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Research Article

Semi-Simple Extension of the (Super) Poincaré Algebra

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A semi-simple tensor extension of the Poincaré algebra is proposed for the arbitrary dimensions D. It is established that this extension is a direct sum of the D-dimensional Lorentz algebra so$(D-1, 1)$ and D-dimensional anti-de Sitter (AdS) algebra so$(D-1, 2)$. A supersymmetric also semi-simple generalization of this extension is constructed in the $D=4$ dimensions. It is shown that this generalization is a direct sum of the 4-dimensional Lorentz algebra so(3, 1) and orthosymplectic algebra osp(1, 4) (super-AdS algebra).

1. Introduction

In the papers [1–7] the Poincaré algebra for the generators of the rotations M_{ab} and translations P_a in D dimensions,

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \left(g_{ad}M_{bc} + g_{bc}M_{ad}\right) - (c \leftrightarrow d),\\
[M_{ab}, P_c] &= g_{bc}P_a - g_{ac}P_b,\\
[P_a, P_b] &= 0,
\end{aligned} \tag{1.1}$$

has been extended by means of the second rank tensor generator Z_{ab} in the following way:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \left(g_{ad}M_{bc} + g_{bc}M_{ad}\right) - (c \leftrightarrow d),\\
[M_{ab}, P_c] &= g_{bc}P_a - g_{ac}P_b,\\
[P_a, P_b] &= cZ_{ab},\\
[M_{ab}, Z_{cd}] &= \left(g_{ad}Z_{bc} + g_{bc}Z_{ad}\right) - (c \leftrightarrow d),\\
[P_a, Z_{bc}] &= 0, \qquad [Z_{ab}, Z_{cd}] = 0
\end{aligned} \tag{1.2}$$

where c is some constant (Note that, to avoid the double count under summation over the pair antisymmetric indices, we adopt the rules which are illustrated by the following example:

$$[P_a, P_b] = cZ_{ab} = \frac{c}{2}\left(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c\right)Z_{cd} = \sum_{c<d} f_{ab}{}^{cd}Z_{cd} = \frac{1}{2}f_{ab}{}^{cd}Z_{cd}, \tag{1.3}$$

where $f_{ab}{}^{cd}$ are structure constants, and so on.)

Such an extension makes common sense, since it is homomorphic to the usual Poincaré algebra (1.1). Moreover, in the limit $c \to 0$ the algebra (1.2) goes to the semidirect sum of the commutative ideal Z_{ab}, and Poincaré algebra (1.1).

It is remarkable enough that the momentum square Casimir operator of the Poincaré algebra under this extension ceases to be the Casimir operator, and it is generalized by adding the term linearly dependent on the angular momentum

$$P^a P_a + cZ^{ab}M_{ba} \stackrel{\text{def}}{=} X_k h^{kl} X_l, \tag{1.4}$$

where $X_k = \{P_a, Z_{ab}, M_{ab}\}$. Due to this fact, an irreducible representation of the extended algebra (1.2) has to contain the fields with the different masses [4, 8]. This extension with noncommuting momenta has also something in common with the ideas of the papers [9–11] and with the noncommutative geometry idea [12].

It is interesting to note that in spite of the fact that the algebra (1.2) is not semi-simple and therefore has a degenerate Cartan-Killing metric tensor nevertheless there exists another nondegenerate invariant tensor h_{kl} in adjoint representation which corresponds to the quadratic Casimir operator (1.4), where the matrix h^{kl} is inverse to the matrix h_{kl}: $h^{kl}h_{lm} = \delta_m^k$.

There are other quadratic Casimir operators

$$c^2 Z^{ab} Z_{ab}, \tag{1.5}$$

$$c^2 \varepsilon^{abcd} Z_{ab} Z_{cd}. \tag{1.6}$$

Note that the Casimir operator (1.6), dependent on the Levi-Civita tensor ϵ^{abcd}, is suitable only for the $D = 4$ dimensions.

It has also been shown that for the dimensions $D = 2, 3, 4$ the extended Poincaré algebra (1.2) allows the following supersymmetric generalization:

$$\begin{aligned} \{Q_\kappa, Q_\lambda\} &= -d(\sigma^{ab}C)_{\kappa\lambda} Z_{ab}, \\ [M_{ab}, Q_\kappa] &= -(\sigma_{ab}Q)_\kappa, \\ [P_a, Q_\kappa] &= 0, \\ [Z_{ab}, Q_\kappa] &= 0, \end{aligned} \tag{1.7}$$

with the help of the supertranslation generators Q_κ. In (1.7) C is a charge conjugation matrix, d is some constant, and $\sigma_{ab} = 1/4[\gamma_a, \gamma_b]$, where γ_a is the Dirac matrix. Under this supersymmetric generalization the quadratic Casimir operator (1.4) is modified into the following form:

$$P^a P_a + cZ^{ab} M_{ba} - \frac{c}{2d} Q_\kappa (C^{-1})^{\kappa\lambda} Q_\lambda, \tag{1.8}$$

while the form of the rest quadratic Casimir operators (1.5), (1.6) remains unchanged.

In the present paper we propose another possible semi-simple tensor extension of the D-dimensional Poincaré algebra (1.1) which turns out a direct sum of the D-dimensional Lorentz algebra so$(D-1,1)$ and D-dimensional anti-de Sitter (AdS) algebra so$(D-1,2)$. For the case $D = 4$ dimensions we give for this extension a supersymmetric generalization which is a direct sum of the 4-dimensional Lorentz algebra so$(3,1)$ and orthosymplectic algebra osp$(1,4)$ (super-AdS algebra). In the limit this supersymmetrically generalized extension go to the Lie superalgebra (1.2), (1.7).

Let us note that the introduction of the semi-simple extension of the (super) Poincaré algebra is very important for the construction of the models, since it is easier to deal with the nondegenerate space-time symmetry.

2. Semi-Simple Tensor Extension

Let us extend the Poincaré algebra (1.1) in the D dimensions by means of the tensor generator Z_{ab} in the following way:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \left(g_{ad} M_{bc} + g_{bc} M_{ad}\right) - (c \leftrightarrow d), \\
[M_{ab}, P_c] &= g_{bc} P_a - g_{ac} P_b, \\
[P_a, P_b] &= cZ_{ab}, \\
[M_{ab}, Z_{cd}] &= \left(g_{ad} Z_{bc} + g_{bc} Z_{ad}\right) - (c \leftrightarrow d), \\
[Z_{ab}, P_c] &= \frac{4a^2}{c}\left(g_{bc} P_a - g_{ac} P_b\right), \\
[Z_{ab}, Z_{cd}] &= \frac{4a^2}{c}\left[\left(g_{ad} Z_{bc} + g_{bc} Z_{ad}\right) - (c \leftrightarrow d)\right],
\end{aligned} \tag{2.1}$$

where a and c are some constants. This Lie algebra, when the quantities P_a and Z_{ab} are taken as the generators of a homomorphism kernel, is homomorphic to the usual Lorentz algebra. It is remarkable that the Lie algebra (2.1) is *semi-simple* in contrast to the Poincaré algebra (1.1) and extended Poincaré algebra (1.2).

The extended Lie algebra (2.1) has the following quadratic Casimir operators:

$$C_1 = P^a P_a + cZ^{ab} M_{ba} + 2a^2 M^{ab} M_{ab} \overset{\text{def}}{=} X_k H_1^{kl} X_l, \tag{2.2}$$

$$C_2 = c^2 Z^{ab} Z_{ab} + 8a^2\left(cZ^{ab} M_{ba} + 2a^2 M^{ab} M_{ab}\right) \overset{\text{def}}{=} X_k H_2^{kl} X_l, \tag{2.3}$$

$$C_3 = \epsilon^{abcd}\left[c^2 Z_{ab} Z_{cd} + 8a^2\left(cZ_{ba} M_{cd} + 2a^2 M_{ab} M_{cd}\right)\right]. \tag{2.4}$$

Note that in the limit $a \rightarrow 0$ the algebra (2.1) tends to the algebra (1.2) and the quadratic Casimir operators (2.2), (2.3), and (2.4) are turned into (1.4), (1.5), and (1.6), respectively.

The symmetric tensor

$$H^{kl} = sH_1^{kl} + tH_2^{kl} = H^{lk} \tag{2.5}$$

with arbitrary constants s and t is invariant with respect to the adjoint representation

$$H^{kl} = H^{mn}U_m{}^kU_n{}^l. \tag{2.6}$$

Conversely, if we demand the invariance with respect to the adjoint representation of the second rank contravariant symmetric tensor, then we come to the structure (2.5) (see also the relation (32) in [6]).

The semi-simple algebra (2.1)

$$[X_k, X_l] = f_{kl}{}^mX_m \tag{2.7}$$

has the nondegenerate Cartan-Killing metric tensor

$$g_{kl} = f_{km}{}^nf_{ln}{}^m, \tag{2.8}$$

which is invariant with respect to the coadjoint representation

$$g_{kl} = U_k{}^mU_l{}^ng_{mn}. \tag{2.9}$$

With the help of the inverse metric tensor g^{kl}: $g^{kl}g_{lm} = \delta_m^k$ we can construct the quadratic Casimir operator which, as it turned out, has the following expression in terms of the quadratic Casimir operators (2.2) and (2.3):

$$X_kg^{kl}X_l = \frac{1}{8a^2(D-1)}\left[C_1 + \frac{3-2D}{8a^2(D-2)}C_2\right], \tag{2.10}$$

that corresponds to the particular choice of the constants s and t in (2.5).

The extended Poincaré algebra (2.1) can be rewritten in the form

$$[N_{ab}, N_{cd}] = \left(g_{ad}N_{bc} + g_{bc}N_{ad}\right) - (c \leftrightarrow d), \tag{2.11}$$

$$[L_{AB}, L_{CD}] = \left(g_{AD}L_{BC} + g_{BC}L_{AD}\right) - (C \leftrightarrow D), \tag{2.12}$$

$$[N_{ab}, L_{CD}] = 0, \tag{2.13}$$

where the metric tensor g_{AB} has the following nonzero components:

$$g_{AB} = \left\{g_{ab}, g_{D+1D+1} = -1\right\}. \tag{2.14}$$

The generators

$$N_{ab} = M_{ab} - \frac{c}{4a^2} Z_{ab} \tag{2.15}$$

form the Lorentz algebra so$(D-1,1)$, and the generators

$$L_{AB} = \left\{ L_{ab} = \frac{c}{4a^2} Z_{ab}, L_{aD+1} = -L_{D+1a} = \frac{1}{2a} P_a, L_{D+1D+1} = 0 \right\} \tag{2.16}$$

form the algebra so$(D-1,2)$(Note that in the case $D = 4$ we obtain the anti-de Sitter algebra so$(3,2)$.) . The algebra (2.11)–(2.13) is a direct sum so$(D-1,1) \oplus$ so$(D-1,2)$ of the D-dimensional Lorentz algebra and D-dimensional anti-de Sitter algebra, correspondingly.

The quadratic Casimir operators $N_{ab}N^{ab}$, $L_{AB}L^{AB}$, and $\epsilon^{abcd}N_{ab}N_{cd}$ of the algebra (2.11)–(2.13) are expressed in terms of the operators C_1 (2.2), C_2 (2.3), and C_3 (2.4) in the following way:

$$N_{ab}N^{ab} - L_{AB}L^{AB} = \frac{1}{2a^2} C_1, \tag{2.17}$$

$$N_{ab}N^{ab} = \frac{1}{16a^4} C_2, \tag{2.18}$$

$$\epsilon^{abcd} N_{ab} N_{cd} = \frac{1}{16a^4} C_3. \tag{2.19}$$

3. Supersymmetric Generalization

In the case $D = 4$ dimensions the extended Poincaré algebra (2.1) admits the following supersymmetric generalization:

$$\begin{aligned}
\{Q_\kappa, Q_\lambda\} &= -d\left[\frac{2a}{c}(\gamma^a C)_{\kappa\lambda} P_a + (\sigma^{ab} C)_{\kappa\lambda} Z_{ab}\right], \\
[M_{ab}, Q_\kappa] &= -(\sigma_{ab} Q)_\kappa, \\
[P_a, Q_\kappa] &= a(\gamma_a Q)_\kappa, \\
[Z_{ab}, Q_\kappa] &= -\frac{4a^2}{c}(\sigma_{ab} Q)_\kappa,
\end{aligned} \tag{3.1}$$

where Q_κ are the supertranslation generators.

Under such a generalization the Casimir operator (2.2) is modified by adding a term quadratic in the supertranslation generators

$$\widetilde{C}_1 = P^a P_a + c Z^{ab} M_{ba} + 2a^2 M^{ab} M_{ab} - \frac{c}{2d} Q_\kappa (C^{-1})^{\kappa\lambda} Q_\lambda \overset{\text{def}}{=} X_K H_1^{KL} X_L, \tag{3.2}$$

whereas the form of the rest quadratic Casimir operators (2.3) and (2.4) is not changed. In (3.2) $X_K = \{P_a, Z_{ab}, M_{ab}, Q_\kappa\}$ is a set of the generators for also the semi-simple extended superalgebra (2.1), (3.1).

The tensor

$$H^{KL} = vH_1^{KL} + wH_2^{KL} = (-1)^{p_K p_L + p_K + p_L} H^{LK} \tag{3.3}$$

is invariant with respect to the adjoint representation

$$H^{KL} = (-1)^{(p_K+p_M)(p_L+1)} H^{MN} U_N{}^L U_M{}^K, \tag{3.4}$$

where $p_K = p(K)$ is a Grassmann parity of the quantity K. In (3.4) v and w are arbitrary constants and nonzero elements of the matrix H_2^{KL} equal to the elements of the matrix H_2^{kl} followed from (2.3). Again, by demanding the invariance with respect to the adjoint representation of the second rank contravariant tensor $H^{KL} = (-1)^{p_K p_L + p_K + p_L} H^{LK}$, we come to the structure (3.4) (see also the relation (32) in [6]).

The semi-simple Lie superalgebra (2.1) (3.1) has the nondegenerate Cartan-Killing metric tensor G_{KL} (see the relation (A.6) in the Appendix A) which is invariant with respect to the coadjoint representation

$$G_{KL} = (-1)^{p_K(p_L+p_N)} U_L{}^N U_K{}^M G_{MN}. \tag{3.5}$$

With the use of the inverse metric tensor G^{KL},

$$G^{KL} G_{LM} = \delta_M^K, \tag{3.6}$$

we can construct the quadratic Casimir operator (see the relation (A.11) in the Appendix A) which takes the following expression in terms of the Casimir operators (2.3) and (3.2):

$$X_K G^{KL} X_L = \frac{1}{20a^2}\left(\tilde{C}_1 - \frac{9}{32a^2} C_2\right), \tag{3.7}$$

that meets the particular choice of the constants v and w in (3.4).

In the $D = 4$ case the extended superalgebra (2.1), (3.1) can be rewritten in the form of the relations (2.11)–(2.13) and the following ones:

$$\{Q_\kappa, Q_\lambda\} = -\frac{4a^2 d}{c} (\Sigma^{AB} C)_{\kappa\lambda} L_{AB}, \tag{3.8}$$

$$[L_{AB}, Q_\kappa] = -(\Sigma_{AB} Q)_\kappa, \tag{3.9}$$

$$[N_{ab}, Q_\kappa] = 0, \tag{3.10}$$

where

$$\Sigma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B], \quad \Gamma_A = \{\gamma_a\gamma_5, \gamma_5\},$$
$$\{\gamma_a, \gamma_b\} = 2g_{ab}, \quad g_{ab} = \mathrm{diag}(-1,1,1,1), \tag{3.11}$$
$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3.$$

The generators N_{ab} (2.15) form the Lorentz algebra so$(3,1)$ and the generators L_{AB} (2.16), Q_κ form the orthosymplectic algebra osp$(1,4)$. We see that superalgebra (2.11)–(2.13), (3.8)–(3.10) is a direct sum so$(3,1) \oplus$ osp$(1,4)$ of the 4-dimensional Lorentz algebra and 4-dimensional super-AdS algebra, respectively.

In this case the Casimir operator (2.17) is modified by adding a term quadratic in the supertranslation generators

$$N_{ab}N^{ab} - L_{AB}L^{AB} - \frac{c}{4a^2d}Q_\kappa\left(C^{-1}\right)^{\kappa\lambda}Q_\lambda = \frac{1}{2a^2}\widetilde{C}_1, \tag{3.12}$$

while the form of the quadratic Casimir operators (2.18) and (2.19) is not changed.

4. Conclusion

Thus, we proposed the semi-simple second rank tensor extension of the Poincaré algebra in the arbitrary dimensions D and super-Poincaré algebra in the $D = 4$ dimensions. It is very important, since under construction of the models, it is more convenient to deal with the nondegenerate space-time symmetry. We also constructed the quadratic Casimir operators for the semi-simple extended Poincaré and super Poincaré algebra.

It is interesting to develop the models based on these extended algebra. The work in this direction is in progress.

Appendix

A. Properties of Lie Superalgerbra

Permutation relations for the generators X_K of Lie superalgebra are

$$[X_K, X_L\} \stackrel{\mathrm{def}}{=} X_KX_L - (-1)^{pKpL}X_LX_K = f_{KL}{}^{M}X_M. \tag{A.1}$$

Structure constants $f_{KL}{}^{M}$ have the Grassmann parity

$$p\left(f_{KL}{}^{M}\right) = p_K + p_L + p_M = 0 \pmod 2, \tag{A.2}$$

following symmetry property:

$$f_{KL}{}^{M} = -(-1)^{pKpL}f_{LK}{}^{M} \tag{A.3}$$

and obey the Jacobi identities

$$\sum_{(KLM)} (-1)^{p_K p_M} f_{KN}{}^{P} f_{LM}{}^{N} = 0, \tag{A.4}$$

where the symbol (KLM) means a cyclic permutation of the quantities K, L, and M. In the relations (A.1)–(A.4) every index K takes either a Grassmann-even value $k(p_k = 0)$ or a Grassmann-odd one $\kappa(p_\kappa = 1)$. The relations (A.1) have the following components:

$$\begin{aligned} [X_k, X_l] &= f_{kl}{}^{m} X_m, \\ \{X_\kappa, X_\lambda\} &= f_{\kappa\lambda}{}^{m} X_m, \\ [X_k, X_\lambda] &= f_{k\lambda}{}^{\mu} X_\mu. \end{aligned} \tag{A.5}$$

The Lie superalgebra possesses the Cartan-Killing metric tensor

$$G_{KL} = (-1)^{p_N} f_{KM}{}^{N} f_{LN}{}^{M} = (-1)^{p_K p_L} G_{LK} = (-1)^{p_K} G_{LK} = (-1)^{p_L} G_{LK}, \tag{A.6}$$

which components are

$$\begin{aligned} G_{kl} &= f_{km}{}^{n} f_{ln}{}^{m} - f_{k\mu}{}^{\nu} f_{l\nu}{}^{\mu}, \\ G_{\kappa\lambda} &= f_{\kappa\mu}{}^{m} f_{\lambda m}{}^{\mu} - f_{\kappa m}{}^{\mu} f_{\lambda\mu}{}^{m}, \\ G_{k\lambda} &= 0. \end{aligned} \tag{A.7}$$

As a consequence of the relations (A.3) and (A.4) the tensor with low indices

$$f_{KLM} = f_{KL}{}^{N} G_{NM} \tag{A.8}$$

has the following symmetry properties:

$$f_{KLM} = -(-1)^{p_K p_L} f_{LKM} = -(-1)^{p_K p_M} f_{KML}. \tag{A.9}$$

For a semi-simple Lie superalgebra the Cartan-Killing metric tensor is nondegenerate and therefore there exists an inverse tensor G^{KL},

$$G_{KL} G^{LM} = \delta_K^M. \tag{A.10}$$

In this case, as a result of the symmetry properties (A.9), the quantity

$$X_K G^{KL} X_L \tag{A.11}$$

is a Casimir operator

$$\left[X_K G^{KL} X_L,\ X_M\right] = 0. \tag{A.12}$$

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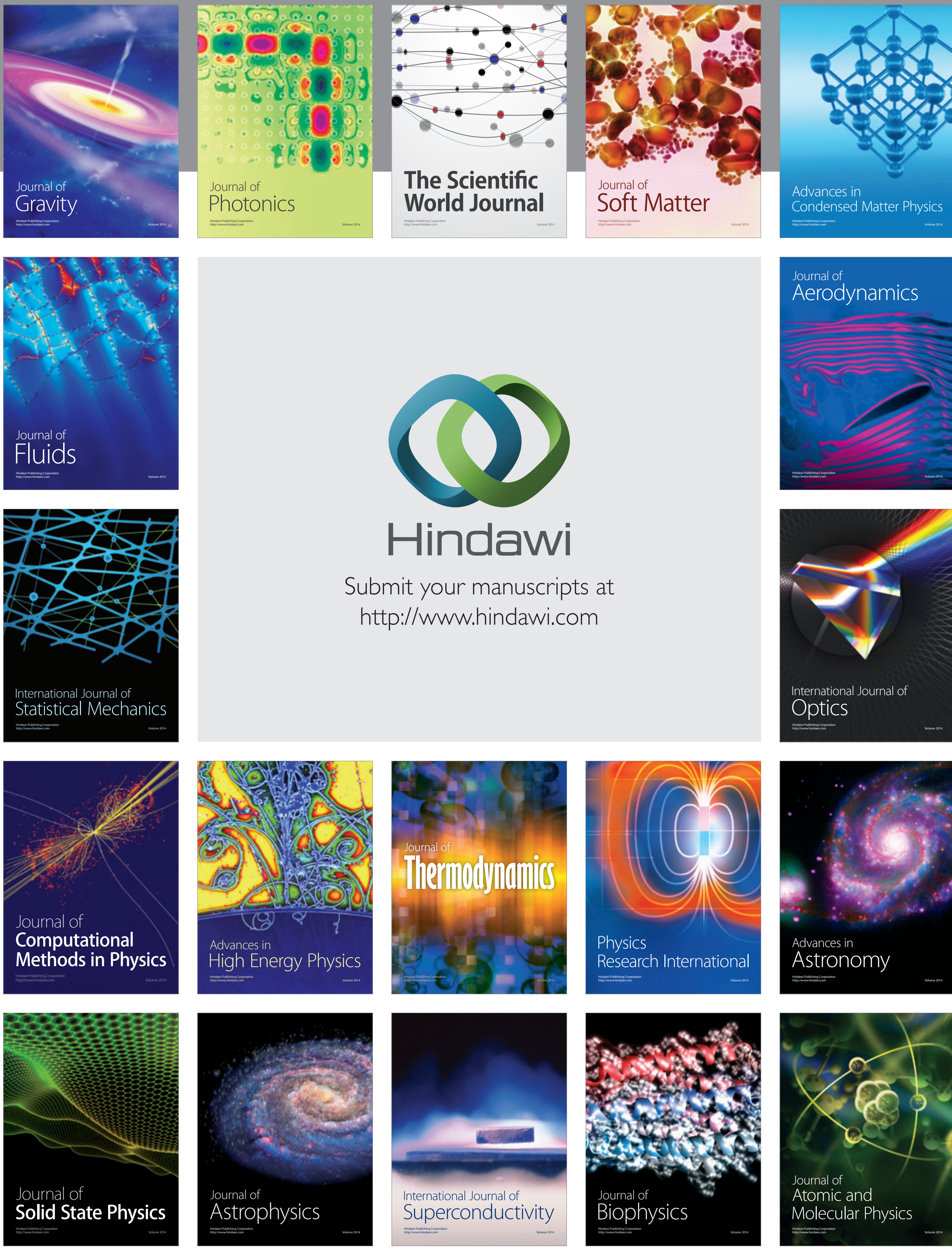
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