^{(\varnothing,[0,1])}=\mathrm{CH}_Y$ and $\chi_Y^{(\bar{0},\bar{1})}=\mathrm{CI}_Y$. Julatha and Iampan [38] gave conditions for a nonempty subset Y of G to be a ΓId by using the CIvFS CI_Y, the CHFS CH_Y, and $\chi_Y^{(\Delta,\nabla)}$ as the following theorem.

Theorem 4 (see [38]). *For a nonempty subset Y of G, the following are equivalent:*

(1) Y is a ΓId of G

(2) The CIvFS CI_Y is a sup-HFΓId of G

(3) The CHFS CH_Y is a sup-HFΓId of G

(4) $\chi_Y^{(\Delta,\nabla)}$ is a sup-HFΓId of G for all $\Delta,\nabla\in\mathscr{P}([0,1])$ with $SUP\Delta<SUP\nabla$

3. inf-Hesitant Fuzzy Γ-Ideals

For a HFS $\hat{\psi}$ on X and an element $\nabla\in\mathscr{P}([0,1])$, define INF$\nabla$ and $[\hat{\psi};\nabla]_{INF}$ by

$$\mathrm{INF}\nabla=\begin{cases}\inf\nabla, & \text{if } \nabla\neq\varnothing,\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$
$$[\hat{\psi};\nabla]_{INF}=\{x\in X\mid \mathrm{INF}\hat{\psi}(x)\geq\mathrm{INF}\nabla\}.$$

Note that for all $x\in X$ and for all $\tilde{\omega}\in\mathrm{IvFS}(X)$, we have $\mathrm{INF}\tilde{\omega}(x)=\inf\tilde{\omega}(x)=\min\tilde{\omega}(x)=\omega^-(x)$. Now, we introduce the notion of an inf-hesitant fuzzy Γ-ideal of a Γ-semigroup in the following definition.

Definition 2. A HFS $\hat{\psi}$ on G is called an inf-hesitant fuzzy Γ-ideal (inf-HFΓId) of G if the set $[\hat{\psi};\nabla]_{INF}$ is a ΓId of G for all $\nabla\in\mathscr{P}([0,1])$ when $[\hat{\psi};\nabla]_{INF}\neq\varnothing$.

Example 1. Let $\mathbb{Z}^-$ be the set of all negative integers, $G=\mathbb{Z}^-\cup\{0\}$, and $\Gamma=2G$. Then, G is a Γ-semigroup with respect to usual multiplication.

(1) Define a HFS $\hat{\psi}$ on G by

$$\hat{\psi}(u)=\begin{cases}\{0.5,0.6,0.7\}, & \text{if } u=0,\\ [0.3,0.8], & \text{if } u\in 2\mathbb{Z}^-,\\ \varnothing, & \text{otherwise,}\end{cases} \tag{7}$$

for all $u\in G$. Then, $\hat{\psi}$ is an inf-HFΓId of G but not a sup-HFΓId of G because

$$\begin{aligned}\mathrm{SUP}\hat{\psi}((-5)(0)(-6))&=\mathrm{SUP}\hat{\psi}(0)=0.7<0.8\\ &=\max\{\mathrm{SUP}\hat{\psi}(-5),\mathrm{SUP}\hat{\psi}(-6)\}.\end{aligned} \tag{8}$$

(2) Define a HFS $\hat{\psi}$ on G by

$$\hat{\psi}(u)=\begin{cases}[0,1], & \text{if } u=0,\\ (0.4,0.8), & \text{if } u\in 2\mathbb{Z}^-,\\ \{0.5,0.6,0.7\}, & \text{otherwise,}\end{cases} \tag{9}$$

for all $u\in G$. Then, $\hat{\psi}$ is a HFΓId of G but not an inf-HFΓId of G because the nonempty subset $[\hat{\psi};[0.5,0.6]]_{INF}$ of G is not a ΓId of G, that is,

$$-1\in[\hat{\psi};[0.5,0.6]]_{INF},\ (-1)(-2)(-3)=-6\notin[\hat{\psi};[0.5,0.6]]_{INF}. \tag{10}$$

By Example 1 and Lemma 2, we obtain that an inf-HFΓId of G is not a sup-HFΓId and a HFΓId of G and a sup-HFΓId of G is not an inf-HFΓId of G.

Lemma 3. *Every IvFΓId of G is an* inf*-HFΓId of G.*

Proof. Suppose that $\tilde{\omega}$ is an IvFΓId of G and $\nabla \in \mathscr{P}([0,1])$ such that $[\tilde{\omega};\nabla]_{\mathrm{INF}}$ is a nonempty set. Let $u \in G$, $v \in [\tilde{\omega};\nabla]_{\mathrm{INF}}$, and $\gamma \in \Gamma$. Since $\tilde{\omega}$ is an IvFΓId of G, we get $\tilde{\omega}(v) \prec \tilde{\omega}(u\gamma v)$ and $\tilde{\omega}(v) \prec \tilde{\omega}(v\gamma u)$. Thus,

$$\begin{aligned}\mathrm{INF}\nabla \le \mathrm{INF}\tilde{\omega}(v) &= \omega^{-}(v) \le \min\{\omega^{-}(u\gamma v), \omega^{-}(v\gamma u)\} \\ &= \min\{\mathrm{INF}\tilde{\omega}(u\gamma v), \mathrm{INF}\tilde{\omega}(v\gamma u)\},\end{aligned} \tag{11}$$

which implies that $u\gamma v, v\gamma u \in [\tilde{\omega};\nabla]_{\mathrm{INF}}$. Hence, $[\tilde{\omega};\nabla]_{\mathrm{INF}}$ is a ΓId of G. Therefore, $\tilde{\omega}$ is an inf-HFΓId of G. □

In the following example, it is shown that the converse of Lemma 3 is not generally true.

Example 2. Let G be a Γ-semigroup defined in Example 1. Define an IvFS $\tilde{\omega}$ on G by for all $u \in G$,

$$\tilde{\omega}(u) = \begin{cases} \bar{1}, & \text{if } u = 0, \\ [0.5, 0.7], & \text{if } u \in 2\mathbb{Z}^{-}, \\ [0.3, 0.8], & \text{otherwise.} \end{cases} \tag{12}$$

Then, $\tilde{\omega}$ is an inf-HFΓId of G but not an IvFΓId of G because

$$\begin{aligned} r\max\{\tilde{\omega}(-1), \tilde{\omega}(-3)\} = [0.3, 0.8] &\nprec [0.5, 0.7] = \tilde{\omega}(-12) \\ &= \tilde{\omega}((-1)(-4)(-3)).\end{aligned} \tag{13}$$

By Lemma 3 and Example 2, we obtain that an inf-HFΓId of a Γ-semigroup G is a general concept of an IvFΓId of G.

For every HFS $\hat{\psi}$ on X, define the FS $\mathscr{F}_{\hat{\psi}}$ of X by $\mathscr{F}_{\hat{\psi}}(x) = \mathrm{INF}\hat{\psi}(x)$ for all $x \in X$. A HFS $\hat{\vartheta}$ on X is called an *infimum complement* of $\hat{\psi}$ on X if $\mathrm{INF}\hat{\vartheta}(x) = 1 - \mathrm{INF}\hat{\psi}(x)$ for all $x \in X$. Let $\mathrm{IC}(\hat{\psi})$ be the set of all infimum complements of $\hat{\psi}$. Define the HFS $\hat{\psi}^{*}$ by $\hat{\psi}^{*}(x) = \{1 - \mathrm{INF}\hat{\psi}(x)\}$ for all $x \in X$, and then we have $\hat{\psi}^{*} \in \mathrm{IC}(\hat{\psi})$, $\mathscr{F}_{\hat{\psi}^{*}}(x) = 1 - \mathrm{INF}\hat{\psi}(x)$, and $\mathrm{INF}(\hat{\psi}^{*})^{*}(x) = \mathrm{INF}\hat{\psi}(x) = \mathscr{F}_{\hat{\psi}}(x)$ for all $x \in X$. Note that $\min\{1 - t_1, 1 - t_2\} = 1 - \max\{t_1, t_2\}$ for all $t_1, t_2 \in [0,1]$.

Lemma 4. *If $\hat{\psi} \in HFS^{*}(G)$ is a HFΓId of G, then $\hat{\vartheta}$ is an* inf*-HFΓId of G for all $\hat{\vartheta} \in IC(\hat{\psi})$.*

Proof. Suppose that $\hat{\psi} \in \mathrm{HFS}^{*}(G)$ is a HFΓId of G and $\hat{\vartheta} \in \mathrm{IC}(\hat{\psi})$. Let $\nabla \in \mathscr{P}([0,1])$, $u \in G$, $v \in [\hat{\vartheta};\nabla]_{\mathrm{INF}}$, and $\gamma \in \Gamma$. Then, $\hat{\psi}(v) \subseteq \hat{\psi}(u\gamma v)$ and $\hat{\psi}(v) \subseteq \hat{\psi}(v\gamma u)$, and since $\hat{\psi} \in \mathrm{HFS}^{*}(G)$, we get

$$\mathrm{INF}\hat{\psi}(v) \ge \max\{\mathrm{INF}\hat{\psi}(u\gamma v), \mathrm{INF}\hat{\psi}(v\gamma u)\}, \tag{14}$$

$$\begin{aligned}\mathrm{INF}\nabla &\le \mathrm{INF}\hat{\vartheta}(v) \\ &= 1 - \mathrm{INF}\hat{\psi}(v) \\ &\le 1 - \max\{\mathrm{INF}\hat{\psi}(u\gamma v), \mathrm{INF}\hat{\psi}(v\gamma u)\} \\ &= \min\{1 - \mathrm{INF}\hat{\psi}(u\gamma v), 1 - \mathrm{INF}\hat{\psi}(v\gamma u)\} \\ &= \min\{\mathrm{INF}\hat{\vartheta}(u\gamma v), \mathrm{INF}\hat{\vartheta}(v\gamma u)\}.\end{aligned} \tag{15}$$

Hence, we have $u\gamma v, v\gamma u \in [\hat{\vartheta};\nabla]_{\mathrm{INF}}$. Therefore, we obtain that $[\hat{\vartheta};\nabla]_{\mathrm{INF}}$ is a ΓId of G. Consequently, $\hat{\vartheta}$ is an inf-HFΓId of G. □

Lemma 5. *For $\hat{\psi} \in HFS(G)$, the following are equivalent:*

(1) $\hat{\psi}$ is an inf*-HFΓId of G*
(2) $\mathscr{F}_{\hat{\psi}}$ is a FΓId of G
(3) $INF\hat{\psi}(u\gamma v) \ge \max\{INF\hat{\psi}(u), INF\hat{\psi}(v)\}$ for all $u, v \in G$ and $\gamma \in \Gamma$
(4) $INF\hat{\vartheta}(u\gamma v) \le \min\{INF\hat{\vartheta}(u), INF\hat{\vartheta}(v)\}$ for all $u, v \in G$, $\gamma \in \Gamma$, and $\hat{\vartheta} \in IC(\hat{\psi})$
(5) $INF\hat{\psi}^{*}(u\gamma v) \le \min\{INF\hat{\psi}^{*}(u), INF\hat{\psi}^{*}(v)\}$ for all $u, v \in G$ and $\gamma \in \Gamma$

Proof. (1)⇒(3). Let $u, v \in G$ and $\gamma \in \Gamma$. Then, $u \in [\hat{\psi};\hat{\psi}(u)]_{\mathrm{INF}}$ and $v \in [\hat{\psi};\hat{\psi}(v)]_{\mathrm{INF}}$. By assumption (1), we get $u\gamma v \in [\hat{\psi};\hat{\psi}(u)]_{\mathrm{INF}}$ and $u\gamma v \in [\hat{\psi};\hat{\psi}(v)]_{\mathrm{INF}}$. Thus, $\mathrm{INF}\hat{\psi}(u\gamma v) \ge \max\{\mathrm{INF}\hat{\psi}(u), \mathrm{INF}\hat{\psi}(v)\}$.

(3)⇒(2) and (4)⇒(5). They are clear.

(2)⇒(1). Let $\nabla \in \mathscr{P}([0,1])$, $u \in G$, $v \in [\hat{\psi};\nabla]_{\mathrm{INF}}$, and $\gamma \in \Gamma$. By assumption (2), we have

$$\begin{aligned}\min\{\mathrm{INF}\hat{\psi}(u\gamma v), \mathrm{INF}\hat{\psi}(v\gamma u)\} &= \min\{\mathscr{F}_{\hat{\psi}}(u\gamma v), \mathscr{F}_{\hat{\psi}}(v\gamma u)\} \\ &\ge \mathscr{F}_{\hat{\psi}}(v) = \mathrm{INF}\hat{\psi}(v) \ge \mathrm{INF}\nabla.\end{aligned} \tag{16}$$

Then, $u\gamma v, v\gamma u \in [\hat{\psi};\nabla]_{\mathrm{INF}}$. Thus, $[\hat{\psi};\nabla]_{\mathrm{INF}}$ is a ΓId of G. Therefore, we have that $\hat{\psi}$ is an inf-HFΓId of G.

(3)⇒(4). Let $u, v \in G$, $\gamma \in \Gamma$, and $\hat{\vartheta} \in \mathrm{IC}(\hat{\psi})$. By assumption (3), we have

$$\mathrm{INF}\hat{\psi}(u\gamma v) \ge \max\{\mathrm{INF}\hat{\psi}(u), \mathrm{INF}\hat{\psi}(v)\}, \tag{17}$$

and then,

$$\begin{aligned}\min\{\mathrm{INF}\hat{\vartheta}(u), \mathrm{INF}\hat{\vartheta}(v)\} &= \min\{1 - \mathrm{INF}\hat{\psi}(u), 1 - \mathrm{INF}\hat{\psi}(v)\} \\ &= 1 - \max\{\mathrm{INF}\hat{\psi}(u), \mathrm{INF}\hat{\psi}(v)\} \\ &\ge 1 - \mathrm{INF}\hat{\psi}(u\gamma v) \\ &= \mathrm{INF}\hat{\vartheta}(u\gamma v).\end{aligned} \tag{18}$$

(5)⇒(3). Let $u, v \in G$ and $\gamma \in \Gamma$. By using assumption (5), we get $\mathrm{INF}\hat{\psi}^{*}(u\gamma v) \le \min\{\mathrm{INF}\hat{\psi}^{*}(u), \mathrm{INF}\hat{\psi}^{*}(v)\}$ and then

$$\begin{aligned}\mathrm{INF}\widehat{\psi}(u\gamma v) &= 1-(1-\mathrm{INF}\widehat{\psi}(u\gamma v))\\ &= 1-\mathrm{INF}\widehat{\psi}^{*}(u\gamma v)\\ &\geq 1-\min\{\mathrm{INF}\widehat{\psi}^{*}(u),\mathrm{INF}\widehat{\psi}^{*}(v)\}\\ &= 1-\min\{1-\mathrm{INF}\widehat{\psi}(u),1-\mathrm{INF}\widehat{\psi}(v)\}\\ &= 1-(1-\max\{\mathrm{INF}\widehat{\psi}(u),\mathrm{INF}\widehat{\psi}(v)\})\\ &= \max\{\mathrm{INF}\widehat{\psi}(u),\mathrm{INF}\widehat{\psi}(v)\}.\end{aligned} \tag{19}$$

□

Theorem 5. *If an IvFS $\widetilde{\omega}$ of G is an IvFΓId of G, then $\mathscr{F}_{\widetilde{\omega}}$ is a FΓId of G.*

Proof. It follows from Lemmas 3 and 5. □

Theorem 6. *If $\widehat{\psi}\in HFS^{*}(G)$ is a HFΓId of G, then $\mathscr{F}_{\widehat{\vartheta}}$ is a FΓId of G for all $\widehat{\vartheta}\in IC(\widehat{\psi})$.*

Proof. It follows from Lemmas 4 and 5. □

For $\widehat{\psi}\in \mathrm{HFS}(X)$ and $t\in[0,1]$, the sets

$$\begin{aligned}U_{\mathrm{INF}}(\widehat{\psi};t) &= \{x\in X \mid \mathrm{INF}\widehat{\psi}(x)\geq t\},\\ L_{\mathrm{INF}}(\widehat{\psi};t) &= \{x\in X \mid \mathrm{INF}\widehat{\psi}(x)\leq t\},\end{aligned} \tag{20}$$

are called an inf-*upper t-level subset* and an inf-*lower t-level subset* of $\widehat{\psi}$, respectively.

Theorem 7. *A HFS $\widehat{\psi}$ of G is an* inf*-HFΓId of G if and only if for all $t\in[0,1]$, a nonempty subset $U_{INF}(\widehat{\psi};t)$ of G is a ΓId of G.*

Proof. Let $t\in[0,1]$ and $U_{\mathrm{INF}}(\widehat{\psi};t)\neq\varnothing$. Choose $\nabla\in\mathscr{P}([0,1])$ such that $\mathrm{INF}\nabla=t$, and we get $[\widehat{\psi};\nabla]_{\mathrm{INF}}=U_{\mathrm{INF}}(\widehat{\psi};t)$. Since $\widehat{\psi}$ is an inf-HFΓId of G, we get that $U_{\mathrm{INF}}(\widehat{\psi};t)=[\widehat{\psi};\nabla]_{\mathrm{INF}}$ is a ΓId of G.

Conversely, let $\nabla\in\mathscr{P}([0,1])$ and $[\widehat{\psi};\nabla]_{\mathrm{INF}}\neq\varnothing$. Choose $t:=\mathrm{INF}\nabla$, and by the assumption, we obtain that $[\widehat{\psi};\nabla]_{\mathrm{INF}}=U_{\mathrm{INF}}(\widehat{\psi};t)$ is a ΓId of G. Therefore, $\widehat{\psi}$ is an inf-HFΓId of G. □

Corollary 1. *Let $\widetilde{\omega}$ be an IvFΓId of G. Then, for all $t\in[0,1]$, a nonempty subset $U_{INF}(\widetilde{\omega};t)$ of G is a ΓId of G.*

Proof. It follows from Lemma 3 and Theorem 7. □

Theorem 8. *Let $\widehat{\psi}\in HFS(G)$ and $\widehat{\vartheta}\in IC(\widehat{\psi})$. Then, $\widehat{\vartheta}$ is an* inf*-HFΓId of G if and only if for all $t\in[0,1]$, a nonempty subset $L_{INF}(\widehat{\psi};t)$ of G is a ΓId of G.*

Proof. Let $t\in[0,1]$ and $L_{\mathrm{INF}}(\widehat{\psi};t)\neq\varnothing$. There exists $\nabla\in\mathscr{P}([0,1])$ such that $\mathrm{INF}\nabla=1-t$ and then $[\widehat{\vartheta};\nabla]_{\mathrm{INF}}=L_{\mathrm{INF}}(\widehat{\psi};t)$. Since $\widehat{\vartheta}$ is an inf-HFΓId of G, we obtain that $L_{\mathrm{INF}}(\widehat{\psi};t)=[\widehat{\vartheta};\nabla]_{\mathrm{INF}}$ is an ΓId of G.

Conversely, let $\nabla\in\mathscr{P}([0,1])$ be such that $[\widehat{\vartheta};\nabla]_{\mathrm{INF}}\neq\varnothing$. Choose $t:\ =1-\mathrm{INF}\nabla$, and by the assumption, we obtain that $[\widehat{\vartheta};\nabla]_{\mathrm{INF}}=L_{\mathrm{INF}}(\widehat{\psi};t)$ is a ΓId of G. Hence, $\widehat{\vartheta}$ is an inf-HFΓId of G. □

Corollary 2. *If $\widehat{\psi}\in HFS^{*}(G)$ is a HFΓId of G, then for all $t\in[0,1]$, a nonempty subset $L_{INF}(\widehat{\psi};t)$ of G is a ΓId of G.*

Proof. It follows from Lemma 4 and Theorem 8. □

In the following theorem, we give conditions for a HFS of a Γ-semigroup to be an inf-HFΓId via IFSs.

Theorem 9. *For $\widehat{\psi}\in HFS(G)$, the following are equivalent:*

(1) $\widehat{\psi}$ is an inf*-HFΓId of G*
(2) $(\mathscr{F}_{\widehat{\psi}},\mathscr{F}_{\widehat{\vartheta}})$ is an IFΓId of G for all $\widehat{\vartheta}\in IC(\widehat{\psi})$
(3) $(\mathscr{F}_{\widehat{\psi}},\mathscr{F}_{\widehat{\psi}^{*}})$ is an IFΓId of G

Proof. It follows from Lemma 5. □

Corollary 3. *If an IvFS $\widetilde{\omega}$ of G is an IvFΓId of G, then $(\mathscr{F}_{\widetilde{\omega}},\mathscr{F}_{\widehat{\vartheta}})$ is an IFΓId of G for all $\widehat{\vartheta}\in IC(\widetilde{\omega})$.*

Proof. It follows from Lemma 3 and Theorem 9. □

Corollary 4. *If $\widehat{\psi}\in HFS^{*}(G)$ is a HFΓId of G, then $(\mathscr{F}_{\widehat{\vartheta}},\mathscr{F}_{\widehat{\psi}})$ is an IFΓId of G for all $\widehat{\vartheta}\in IC(\widehat{\psi})$.*

Proof. It follows from Lemma 4 and Theorem 9. □

For $\widehat{\psi}\in\mathrm{HFS}(X)$ and $\nabla\in\mathscr{P}([0,1])$, we define the HFS $\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)$ on X by

$$\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(x)=\{t\in\nabla \mid \mathrm{INF}\widehat{\psi}(x)\geq t\}\ \text{for all}\ x\in X, \tag{21}$$

and we denote $\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};[0,1])$ by $\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}$. Then, the following statements hold:

(1) $\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(x)\subseteq\nabla$ for all $x\in X$
(2) $\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}\in\mathrm{IvFS}(X)$
(3) $0=\min\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(x)\leq\mathrm{INF}\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(x)\leq \mathrm{INF}\widehat{\psi}(x)=\max\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(x)$ for all $x\in X$

In the following theorem, we give conditions for a HFS of a Γ-semigroup to be an inf-HFΓId in terms of IvFSs and HFSs.

Theorem 10. *For $\widehat{\psi}\in HFS(G)$, the following are equivalent:*

(1) $\widehat{\psi}$ is an inf*-HFΓId of G*
(2) $\mathscr{H}_{INF}(\widehat{\psi};\nabla)$ is a HFΓId of G for all $\nabla\in\mathscr{P}([0,1])$
(3) $\mathscr{H}^{\widehat{\psi}}_{INF}$ is a HFΓId of G
(4) $\mathscr{H}^{\widehat{\psi}}_{INF}$ is an IvFΓId of G

Proof. (1)⇒(2). Let $u,v\in G$, $\gamma\in\Gamma$, $\nabla\in\mathscr{P}([0,1])$, and $t\in\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(u)\cup\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(v)$. Then, $t\in\nabla$ and $t\leq\max\{\mathrm{INF}\widehat{\psi}(u),\mathrm{INF}\widehat{\psi}(v)\}$. By assumption (1) and Lemma 5, we get

$$t\leq\max\{\mathrm{INF}\widehat{\psi}(u),\mathrm{INF}\widehat{\psi}(v)\}\leq\mathrm{INF}\widehat{\psi}(u\gamma v). \tag{22}$$

Thus $t \in \mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(u\gamma v)$. Hence, $\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(u) \cup \mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(v) \subseteq \mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)(u\gamma v)$. Therefore, we have that $\mathscr{H}_{\mathrm{INF}}(\widehat{\psi};\nabla)$ is a HFΓId of G.

(2)⇒(3). It is clear.

(3)⇒(4). Let $u, v \in G$ and $\gamma \in \Gamma$. Then, $\mathrm{INF}\widehat{\psi}(u) \in \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u)$ and $\mathrm{INF}\widehat{\psi}(v) \in \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)$. By assumption (3), we get $\mathrm{INF}\widehat{\psi}(u), \mathrm{INF}\widehat{\psi}(v) \in \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v)$. Thus,

$$\begin{aligned}\max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v) &= \mathrm{INF}\widehat{\psi}(u\gamma v)\\ &\geq \max\{\mathrm{INF}\widehat{\psi}(u), \mathrm{INF}\widehat{\psi}(v)\}\\ &= \max\left\{\max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\}.\end{aligned} \tag{23}$$

Since $\min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(x) = 0$ for all $x \in G$, we have $\min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v) \geq \max\left\{\min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\}$ and so

$$\begin{aligned}r\max\left\{\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\} &= \Big[\max\left\{\min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\},\\ &\quad \max\left\{\max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\}\Big]\\ &\preceq \Big[\min \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v), \max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v)\Big]\\ &= \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v).\end{aligned} \tag{24}$$

Therefore, $\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}$ is an IvFΓId of G.

(4)⇒(1). Let $u, v \in G$ and $\gamma \in \Gamma$. By assumption (4), we get $r\max\left\{\mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\} \prec \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v)$. Then,

$$\begin{aligned}\mathrm{INF}\widehat{\psi}(u\gamma v) &= \max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u\gamma v)\\ &\geq \max\left\{\max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(u), \max \mathscr{H}^{\widehat{\psi}}_{\mathrm{INF}}(v)\right\}\\ &= \max\{\mathrm{INF}\widehat{\psi}(u), \mathrm{INF}\widehat{\psi}(v)\}.\end{aligned} \tag{25}$$

Therefore, it follows from Lemma 5 that $\widehat{\psi}$ is an inf-HFΓId of G. □

Corollary 5. *Let $\widetilde{\omega}$ be an IvFΓId of G. Then, the following hold:*

(1) $\mathscr{H}_{INF}(\widetilde{\omega};\nabla)$ is a HFΓId of G for all $\nabla \in \mathscr{P}([0,1])$

(2) $\mathscr{H}^{\widetilde{\omega}}_{INF}$ is both a HFΓId and an IvFΓId of G

Proof. It follows from Lemma 3 and Theorem 10. □

Corollary 6. *Let $\widehat{\psi} \in HFS^*(G)$ be a HFΓId of G. Then, the following hold:*

(1) $\mathscr{H}_{INF}(\widehat{\vartheta};\nabla)$ is a HFΓId of G for all $\nabla \in \mathscr{P}([0,1])$ and $\widehat{\vartheta} \in IC(\widehat{\psi})$

(2) $\mathscr{H}^{\widehat{\vartheta}}_{INF}$ is both a HFΓId and an IvFΓId of G for all $\widehat{\vartheta} \in IC(\widehat{\psi})$

Proof. It follows from Lemma 4 and Theorem 10. □

In the following theorem, we give one characterization of a ΓId of a Γ-semigroup in terms of a HFS.

Theorem 11. *Let Y be a nonempty subset of G and $\Delta, \nabla \in \mathscr{P}([0,1])$ with $INF\Delta < INF\nabla$. Then, Y is a ΓId of G if and only if $\chi_Y^{(\Delta,\nabla)}$ is an* inf*-HFΓId of G.*

Proof. Assume that Y is a ΓId of G. Suppose that $\chi_Y^{(\Delta,\nabla)}$ is not an inf-HFΓId of G. By Lemma 5, there exist $u, v \in G$ and $\gamma \in \Gamma$ such that

$$\max\left\{\mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u), \mathrm{INF}\chi_Y^{(\Delta,\nabla)}(v)\right\} > \mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u\gamma v). \tag{26}$$

Thus, $\max\left\{\mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u), \mathrm{INF}\chi_Y^{(\Delta,\nabla)}(v)\right\} = \mathrm{INF}\nabla$, which implies that $u \in Y$ or $v \in Y$. Since Y is a ΓId of G, we get $u\gamma v \in Y$ and so

$$\max\left\{\mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u), \mathrm{INF}\chi_Y^{(\Delta,\nabla)}(v)\right\} = \mathrm{INF}\nabla = \mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u\gamma v), \tag{27}$$

which is a contradiction. Therefore, $\chi_Y^{(\Delta,\nabla)}$ is an inf-HFΓId of G.

Conversely, let $u \in Y, v \in G$, and $\gamma \in \Gamma$. Then, $\chi_Y^{(\Delta,\nabla)}(u) = \nabla$ and so $\max\left\{\mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u), \mathrm{INF}\chi_Y^{(\Delta,\nabla)}(v)\right\} = \mathrm{INF}\nabla$. Since $\chi_Y^{(\Delta,\nabla)}$ is an inf-HFΓId of G and Lemma 5, we get $\mathrm{INF}\chi_Y^{(\Delta,\nabla)}(u\gamma v) \geq \mathrm{INF}\nabla$ and $\mathrm{INF}\chi_Y^{(\Delta,\nabla)}(v\gamma u) \geq \mathrm{INF}\nabla$. Thus, $u\gamma v, v\gamma u \in Y$. Therefore, Y is a ΓId of G. □

Theorem 12. *A nonempty subset Y of G is a ΓId of G if and only if the CIvFS CI_Y is an* inf*-HFΓId of G.*

Proof. It follows from Theorem 11. □

Remark 1. If Y is a subset of G, then the CHFS CH_Y is an inf-HFΓId of G.

Definition 3. A HFS $\widehat{\psi}$ on G is called a (sup, inf)*-hesitant fuzzy Γ-ideal* ((sup, inf)-HFΓId) of G if $\widehat{\psi}$ is both an inf-HFΓId and a sup-HFΓId of G.

In Theorem 13, equivalent conditions for a hesitant fuzzy set to be a (sup, inf)-HFΓId are given in terms of level sets, FSs, IFSs, IvFSs, and HFSs.

Theorem 13. *For $\widehat{\psi} \in HFS(G)$, the following are equivalent:*

(1) $\widehat{\psi}$ is a (sup, inf)*-HFΓId of G*

(2) $\mathscr{F}_{\widehat{\psi}}$ and $\mathscr{F}^{\widehat{\psi}}$ are FΓIds of G

(3) $\mathscr{H}^{\widehat{\psi}}_{INF}$ and $\mathscr{H}^{\widehat{\psi}}_{SUP}$ are HFΓIds of G

(4) $\mathscr{H}^{\widehat{\psi}}_{INF}$ and $\mathscr{H}^{\widehat{\psi}}_{SUP}$ are IvFΓIds of G

(5) $(\mathscr{F}_{\widehat{\psi}}, \mathscr{F}^{\widehat{\psi}^*})$ is an IFΓId of G

(6) $((\mathscr{F}^{\widehat{\psi}}/2), (\mathscr{F}_{\widehat{\psi}^*}/2))$ is an IFΓId of G

(7) $((\mathscr{F}^{\widehat{\psi}}/2), (\mathscr{F}_{\widehat{\vartheta}}/2))$ is an IFΓId of G for all $\widehat{\vartheta} \in IC(\widehat{\psi})$

(8) $\mathscr{H}_{INF}(\widehat{\psi};\nabla)$ and $\mathscr{H}_{SUP}(\widehat{\psi};\Delta)$ are HFΓIds of G for all $\Delta, \nabla \in \mathscr{P}([0,1])$

(9) *For all* $t_1, t_2 \in [0,1]$, *nonempty subsets* $U_{INF}(\widehat{\psi}; t_1)$ *and* $U_{SUP}(\widehat{\psi}; t_2)$ *of G are ΓIds of G*

(10) $SUP\widehat{\psi}(u\gamma v) \geq \max\{SUP\widehat{\psi}(u), SUP\widehat{\psi}(v)\}$ *and* $INF\widehat{\psi}(u\gamma v) \geq \max\{INF\widehat{\psi}(u), INF\widehat{\psi}(v)\}$ *for all* $u, v \in G$ *and* $\gamma \in \Gamma$

Proof. It follows from Theorems 1, 2, 3, 7, and 10 and Lemma 5. □

Example 3. Let $\mathbb{Z}^-$ be the set of all negative integers and $G = \Gamma = \mathbb{Z}^-$. Then, G forms a Γ-semigroup with the usual multiplication. Define a HFS $\widehat{\psi}$ on G by $\widehat{\psi}(u) = \{(u + 1/3u), (3u + 1/3u)\}$ for all $u \in G$. Then,

$$\begin{aligned} \mathrm{SUP}\widehat{\psi}(u\gamma v) &= \frac{3(u\gamma v)+1}{3(u\gamma v)} \geq \max\left\{\frac{3u+1}{3u}, \frac{3v+1}{3v}\right\} \\ &= \max\{\mathrm{SUP}\widehat{\psi}(u), \mathrm{SUP}\widehat{\psi}(v)\}, \\ \mathrm{INF}\widehat{\psi}(u\gamma v) &= \frac{u\gamma v+1}{3(u\gamma v)} \geq \max\left\{\frac{u+1}{3u}, \frac{v+1}{3v}\right\} \\ &= \max\{\mathrm{INF}\widehat{\psi}(u), \mathrm{INF}\widehat{\psi}(v)\}, \end{aligned} \tag{28}$$

for all $u, v \in G$ and $\gamma \in \Gamma$. Hence, it follows from Theorem 13 that $\widehat{\psi}$ is a (sup, inf)-HFΓId of G. Since $\widehat{\psi}$ is not an IvFS of G, we get that it is not an IvFΓId of G.

Lemma 6. *Every IvFΓId of G is a* (sup, inf)*-HFΓId of G.*

Proof. It follows from Lemmas 1 and 3. □

By Example 3 and Lemma 6, we see that a (sup, inf)-HFΓId of G is a general concept of an IvFΓId of G.

Lemma 7. *Let* $\widetilde{\omega}$ *be an IvFS of G. Then,* $\widetilde{\omega}$ *is an IvFΓId of G if and only if* $\widetilde{\omega}$ *is a* (sup, inf)*-HFΓId of G.*

Proof. It follows from Lemma 6.

Conversely, assume that $\widetilde{\omega}$ is a (sup, inf)-HFΓId of G. By Theorem 13, we get $\mathrm{SUP}\widetilde{\omega}(u\gamma v) \geq \max\{\mathrm{SUP}\widetilde{\omega}(u), \mathrm{SUP}\widetilde{\omega}(v)\}$ and $\mathrm{INF}\widetilde{\omega}(u\gamma v) \geq \max\{\mathrm{INF}\widetilde{\omega}(u), \mathrm{INF}\widetilde{\omega}(v)\}$ for all $u, v \in G$ and $\gamma \in \Gamma$. Thus,

$$\begin{aligned} r\max\{\widetilde{\omega}(u), \widetilde{\omega}(v)\} &= [\max\{\mathrm{INF}\widetilde{\omega}(u), \mathrm{INF}\widetilde{\omega}(v)\}, \\ &\quad \max\{\mathrm{SUP}\widetilde{\omega}(u), \mathrm{SUP}\widetilde{\omega}(v)\}] \\ &\preceq [\mathrm{INF}\widetilde{\omega}(u\gamma v), \mathrm{SUP}\widetilde{\omega}(u\gamma v)] \\ &= \widetilde{\omega}(u\gamma v), \end{aligned} \tag{29}$$

for all $u, v \in G$ and $\gamma \in \Gamma$. Therefore, $\widetilde{\omega}$ is an IvFΓId of G. □

In Theorem 14, equivalent conditions for an IvFS to be an IvFΓId are given in terms of level sets, FSs, IFSs, IvFSs, and HFSs.

Theorem 14. *For* $\widetilde{\omega} \in IvFS(G)$*, the following are equivalent:*

(1) $\widetilde{\omega}$ *is an IvFΓId of G*

(2) $\widetilde{\omega}$ *is a* (sup, inf)*-HFΓId of G*

(3) $\mathscr{F}_{\widetilde{\omega}}$ *and* $\mathscr{F}^{\widetilde{\omega}}$ *are FΓIds of G*

(4) $\mathscr{H}^{\widetilde{\omega}}_{INF}$ *and* $\mathscr{H}^{\widetilde{\omega}}_{SUP}$ *are HFΓIds of G*

(5) $\mathscr{H}^{\widetilde{\omega}}_{INF}$ *and* $\mathscr{H}^{\widetilde{\omega}}_{SUP}$ *are IvFΓIds of G*

(6) $(\mathscr{F}_{\widetilde{\omega}}, \mathscr{F}^{\widetilde{\omega}^*})$ *is an IFΓId of G*

(7) $((\mathscr{F}^{\widetilde{\omega}}/2), (\mathscr{F}_{\widetilde{\omega}^*}/2))$ *is an IFΓId of G*

(8) $((\mathscr{F}^{\widetilde{\omega}}/2), (\mathscr{F}_{\widehat{\vartheta}}/2))$ *is an IFΓId of G for all* $\widehat{\vartheta} \in IC(\widetilde{\omega})$

(9) $\mathscr{H}_{INF}(\widetilde{\omega}; \nabla)$ *and* $\mathscr{H}_{SUP}(\widetilde{\omega}; \Delta)$ *are HFΓIds of G for all* $\Delta, \nabla \in \mathscr{P}([0,1])$

(10) *For all* $t_1, t_2 \in [0,1]$, *nonempty subsets* $U_{INF}(\widetilde{\omega}; t_1)$ *and* $U_{SUP}(\widetilde{\omega}; t_2)$ *of G are ΓIds of G*

Proof. It follows from Lemma 7 and Theorem 13. □

Theorem 15. *For a nonempty subset Y of G, the following are equivalent:*

(1) *Y is a ΓId of G*

(2) *The CIvFS* CI_Y *is a* (sup, inf)*-HFΓId of G*

(3) *The CHFS* CH_Y *is a* (sup, inf)*-HFΓId of G*

(4) $\chi_Y^{(\Delta,\nabla)}$ *is a* (sup, inf)*-HFΓId of G for all* $\Delta, \nabla \in \mathscr{P}([0,1])$ *with* $INF\Delta < INF\nabla$ *and* $SUP\Delta < SUP\nabla$

Proof. It follows from Theorems 4, 11, and 12 and Remark 1. □

4. Conclusions

In this paper, we have introduced the notions of an inf-HFΓId and a (sup, inf)-HFΓId, which are a generalization of an IvFΓId, of a Γ-semigroup and examined their characterizations in terms of sets, FSs, IFSs, IvFSs, and HFSs. Furthermore, we have discussed the relation between a ΓId and a generalization of the CHFS and CIvFS. From the study results, we found that the following conditions are all equivalent in a Γ-semigroup: a nonempty subset Y is a ΓId, CI_Y is an inf-HFΓId, CI_Y is a (sup, inf)-HFΓId, and CH_Y is a (sup, inf)-HFΓId.

In the future, we will study an inf-HFΓId and a (sup, inf)-HFΓId in LA-semigroups and *UP*-algebras and examine their characterizations in terms of sets, FSs, IFSs, IvFSs, and HFSs.

Data Availability

No data were used to support this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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