Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2011, Article ID 479038, 18 pages
doi:10.1155/2011/479038

Research Article

On $\mathcal{R}^4$ Terms and MHV Amplitudes in $\mathcal{N} = 5, 6$ Supergravity Vacua of Type II Superstrings

Massimo Bianchi[1, 2]

[1] *Dipartimento di Fisica, INFN Sezione di Roma "Tor Vergata", Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy*
[2] *Physics Department, Theory Unit, CERN, 1211 Geneva 23, Switzerland*

Correspondence should be addressed to Massimo Bianchi, massimo.bianchi@roma2.infn.it

Received 14 April 2011; Accepted 23 June 2011

Academic Editor: Yang-Hui He

We compute one-loop threshold corrections to $\mathcal{R}^4$ terms in $\mathcal{N} = 5, 6$ supergravity vacua of Type II superstrings. We then discuss nonperturbative corrections generated by asymmetric D-brane instantons. Finally we derive generating functions for MHV amplitudes at tree level in $\mathcal{N} = 5, 6$ supergravities.

1. Introduction

$\mathcal{N} = 5, 6$ supergravities in $D = 4$ enjoy many of the remarkable properties of $\mathcal{N} = 8$ supergravity. Their massless spectra are unique and consist solely of the supergravity multiplets. Their R-symmetries are not anomalous [1]. Regular BH solutions can be found whereby the scalars are stabilized at the horizon by the attractor mechanism (for a recent review see, e.g., [2]). It is thus tempting to conjecture that if pure $\mathcal{N} = 8$ supergravity turned out to be UV finite [3–7] then $\mathcal{N} = 5, 6$ supergravities should be so, too.

As shown in [8–10], Type II superstrings or M-theory accommodate $\mathcal{N} = 8$ supergravity in such a way as to include nonperturbative states that correspond to singular BH solutions in $D = 4$. The same is true for $\mathcal{N} = 5, 6$ supergravities. While the embedding of $\mathcal{N} = 8$ supergravity corresponds to simple toroidal compactifications, the embedding of $\mathcal{N} = 5, 6$ supergravities, pioneered by Ferrara and Kounnas in [11] and recently reviewed in [12], requires asymmetric orbifolds [13, 14] or free fermion constructions [15–20].

The inclusion of BPS states, whose possible singular behavior from a strict 4D viewpoint is resolved from a higher-dimensional perspective, generates higher derivative corrections to the low-energy effective action. In particular a celebrated $\mathcal{R}^4$ term appears that spoils the continuous noncompact symmetry of "classical" supergravity. Absence of such a term has been recently shown for pure $\mathcal{N} = 8$ supergravity in [21]. In superstring theory, the $\mathcal{R}^4$ term receives contribution at tree level, one loop, and from nonperturbative effects associated to D-instantons [22] and other wrapped branes [23]. Proposals for the relevant modular form of the $E_{7(7)}(Z)$ U-duality group have been recently put forward in [24–26] that seem to satisfy all the checks.

In this paper we consider one-loop threshold corrections to the same kind of terms in superstring models with $\mathcal{N} = 5, 6$ supersymmetry in $D = 4$ and $\mathcal{N} = 6$ in $D = 5$. After excluding $\mathcal{R}^2$ terms ($\mathcal{R}^3$ terms cannot be supersymmetrized on shell when all particles are in the supergravity multiplet [21]), we will derive formulae for the "perturbative" threshold corrections. In $D = 4$ we will also discuss other MHV amplitudes (for a recent review see, e.g., [27]) that can be obtained by orbifold techniques from the generating function of $\mathcal{N} = 8$ supergravity amplitudes [28].

Aim of the analysis is threefold. First, we would like to show that $\mathcal{N} = 5, 6$ supersymmetric models in $D = 4$ behave very much as their common $\mathcal{N} = 8$ supersymmetric parent. The threshold corrections that we find may be taken as evidence that, as in the $\mathcal{N} = 8$ case, superstring calculations do not reproduce field theory results, where such $\mathcal{R}^4$ corrections are absent as a result of the unbroken (anomaly free) continuous U-duality symmetry as in the $\mathcal{N} = 8$ case [1]. This is in line with the nondecoupling in Type II superstrings of BPS states that are singular from the strict 4-dimensional supergravity perspective [8–10].

Second, (gauged) $\mathcal{N} = 5, 6$ supergravities have played a crucial role in the recent understanding of M2-brane dynamics [29–32], and nonperturbative tests may be refined by considering the effects of world-sheet instantons in CP^3 [33–36] along the lines of our present (ungauged) analysis. Finally, in addition to world-sheet instantons, D-brane instantons corresponding to Euclidean bound states of "exotic" D-branes should contribute to generalize "standard" D-brane instanton calculus to Left-Right asymmetric backgrounds.

Plan of the paper is as follows. In Section 1, we briefly review $\mathcal{N} = 5, 6$ supergravities in $D = 4, 5$ and their embedding in Type II superstrings. We then pass to consider in Section 2 a 4-graviton amplitude at one loop which allows to derive the "perturbative" threshold corrections to $\mathcal{R}^4$ terms, thus excluding $\mathcal{R}^2$ terms. For simplicity, we only give the explicit result for $\mathcal{N} = 6$ in $D = 5$ in Section 3 and sketch how to complete the nonperturbative analysis by including asymmetric D-brane instantons [12] in Section 4. Finally, in Section 5 we consider MHV amplitudes in $\mathcal{N} = 5, 6$ supergravities in $D = 4$ and show how they can be obtained at tree level by orbifold techniques from the generating function for MHV amplitudes in $\mathcal{N} = 8$ supergravity [28]. Section 6 contains a summary of our results and directions for further investigation.

2. Type II Superstring Models with $\mathcal{N} = 5, 6$ in $D = 4, 5$

Let us briefly recall how $\mathcal{N} = 5, 6$ supergravities can be embedded in String Theory. The highest dimension where classical $\mathcal{N} = 6$ supergravity with 24 supercharges can be defined is $D = 6$. However the resulting $\mathcal{N} = (2, 1)$ theory is anomalous and thus inconsistent at the quantum level [37]. So we are led to consider $D = 5$ and then reduce to $D = 4$. $\mathcal{N} = 5$ supergravity with 20 supercharges can only be defined as $D = 4$ and lower. Although we will only focus on $\mathcal{R}^4$ terms in $D = 4$ the parent $D = 5$ theory is instrumental to the identification of the relevant BPS instantons.

2.1. $\mathcal{N} = 6 = 2_L + 4_R$ *Supergravity in* $D = 5$

The simplest way to embed $\mathcal{N} = 6$ in Type II superstrings is to quotient a toroidal compactification $T^5 = T^4 \times S^1$ by a chiral Z_2 twist of the L-movers (T-duality) on four internal directions

$$X_L^i \longrightarrow -X_L^i, \quad \Psi_L^i \longrightarrow -\Psi_L^i, \quad i = 6, 7, 8, 9 \tag{2.1}$$

accompanied by an order-two shift that makes twisted states massive. As a result half of the supersymmetries in the L-moving sector are broken. The perturbative spectrum is coded in the one-loop torus partition function.

In the untwisted sector, one finds

$$\mathcal{T}_u = \frac{1}{2}\left\{ (Q_o + Q_v)\overline{Q}\Lambda_{5,5}\begin{bmatrix}0\\0\end{bmatrix} + (Q_o - Q_v)(X_o - X_v)\overline{Q}\Lambda_{1,5}\begin{bmatrix}0\\1\end{bmatrix} \right\}, \tag{2.2}$$

where $X_o - X_v = 4\eta^2/\theta_2^2$ (with η denoting Dedekind's function and $\theta_{1,\ldots 4}$ denoting Jacobi's elliptic functions) describes the effect of the Z_2 projection on four internal L-moving bosons, while

$$\Lambda_{l,r}\begin{bmatrix}a\\b\end{bmatrix} = \sum_{p_L, p_R} e^{i\pi[a_L p_L - a_R p_R]} q^{(1/2)(p_L + (1/2)b_L)^2} \overline{q}^{(1/2)(p_R + (1/2)b_R)^2} \tag{2.3}$$

are (shifted) Lorentzian lattice sums of signature (l, r) and $Q = V_8 - S_8$, $Q_o = V_4 O_4 - S_4 S_4$, $Q_v = O_4 V_4 - C_4 C_4$, with O_n, V_n, S_n, C_n the characters of $SO(n)$ at level $\kappa = 1$ (for n odd S_n coincides with C_n and will be denoted by Σ_n).

At the massless level, in $D = 5$ notation with $SO(3)$ little group, one finds

$$\begin{aligned}(V_3 + O_3 - 2\Sigma_3) \times \left(\overline{V}_3 + 5\overline{O}_3 - 4\overline{\Sigma_3}\right)& \\ \longrightarrow (g + b_2 + \phi)_{\text{NS-NS}} + 6A_{\text{NS-NS}} + 5\phi_{\text{NS-NS}} + 8A_{R\text{-}R} + 8\phi_{R\text{-}R} - \text{Fermi}&\end{aligned} \tag{2.4}$$

that form the $\mathcal{N} = 6$ supergravity multiplet in $D = 5$

$$\text{SG}^{D=5}_{\mathcal{N}=6} = \{g_{\mu\nu}, 6\psi_\mu, 15A_\mu, 20\chi, 14\varphi\}. \tag{2.5}$$

The R-symmetry is $Sp(6)$ while the "hidden" noncompact symmetry is $SU^*(6)$, of dimension 35 and rank 3 generated by 6×6 matrices of the form $Z = (Z_1, Z_2; -\overline{Z}_2, \overline{Z}_1)$ with $\text{Tr}(Z_1 + \overline{Z}_1) = 0$.

For later purposes, let us observe that the 128 massless states of $\mathcal{N} = 6$ supergravity in $D = 5$ are given by the tensor product of the 8 massless states of $\mathcal{N} = 2$ SYM (for the Left-movers) and the 16 massless states of $\mathcal{N} = 4$ SYM (for the Right-movers), namely,

$$\mathrm{SG}^{D=5}_{\mathcal{N}=6} = \mathrm{SYM}^{D=5}_{\mathcal{N}=2} \otimes \mathrm{SYM}^{D=5}_{\mathcal{N}=4} = \{A_\mu, 2\lambda, \phi\}_L \otimes \left\{\widetilde{A}_\nu, 4\widetilde{\lambda}, 5\widetilde{\phi}\right\}_R. \tag{2.6}$$

After dualizing all massless 2 forms into vectors, the $\mathbf{15} = 7_{\text{NS-NS}} + 8_{R\text{-}R}$ vectors transform according to the antisymmetric tensor of $SU^*(6)$. The $14 = 1_{\text{NS-NS}} + 5_{\text{NS-NS}} + 8_{R\text{-}R}$ scalars parameterize the moduli space

$$\mathcal{M}^{D=5}_{\mathcal{N}=6} = \frac{SU^*(6)}{Sp(6)}. \tag{2.7}$$

By world-sheet modular transformations (first S and then T) one finds the contribution of the twisted sector

$$\mathcal{T}_t = \frac{1}{2}\left\{(Q_s + Q_c)(X_s + X_c)\overline{Q}\Lambda_{1,5}\begin{bmatrix}1\\0\end{bmatrix} + (Q_s - Q_c)(X_s - X_c)\overline{Q}\Lambda_{1,5}\begin{bmatrix}1\\1\end{bmatrix}\right\}, \tag{2.8}$$

where $X_s + X_c = 4\eta^2/\theta_4^2$, $X_s - X_c = 4\eta^2/\theta_3^2$, $Q_s = O_4S_4 - C_4O_4$ (massless), $Q_c = V_4C_4 - S_4V_4$ (massive). Due to the (L-R symmetric) Z_2 shift, the massless spectrum receives no contribution from the twisted sector. Nonperturbative states associated to L-R asymmetric bound states of D-branes were studied in [12]. There are several other ways to embed $\mathcal{N} = 6$ supergravity in Type II superstrings, reviewed in [12].

2.2. $\mathcal{N} = 6$ Supergravities in $D = 4$

Reducing on another circle with or without further shifts yields $\mathcal{N} = 6$ supergravity in $D = 4$ [11].

The massless spectrum is given by

$$\begin{aligned}&(V_2 + 2O_2 - 2S_2 - 2C_2) \times \left(\overline{V}_2 + 6\overline{O}_2 - 4\overline{S}_2 - 4\overline{C}_2\right)\\ &\qquad \longrightarrow (g + b + \phi)_{\text{NS-NS}} + 8A_{\text{NS-NS}} + 12\phi_{\text{NS-NS}} + 8A_{R\text{-}R} + 16\phi_{R\text{-}R} - \text{Fermi}\end{aligned} \tag{2.9}$$

and gives rise to the $\mathcal{N} = 6$ supergravity multiplet in $D = 4$

$$\mathrm{SG}^{D=4}_{\mathcal{N}=6} = \{g_{\mu\nu}, 6\psi_\mu, 16A_\mu, 26\chi, 30\varphi\}. \tag{2.10}$$

For later purposes, let us observe that the 128 massless states of $\mathcal{N} = 6$ supergravity in $D = 4$ are given by the tensor product of the 8 massless states of $\mathcal{N} = 2$ SYM (for the Left-movers) and the 16 massless states of $\mathcal{N} = 4$ SYM (for the Right-movers), namely,

$$\mathrm{SG}^{D=4}_{\mathcal{N}=6} = \mathrm{SYM}^{D=4}_{\mathcal{N}=2} \otimes \mathrm{SYM}^{D=4}_{\mathcal{N}=4} = \{A_\mu, 2\lambda, 2\phi\}_L \otimes \left\{\widetilde{A}_\nu, 4\widetilde{\lambda}, 6\widetilde{\phi}\right\}_R. \tag{2.11}$$

The hidden noncompact symmetry is $SO^*(12)$, of dimension 66 and rank 3 generated by 12×12 matrices of the form $Z = (Z_1, Z_2; -\overline{Z}_2, \overline{Z}_1)$ with $Z_1 = -Z_1^t$ and Z_2 hermitean. They satisfy $L^\dagger \mathcal{Q} L = \mathcal{Q}$ with $\mathcal{Q} = -\mathcal{Q}^t = -\mathcal{Q}^\dagger$ the symplectic metric in 12D. After dualizing all massless 2 forms into axions, the $30 = 2_{\text{NS-NS}} + 12_{\text{NS-NS}} + 16_{R\text{-}R}$ scalar parameterize the moduli space

$$\mathcal{M}^{D=4}_{\mathcal{N}=6} = \frac{SO^*(12)}{U(6)}. \tag{2.12}$$

The $16 = 8_{\text{NS-NS}} + 8_{R\text{-}R}$ vectors together with their magnetic duals transform according to the **32**-dimensional chiral spinor representation of $SO^*(12)$.

Due to the (L-R symmetric) Z_2 shift, the massless spectrum receives no contribution from the twisted sector. Nonperturbative states associated to L-R asymmetric bound states of D-branes were studied in [12].

2.3. $\mathcal{N} = 5 = 1_L + 4_R$ Supergravity in $D = 4$

The highest dimension where $\mathcal{N} = 5$ supergravity exists is $D = 4$. In $D = 5$ because one cannot impose a symplectic Majorana condition on an odd number of spinors. A simple way to realize $\mathcal{N} = 5 = 1_L + 4_R$ supergravity in $D = 4$ is to combine $Z_2^L \times Z_2^L$ twists, acting by T-duality along T^4_{6789} and T^4_{4589}, with order two shifts, that eliminate massless twisted states. In [11], "minimal" $\mathcal{N} = 5$ superstring solutions of this kind have been classified into four classes which correspond to different choices of the basis sets of free fermions or inequivalent choices of shifts in the orbifold language.

Due to the uniqueness of $\mathcal{N} = 5$ supergravity in $D = 4$, all models display the same massless spectrum

$$\text{SG}^{D=4}_{\mathcal{N}=5} = \{g_{\mu\nu}, 5\psi_\mu, 10A_\mu, 11\chi, 10\phi\}. \tag{2.13}$$

For later purposes, let us observe that the 64 massless states of $\mathcal{N} = 5$ supergravity in $D = 4$ are given by the tensor product of the 4 massless states of $\mathcal{N} = 1$ SYM (for the Left-movers) and the 16 massless states of $\mathcal{N} = 4$ SYM (for the Right-movers), namely,

$$\text{SG}^{D=4}_{\mathcal{N}=5} = \text{SYM}^{D=4}_{\mathcal{N}=1} \otimes \text{SYM}^{D=4}_{\mathcal{N}=4} = \{A_\mu, \lambda\}_L \otimes \left\{\widetilde{A}_\nu, 4\widetilde{\lambda}, 6\widetilde{\phi}\right\}_R. \tag{2.14}$$

The massless scalars parameterize the moduli space

$$\mathcal{M}^{D=4}_{\mathcal{N}=5} = \frac{SU(5,1)}{U(5)}. \tag{2.15}$$

The graviphotons together with their magnetic duals transform according to the **20** complex (3-index totally antisymmetric tensor) representation of $SU(5,1)$.

3. Four-Graviton One-Loop Amplitude

Since $\mathcal{N} = 5, 6$ supergravities can be obtained as asymmetric orbifolds of tori, tree-level scattering amplitudes of untwisted states such as gravitons are identical to the corresponding amplitudes in the parent $\mathcal{N} = 8$ theory. In particular, denoting by $f_{\mathcal{R}^4}^{\mathcal{N}=5,6}(\varphi)$ the moduli dependent coefficient function of the $\mathcal{R}^4$ term, one has

$$f_{\mathcal{R}^4}^{\mathcal{N}=5,6} = \frac{2}{n}\zeta(3)\frac{V(\mathbf{T}^d)}{g_s^2\ell_s^2} + \frac{\mathcal{I}_{d,d}^{\mathcal{N}=8}}{n\ell_s^2} + \cdots, \tag{3.1}$$

where n is the order of the orbifold group, that reduces the volume of $\mathbf{T}^d$ with $d = 5, 6$ to the volume of the orbifold, $\ell_s^2 = \alpha'$ and $\cdots$ stands for nonperturbative terms. The one-loop threshold integral is given by

$$\mathcal{I}_{d,d}^{\mathcal{N}=8} = (2\pi)^d \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\left[\tau_2^{d/2}\Gamma_{d,d}(G, B; \tau) - \tau_2^{d/2}\right] = 2\pi^{2-d/2}\Gamma\left(\frac{d}{2} - 1\right)\mathcal{E}_{\mathbf{v=2d},s=d/2-1}^{SO(d,d|Z)}, \tag{3.2}$$

where

$$\mathcal{E}_{\mathbf{v=2d},s=d/2-1}^{SO(d,d|Z)} = \sum_{\vec{m},\vec{n}:\vec{m}\cdot\vec{n}=0}\left[(\vec{m} + B\vec{n})^t G^{-1}(\vec{m} + B\vec{n}) + \vec{n}^t G\vec{n}\right]^{-d+2} \tag{3.3}$$

is a constrained Epstein series that encodes the contribution of perturbative $1/2$ BPS, states that is, those satisfying $\vec{m}\cdot\vec{n} = 0$. The subtraction eliminates IR divergences, that is the terms with $\vec{m} = \vec{n} = 0$. For $\mathcal{N} = 5, 6$ the contribution of the $(r, s) = (0, 0)$ "untwisted" sector is up to a factor $1/n$ the same as in toroidal Type II compactifications with restricted metric G_{ij} and antisymmetric tensor B_{ij}.

In the following we will focus on the contribution of the "twisted" sectors (we write "twisted" in quotes, since the terminology includes projections of the untwisted sector, i.e., amplitudes with $r = 0$ and $s = 1, \ldots, n-1$) with $(r, s) \neq (0, 0)$.

Recall that the partition function reads

$$\mathcal{Z} = \overline{Q}\frac{1}{n}\sum_{r,s}^{0,n-1}\sum_{\alpha}\frac{\theta_\alpha(0)}{\eta^3}\prod_{I=1}^{3}\frac{\theta_\alpha(u_{rs}^I)}{\theta_1(u_{rs}^I)}\Gamma\begin{bmatrix} r \\ s \end{bmatrix}, \tag{3.4}$$

where u_{rs}^I encode the effect of the Left-moving twist on the three complex internal directions, while $\Gamma[{}^r_s]$ denote the twisted and shifted lattice sums.

Following the analysis in [38] for one-loop scattering of vector bosons in unoriented D-brane worlds and exploiting the "factorization" of world-sheet correlation functions one has

$$\mathcal{A}_{4h} = \frac{1}{n}\sum_{r,s}^{0,n-1}\int\frac{d^2\tau}{\tau_2^2}\Gamma\begin{bmatrix} r \\ s \end{bmatrix}\mathcal{C}_{4v}^{L}\mathcal{C}_{4v}^{R}. \tag{3.5}$$

Since in both $\mathcal{N} = 5, 6$ cases the orbifold projection only acts by a shift of the lattice on the Left-movers, that is, preserves all four space-time supersymmetries, their contribution is simply

$$\mathcal{C}^L_{4v} = \text{const} \tag{3.6}$$

after summing over spin structures. In the terminology of [38] only terms with 4 fermion pairs contribute. Recall that the graviton vertex in the $q = 0$ superghost picture reads

$$V_h = h_{\mu\nu}\left(\partial X^\mu + ik\psi\psi^\mu\right)\left(\overline{\partial}\widetilde{X}^\nu + ik\widetilde{\psi}\widetilde{\psi}^\nu\right)e^{ikX}, \tag{3.7}$$

and, for fixed graviton helicity (henceforth we use $D = 4$ notation but the analysis is valid in $D = 5$ too), one can exploit "factorization" of the physical polarization tensor

$$h^{(2\sigma)}_{\mu\nu} = a^{(\sigma)}_\mu a^{(\sigma)}_\nu \tag{3.8}$$

in terms of photon polarization vectors.

In the R-moving sector however, the orbifold projection breaks $1/2$ ($\mathcal{N} = 6$) or $3/4$ ($\mathcal{N} = 5$) of the original four space-time supersymmetries. Correlation functions of two and three fermion bilinears will be nonvanishing, too.

For two fermion bilinears one has [38]

$$\left\langle \partial X^{\mu_1}\partial X^{\mu_2} k_3\psi\psi^{\mu_3} k_4\psi\psi^{\mu_4}\right\rangle \left[\eta^{\mu_1\mu_2}\partial_1\partial_2\mathcal{G}_{12} - \sum_{i\neq 1} k_i^{\mu_1}\partial_1\mathcal{G}_{1i}\sum_{j\neq 2} k_j^{\mu_2}\partial_2\mathcal{G}_{1j}\right] = \left[k_3k_4\eta^{\mu_3\mu_4} - k_3^{\mu_4}k_4^{\mu_3}\right], \tag{3.9}$$

where $\mathcal{G}_{ij}$ denotes the scalar propagator on the torus (with $\alpha' = 2$)

$$\mathcal{G}_{z,w} = -\log\frac{|\theta_1(z-w)|}{|\theta_1'(0)|} - \pi\frac{\operatorname{Im}(z-w)^2}{\operatorname{Im}\tau}. \tag{3.10}$$

Similarly, for three fermion bilinears, one finds [38]

$$\left\langle \partial X^{\mu_1} k_2\psi\psi^{\mu_2} k_3\psi\psi^{\mu_3} k_4\psi\psi^{\mu_4}\right\rangle = \sum_{i\neq 1} k_i^{\mu_1}\partial_1\mathcal{G}_{1i}\left[k_2k_3k_4^{\mu_2}\eta^{\mu_3\mu_4} - \cdots\right]\omega_{234} \tag{3.11}$$

with $\omega_{234} = \partial\log\theta_1(z_{23}) + \partial\log\theta_1(z_{34}) + \partial\log\theta_1(z_{42})$.

For four fermion bilinears, disconnected contractions yield [38]

$$\begin{aligned}\left\langle k_1\psi\psi^{\mu_1} k_2\psi\psi^{\mu_2} k_3\psi\psi^{\mu_3} k_4\psi\psi^{\mu_4}\right\rangle_{\text{disc}} = \Big\{ &\left[k_1k_2\eta^{\mu_1\mu_2} - k_1^{\mu_2}k_2^{\mu_1}\right]\left[k_3k_4\eta^{\mu_3\mu_4} - k_3^{\mu_4}k_4^{\mu_3}\right] \\ &\times\left(\wp_{12} + \wp_{34} - \Delta_{rs}\right) + \cdots\Big\},\end{aligned} \tag{3.12}$$

where $\wp$ is Weierstrass function

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{m,n}{}' \frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2} \\ &= -\partial_z^2 \log\theta_1(z) - 2\eta_1 = -2\partial_z^2 \mathcal{G}(z,\bar{z}) - \frac{\pi^2}{3}\widehat{E}_2\end{aligned} \tag{3.13}$$

with $\eta_1 = -\theta_1'''/6\theta_1'$ and $\widehat{E}_2$ the nonholomorphic modular form of weight 2 (Eisenstein series). Weierstrass function satisfies $\wp(1/2) = e_1$, $\wp(\tau/2) = e_2$, $\wp(1/2+\tau/2) = e_3$ with $e_1 + e_2 + e_3 = 0$.

Finally, connected contractions of four fermion bilinears yield [38]

$$\langle k_1\psi\psi^{\mu_1} k_2\psi\psi^{\mu_2} k_3\psi\psi^{\mu_3} k_4\psi\psi^{\mu_4}\rangle_{\text{conn}} = \left[k_1^{\mu_4}k_2^{\mu_1}k_3^{\mu_2}k_4^{\mu_3} \pm \cdots\right](\wp_{13} - \omega_{123}\omega_{143} + \Delta_{rs}), \tag{3.14}$$

where, for $\mathcal{N} = 6$,

$$\Delta_{rs} = \wp(u_{rs}) \tag{3.15}$$

while, for $\mathcal{N} = 5$,

$$\Delta_{rs} = 3\eta_1 + \frac{1}{6}\frac{\mathcal{H}'''(u_{rs})}{\mathcal{H}'(u_{rs})} \tag{3.16}$$

with $\mathcal{H}'/\mathcal{H} = \sum_I \partial \log\theta_1(u^I_{rs})$, which is clearly moduli independent, since no NS-NS moduli survive except for the axio-dilaton. Dependence on R-R moduli and the axio-dilaton is expected to be generated by L-R asymmetric bound states of Euclidean D-branes and NS5-branes.

3.1. World-Sheet Integrations

Worldsheet integrations can be performed with the help of $\int d^2z\partial_z\mathcal{G}_{zw} = 0 = \int d^2z\partial_z^2\mathcal{G}_{zw}$ as well as of

$$\begin{gathered}\int d^2z d^2w (\partial_z\mathcal{G}_{zw})^2 = -\tau_2\widehat{E}_2\frac{\pi^2}{3}, \\ \int d^2z d^2w \left[\eta^{\mu_1\mu_2}\partial_1\partial_2\mathcal{G}_{12}k_1k_2\mathcal{G}_{12} - \sum_{i\neq 1}k_i^{\mu_1}\partial_1\mathcal{G}_{1i}\sum_{j\neq 2}k_j^{\mu_2}\partial_2\mathcal{G}_{1j}\right] = -\tau_2\widehat{E}_2\ \frac{\pi^2}{3}\left[\eta^{\mu_1\mu_2}k_1k_2 - k_1^{\mu_2}k_2^{\mu_1}\right].\end{gathered} \tag{3.17}$$

For $\mathcal{N} = 6 = 4_L + 2_R$, setting $f_{\mu\nu}^{L/R} = k_\mu a_\nu^{L/R} - k_\nu a_\mu^{L/R}$, one has

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{twist}} = \frac{1}{n}\sum_{r,s}{}' \int d^2\tau \Gamma\begin{bmatrix} r \\ s \end{bmatrix} \langle f_1 f_2 f_3 f_4 \rangle_L^{\text{MHV}} \\ \times \Big\{ 4\left[(f_1 f_2)(f_3 f_4) + \cdots\right]_R \frac{\pi^2}{3}\widehat{E}_2 \\ + \left[(f_1 f_2)(f_3 f_4) + \cdots\right]_R \left(-2\frac{\pi^2}{3}\widehat{E}_2 + \wp(u_{rs})\right) \\ + \left[(f_1 f_2 f_3 f_4) + \cdots\right]_R \left(-2\frac{\pi^2}{3}\widehat{E}_2 - \wp(u_{rs})\right)\Big\},\end{aligned} \tag{3.18}$$

where, including all permutations,

$$\begin{aligned}\langle f_1 f_2 f_3 f_4 \rangle^{\text{MHV}} = (f_1 f_2 f_3 f_4) + (f_1 f_3 f_4 f_2) + (f_1 f_4 f_2 f_3) \\ - 2(f_1 f_2)(f_3 f_4) - 2(f_1 f_3)(f_4 f_2) - 2(f_1 f_4)(f_2 f_3)\end{aligned} \tag{3.19}$$

is the structure that appears in 4-pt vector boson amplitudes, that are necessarily MHV (Maximally Helicity Violating) in $D = 4$ (in $D = 5$ there is more than one "helicity," but the tensor structure has the same form [39, 40]).

Combining the R-moving contributions one eventually finds

$$\mathcal{L}_{\text{eff}}^{\text{twist}} = \langle \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \mathcal{R}_4 \rangle^{\text{MHV}} \frac{1}{n}\sum_{r,s}{}' \int d^2\tau \Gamma\begin{bmatrix} r \\ s \end{bmatrix} \left(+2\frac{\pi^2}{3}\widehat{E}_2 - \wp(u_{rs})\right), \tag{3.20}$$

where $\mathcal{R}_i$ denote the linearized Riemann tensors of the four gravitons and

$$\langle \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \mathcal{R}_4 \rangle^{\text{MHV}} = \langle f_1 f_2 f_3 f_4 \rangle_L^{\text{MHV}} \langle f_1 f_2 f_3 f_4 \rangle_R^{\text{MHV}} \tag{3.21}$$

reproduces the expected $\mathcal{R}^4$ structure, which is MHV in $D = 4$, and no lower derivative $\mathcal{R}^2$ and/or $\mathcal{R}^3$ terms [21].

For $\mathcal{N} = 5 = 4_L + 1_R$ in $D = 4$ one gets similar results with $\mathcal{E}_{\mathcal{N}=2_R} = \Gamma[{}^r_s]$ replaced by $\mathcal{E}_{\mathcal{N}=1_R} = \mathcal{I}_{ab}\mathcal{H}'/\mathcal{H}(\alpha'\tau_2)^{-2}$ which is moduli independent.

Henceforth we will focus on the $\mathcal{N} = 6 = 4_L + 2_R$ case and explore NS-NS moduli dependence of the one-loop threshold in $D = 5$.

4. One-Loop Threshold Integrals

One-loop threshold integrals for toroidal compactifications have been briefly reviewed above and shown to represent the contribution of the $(r, s) \neq (0, 0)$ untwisted sector. For $(r, s) \neq (0, 0)$ the threshold integrals involve shifted lattice sums as in heterotic strings with Wilson lines [41–45].

For simplicity let us discuss here the case of $\mathcal{N} = 6$ in $D = 5$. For definiteness we consider $n = 2$ (Z_2 shift orbifold) and start at the special point in the moduli space where $\mathbf{T}^5 = \mathbf{T}^4_{SO(8)} \times \mathbf{S}^1$. Later on we will include off-diagonal moduli that effectively behave as Wilson lines.

In the "twisted" $\left[{0 \atop 1}\right]$ sector, the relevant threshold integral is of the form

$$\begin{aligned}\mathcal{I}^{\mathcal{N}=6}_{1,5}\begin{bmatrix}0\\1\end{bmatrix} &= (2\pi)^5 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\tau_2^{5/2}\Gamma_{1,1}\begin{bmatrix}0\\1\end{bmatrix}(R)\overline{O}_8\left[\frac{2\pi^2}{3}\widehat{E}_2 + \wp\left(\frac{1}{2}\right)\right]\\ &= (2\pi)^5 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\tau_2^2\frac{R}{\sqrt{\alpha'}}\sum_{m,n} e^{-|2m+(2n+1)\tau|^2\pi R^2/4\alpha'\tau_2}\overline{O}_8\left[\frac{2\pi^2}{3}\widehat{E}_2 + \wp\left(\frac{1}{2}\right)\right].\end{aligned} \tag{4.1}$$

Setting $(2m, 2n+1) = (2\ell+1)(2m', 2n'+1)$ and using invariance of $\overline{O}_8$ under $\tau \to \tau+2$ allow to unfold the integral to the double strip

$$(2\pi)^5\frac{R}{\sqrt{\alpha'}}\int_{-1}^{+1} d\tau_1 \int_0^\infty d\tau_2 \sum_\ell e^{-(2\ell+1)^2\pi R^2/4\alpha'\tau_2}\sum_{N,\overline{N}} d_{\overline{N}}\overline{q}^{\overline{N}} c_N \overline{q}^N, \tag{4.2}$$

where $\overline{O}_8 = \sum_{N=|\mathbf{r}|^2/2} d_N\overline{q}^N$ and $(2\pi^2/3)\widehat{E}_2 + \wp(1/2) = \sum_N c_N q^N$. Performing the trivial integral over τ_1 (level matching $\overline{N} = N$) and the less trivial integral over τ_2 by means of

$$\int_0^\infty dy y^{\nu-1} e^{-cy-b/y} = \left(\frac{b}{c}\right)^{\nu/2} K_\nu\left(\sqrt{bc}\right), \tag{4.3}$$

where $K_\nu(z)$ is a Bessel function of second kind, finally yields

$$\mathcal{I}^{\mathcal{N}=6}_{1,5}\begin{bmatrix}0\\1\end{bmatrix}(R, A_i = 0) = (2\pi)^5\left(\frac{R}{\sqrt{\alpha'}}\right)^{3/2}\sum_{\ell=0}^\infty\sum_{N=1}^\infty (2\ell+1)\sqrt{N} d_N \sigma_1(N) K_1\left(4\pi(2\ell+1)\sqrt{\frac{NR}{\sqrt{\alpha'}}}\right), \tag{4.4}$$

where

$$\sigma_1(N) = \sum_{d|N} = \psi(N) - \psi(1) = \frac{c_N}{N} \tag{4.5}$$

from the expansion of $\widehat{E}_2$ in powers of q.

The result can be easily generalized to the other sectors of the Z_2 orbifold under consideration as well as to different (orbifold) constructions that give rise to different shifted lattice sums. Manifest $SO(1,5 \mid Z)$ symmetry can be achieved turning on off-diagonal components of B and G (subject to restrictions). Denoting by $2A_i = G_{i5} + B_{i5}$ and observing that $G_{i5} - B_{i5} = 0$ by construction, one finds

$$\begin{aligned}\mathcal{I}_{1,5}^{\mathcal{N}=6}\begin{bmatrix} r \\ s \end{bmatrix}(R, A_i) &= (2\pi)^5 \frac{R}{2\sqrt{\alpha'}} \sum_{\ell=0}^{\infty} \sum_{\vec{w}\in\Gamma[^r_s]} \frac{c_{\vec{w}^2}}{2} \int_0^{\infty} dy (2\ell+1) e^{-(2\ell+1)^2\pi R^2/4\alpha' y - 2\pi\vec{w}\cdot\vec{w} + 2\pi i\vec{w}\cdot\vec{A}} \\ &= (2\pi)^5 \left(\frac{R}{\sqrt{\alpha'}}\right)^{3/2} \sum_{\ell=0}^{\infty} \sum_{\vec{w}\in\Gamma[^r_s]} \sigma_1\left(\frac{\vec{w}^2}{2}\right)(2\ell+1) \\ &\quad \times \sqrt{\frac{\vec{w}^2}{2}} e^{2\pi i\vec{r}\cdot\vec{A}} K_1\left(4\pi(2\ell+1)\sqrt{\frac{\vec{w}^2 R}{2\sqrt{\alpha'}}}\right).\end{aligned} \tag{4.6}$$

Summing up the contributions of the various sectors, that is, various shifted lattice sums, yields the complete one-loop threshold correction to the $\mathcal{R}^4$ terms for $\mathcal{N} = 6$ superstring vacua in $D = 5$. Clearly only NS-NS moduli (except the dilaton) appear that expose $SO(1,5)$ T-duality symmetry.

The analysis is rather more involved in $D = 4$ where one-loop threshold integrals receive contribution from trivial, degenerate, and nondegenerate orbits [46, 47]. Alternative methods for unfolding the integrals over the fundamental domain have been proposed [48, 49].

Explicit computation is beyond the scope of the present investigation. It proceeds along the lines above and presents close analogy with threshold computations in $\mathcal{N} = 2$ heterotic strings sectors in the present of Wilson lines [41, 42, 44] or, equivalently, $\mathcal{N} = 4$ heterotic strings in $D = 8$ [50]. Rather than focussing on this interesting but rather technical aspect of the problem, let us turn our attention onto the nonperturbative dependence on the other R-R moduli as well as dilaton. This is brought about by the inclusion of asymmetric D-brane instantons.

5. Low-Energy Action and U-Duality

In [12] the conserved charges coupling to the surviving R-R and NS-NS graviphotons were identified as combinations of those appearing in toroidal compactifications. In the case of maximal $\mathcal{N} = 8$ supergravity, the 12 NS-NS graviphotons couple to windings and KK momenta. Their magnetic duals couple to wrapped NS5-branes (H-monoples) and KK monopoles. The 32 R-R graviphotons (including magnetic duals) couple to 6 D1-, 6 D5-, and 20 D3-branes in Type IIB and to 1 D0-, 15 D2-, 15 D4-, and 1 D6-branes in Type IIA.

An analogous statement applies to Euclidean branes inducing instanton effects. In toroidal compactifications with $\mathcal{N} = 8$ supersymmetry, one has 15 kinds of worldsheet instantons (EF1), 1 D(-1), 15 ED1, 15 ED3 and one each of EN5, ED5, EKK5 for Type IIB. For Type IIA superstrings one finds 6 ED0, 20 ED2, 6 ED4 and one each of EN5 and EKK5.

In a series of paper [24, 25], a natural proposal has been made for the nonperturbative completion of the modular form of $E_{d+1}(Z)$ that represent the scalar dependence of the $\mathcal{R}^4$

and higher derivative terms in $\mathcal{N} = 8$ superstring vacua. The explicit formulae are rather simple and elegant. In particular

$$f_{\mathcal{R}^4}^{\mathcal{N}=8}(\Phi) = \mathcal{E}^{E(d+1|Z)}_{[10^d],3/2}(\Phi), \tag{5.1}$$

where $\mathcal{E}^{E(d+1|Z)}_{[10^d],3/2}(\Phi)$ is an Einstein series of the relevant U-duality group. The above proposal satisfies a number of consistency checks including perturbative string limit that is small string coupling in which $E(d+1 \mid Z) \rightarrow SO(d,d \mid Z)$ and $[10\cdots0] \rightarrow \mathbf{2d}$, large radius limit in which $E(d+1 \mid Z) \rightarrow E(d \mid Z)$ and $[10\cdots0] \rightarrow [10\cdots0]$, and M-theory limit in which $E(d+1 \mid Z) \rightarrow SL(d+1 \mid Z)$ and $[10\cdots0] \rightarrow [10\cdots0]'$. Moreover $f_{\mathcal{R}^4}$ only receives contribution from 1/2 BPS states as expected for a supersymmetric invariant that can be written as an integral over half of (on-shell) superspace.

An independent but not necessarily inequivalent proposal has been made in [26].

We expect similar results for $\mathcal{R}^4$ terms in $\mathcal{N} = 5, 6$ superstring vacua with the following caveats. First, in $\mathcal{N} = 5, 6$ superspace $\mathcal{R}^4$ terms are 1/5 and 1/3 BPS, respectively, since they require integrations over 16 Grassman variables. Indeed we have explicitly seen that one-loop threshold correction involves the left-moving sector, in which supersymmetry is partially broken, in an essential way. Second, the U-duality group is not of maximal rank, and the same applies to the T-duality subgroup, present in the $\mathcal{N} = 6$ case. Third, $\mathcal{N} = 5, 6$ only exist in $D \leq 5$ or $D \leq 4$. Some decompactification limits should produce $\mathcal{N} = 8$ vacua in $D = 10$.

Let us try and identify the relevant 1/3 or 1/5 BPS Euclidean D-brane bound states.

5.1. $\mathcal{N} = 6$ *ED-Branes*

In the Type IIB description, the chiral Z_2 projection (T-duality) from $\mathcal{N} = 8$ to $\mathcal{N} = 6$ yields the Euclidean D-brane bound states of the form

$$\begin{gathered} \mathrm{D}(-1) + \mathrm{ED3}_{\widehat{T}^4}, \quad \mathrm{ED1}_{T^2} + \mathrm{ED5}_{T^2\times\widehat{T}^4}, \quad \mathrm{ED1}_{S^1\times\widehat{S}^1} + \mathrm{ED3}_{S^1\times\widehat{T}^3_\perp}, \\ \mathrm{ED1}_{\widehat{T}^2} + \mathrm{ED1}_{\widehat{T}^2_\perp}, \quad \mathrm{ED3}_{T^2\times\widehat{T}^2} + \mathrm{ED3}_{T^2\times\widehat{T}^2_\perp}. \end{gathered} \tag{5.2}$$

The above bound states of Euclidean D-branes are 1/3 BPS since they preserve 8 supercharges out of the 24 supercharges present in the background.

A similar analysis applies to world-sheet and ENS5 instantons.

There are several other superstring realizations of $\mathcal{N} = 6$ supergravity in $D = 4$. Given the uniqueness of the low-energy theory, they all share the same massless spectrum but the massive spectrum and the relevant (Euclidean) D-brane bound-states depend on the choice of model.

5.2. Nonperturbative Threshold Corrections

By analogy with $\mathcal{N} = 8$ one would expect $f_{\mathcal{R}^4} = \Theta_G$, that is, an automorphic form of the U-duality group G, that is, $G = SO^*(12)$ $(SU^*(6))$ for $\mathcal{N} = 6$ in $D = 4$ $(D = 5)$ and $G = SU(5,1)$ for $\mathcal{N} = 5$ in $D = 4$. The relevant "instantons" should be associated to BPS particles in one higher dimension (when possible).

For $\mathcal{N} = 6$, in the decompactification limit the relevant decomposition under $SO^*(12) \rightarrow SU(5,1) \times R^+$ is

$$\mathbf{66} \longrightarrow \mathbf{35}_0 + \mathbf{1}_0 + \mathbf{15}_{+2} + \mathbf{15}'_{-2} \tag{5.3}$$

so that the 15 particle charges in $D = 6$ satisfy 15 1/3 BPS "purity" conditions in $D = 5$

$$\frac{\partial \mathcal{I}_3}{\partial Q^{[ij]}} = 0, \tag{5.4}$$

where $\mathcal{I}_3^{\mathcal{N}=6,D=5} = \varepsilon_{ijklmn} Q^{[ij]} Q^{[kl]} Q^{[mn]}$. The moduli space decomposes according to

$$\frac{SO^*(12)}{U(6)} \supset \frac{SU(5,1)}{Sp(6)} \times R^{15} \times R^+. \tag{5.5}$$

More precisely the 15 charges decompose under $SO(1,5)$ into a 15-dim irrep. The "purity" conditions include $\det Q = 0$, viewed as a 6×6 antisymmetric matrix.

For $\mathcal{N} = 6$, in the string theory limit the relevant decomposition under $SO^*(12) \rightarrow SO(2,6) \times SL(2)_S$ is

$$\mathbf{32} \longrightarrow (\mathbf{8}_v, \mathbf{2})_{\text{NS-NS}} + (\mathbf{8}_s, \mathbf{1})_{R\text{-}R} + (\mathbf{8}_c, \mathbf{1})_{R\text{-}R} \tag{5.6}$$

that yields

$$\mathbf{66} \longrightarrow (\mathbf{28}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{8}_s, \mathbf{2}) + (\mathbf{8}_c, \mathbf{2}) + 3(\mathbf{1}, \mathbf{1}). \tag{5.7}$$

The moduli space decomposes according to

$$\frac{SO^*(12)}{U(6)} \supset \frac{SO(6,2)}{SO(6) \times SO(2)} \times \frac{SL(2)}{U(1)} \times R^{16}. \tag{5.8}$$

Further decomposition under $SL(2)_T \times SL(2)_U \times SL(2)_S$ should allow to get the "non-Abelian" part of the automorphic from from the "Abelian" one by means of $SL(2)_{U=\tau} \equiv SL(2)_B$. In particular the action for a (T-duality invariant) bound state of ED5 and three ED1's into the action of EN5 and EF1's, while the action of (T-duality invariant) bound state of ED(−1) and three ED3's is invariant (singlet). Clearly further detailed analysis is necessary.

5.3. $\mathcal{N} = 5$ ED-Branes

In the Type IIB description, the two chiral Z_2 projections ("T-duality" on T^4_{1234} and T^4_{3456}) from $\mathcal{N} = 8$ to $\mathcal{N} = 5$ yield Euclidean D-brane bound states of the form

$$\begin{gathered}
D(-1) + ED3_{\widehat{T}^4_{1234}} + ED3_{\widehat{T}^4_{3456}} + ED3_{\widehat{T}^4_{1256}}, \\
ED(-1)_{12} + ED5_{123456} + ED1_{34} + ED1_{56}, \\
ED1_{i_1 i_2} + ED3_{i_1 j_2 k_3 l_3} + ED3_{j_1 i_2 k_3 l_3} + ED1_{j_1 j_2}, \\
ED1_{i_1 i_3} + ED3_{i_1 j_2 k_2 l_3} + ED3_{j_1 j_2 k_2 k_3} + ED1_{j_1 k_3}, \\
ED1_{i_2 i_3} + ED1_{j_2 j_3} + ED3_{i_1 j_1 j_2 i_3} + ED3_{i_1 j_1 i_2 j_3}.
\end{gathered} \tag{5.9}$$

Bound states of Euclidean D-branes carrying the above charges are 1/5 BPS since they preserve 4 supercharges out of the 20 supercharges present in the background.

As in the $\mathcal{N} = 6$ case, a different analysis applies to BPS states carrying KK momenta or windings or their magnetic duals. However, at variant with the $\mathcal{N} = 6$, the three massive gravitini cannot form a single complex 2/5 BPS multiplet. One of them, together with its superpartners, should combine with string states which are degenerate in mass at the special rational point in the moduli space where the chiral $Z_2 \times Z_2$ projection is allowed.

6. Generating MHV Amplitudes in $\mathcal{N} = 5, 6$ SG's

Very much like, tree-level amplitudes in $\mathcal{N} = 8$ supergravity in $D = 4$ can be identified with "squares" of tree-level amplitudes in $\mathcal{N} = 4$ SYM theory [3, 4], tree-level amplitudes in $\mathcal{N} = 5, 6$ supergravity in $D = 4$ can be identified with "products" of tree-level amplitudes in $\mathcal{N} = 4$ and $\mathcal{N} = 1, 2$ SYM theory.

As previously observed, a first step in this direction is to show that the spectra of $\mathcal{N} = 5, 6$ supergravity are simply the tensor products of the spectra of $\mathcal{N} = 4$ and $\mathcal{N} = 1, 2$ SYM theory.

The second step is to work in the helicity basis and focus on MHV amplitudes (for a recent review see, e.g., [27]). In $\mathcal{N} = 4$ SYM the generating function for (colour-ordered) n-point MHV amplitudes is given by [51]

$$\mathcal{F}^{\mathcal{N}=4\ \mathrm{SYM}}_{\mathrm{MHV}}\left(\eta_i^a, u_i^\alpha\right) = \frac{\delta^8\left(\sum_i \eta_i^a u_i^\alpha\right)}{\langle u_1 u_2 \rangle \langle u_2 u_3 \rangle \cdots \langle u_n u_1 \rangle}, \tag{6.1}$$

where η_i^a with $i = 1, \dots n$ and $a = 1, \dots 4$ are auxiliary Grassmann variables and u_i are commuting left-handed spinors, such that $p_i = u_i \overline{u}_i$.

Individual amplitudes are obtained by taking derivatives with respect to the Grassman variables η's according to the rules

$$A^+ \longrightarrow 1, \quad \lambda_a^+ \longrightarrow \frac{\partial}{\partial \eta^a}, \dots, A^- \longrightarrow \frac{1}{4!}\varepsilon^{abcd}\frac{\partial^4}{\partial \eta^a \cdots \partial \eta^d}. \tag{6.2}$$

The intermediate derivatives representing scalars ($\varphi \sim \partial^2/\partial\eta^2$) and right-handed gaugini ($\lambda^- \sim \partial^3/\partial\eta^3$).

One can reconstruct all tree-level amplitudes, be they MHV or not, from MHV amplitudes using factorization, recursion relation or otherwise, see for example [27].

One can easily derive (super)gravity MHV amplitudes by simply taking the product of the generating functions for SYM amplitudes

$$\mathcal{G}^{\mathcal{N}=8\text{ SG}}_{\text{MHV}}\left(\eta_i^A, u_i^\alpha\right) = \frac{\mathcal{C}(u_i)\delta^{16}\left(\sum_i \eta_i^A u_i^\alpha\right)}{\langle u_1 u_2\rangle^2 \langle u_2 u_3\rangle^2 \cdots \langle u_n u_1\rangle^2} = \mathcal{C}(u_i)\mathcal{F}^{\mathcal{N}=4\text{ SYM}}_{\text{MHV},L}\left(\eta_i^{a_L}, u_i^\alpha\right)\mathcal{F}^{\mathcal{N}=4\text{ SYM}}_{\text{MHV},R}\left(\eta_i^{a_R}, u_i^\alpha\right), \tag{6.3}$$

where $\eta^A = (\eta_i^{a_L}, \eta_i^{a_R})$ with $A = 1, \ldots 8$ and the correction factor $\mathcal{C}(u_i)$ is only a function of the spinors u_i, actually of the massless momenta $p_i = u_i\overline{u}_i$ [28].

The relevant dictionary would read

$$h^+ \longrightarrow 1, \quad \psi_A^+ \longrightarrow \frac{\partial}{\partial\eta^A}, \ldots, h^- \longrightarrow \frac{1}{\mathcal{N}!}\frac{\partial^8}{\partial\eta^8}. \tag{6.4}$$

In principle one can reconstruct all tree-level amplitudes, be they MHV or not, from MHV amplitudes using factorization, recursion relations, or otherwise, see for example [27]. Unitary methods allow to extend the analysis beyond tree level. If all $\mathcal{N} = 8$ supergravity amplitudes were expressible in terms of squares of $\mathcal{N} = 4$ SYM amplitudes, UV finiteness of the latter would imply UV finiteness of the former. Although support to this conjecture at the level of 4-graviton amplitudes, which are necessarily MHV, seems to exclude the presence of $\mathcal{R}^4$ corrections, which are 1/2 BPS saturated, it would be crucial to explicitly test the absence of $D^8\mathcal{R}^4$ corrections, the first that are not BPS saturated.

Going back to the problem of expressing MHV amplitudes in $\mathcal{N} = 5, 6$ supergravities in terms of SYM amplitudes, one has to resort to "orbifold" techniques.

In the $\mathcal{N} = 6$ case, half of the 4 η's (say η_L^3 and η_L^4) of the "left" $\mathcal{N} = 4$ SYM factor are to be projected out, that is, "odd" under a Z_2 involution. As a result the generating function is the same as in $\mathcal{N} = 8$ supergravity but the dictionary gets reduced to

$$\begin{gathered} h^+ \longrightarrow 1, \quad \psi_{A'}^+ \longrightarrow \frac{\partial}{\partial\eta^{A'}}, \quad A_0^+ = \frac{\partial^2}{\partial\eta_L^3\partial\eta_L^4}, \quad A_{A'B'}^+ = \frac{\partial^2}{\partial\eta^{A'}\partial\eta^{B'}}, \ldots, \\ h^- = \frac{1}{6!}\varepsilon^{A'_1\cdots A'_6}\frac{\partial^{2+6}}{\partial\eta_L^3\partial\eta_L^4\partial\eta^{A'_1}\cdots\partial\eta^{A'_6}}, \end{gathered} \tag{6.5}$$

where $A' = 1, \ldots 6$.

Further reduction is necessary for $\mathcal{N} = 5$ case; 3 of the 4 η's of the "left" $\mathcal{N} = 4$ SYM factor are to be projected out. For instance, they may acquire a phase $\omega = \exp(i2\pi/3)$ under a Z_3 projection.

The same projections should be implemented on the intermediate states flowing around the loops. Although tree-level amplitudes in $\mathcal{N} = 5, 6$ supergravity are simply a subset of the ones in $\mathcal{N} = 8$ supergravity, naive extension of the argument at loop order does not immediately work [52–54]. Several cancellations are not expected to take place despite the residual supersymmetry of the left SYM factor. However, in view of the recent observations on the factorization of $\mathcal{N} = 4$ SYM into a kinematical part and a group theory part, where the latter satisfies identities similar to the former [55–57] and can thus be consistently replaced with the former giving rise to consistent and UV finite $\mathcal{N} = 8$ SG amplitudes, it may well be the case that a similar decomposition can be used to produce, possibly UV finite, $\mathcal{N} = 5, 6$ SG amplitudes. Our results on $\mathcal{R}^4$ lend some support to this viewpoint.

7. Conclusions

Let us summarize our results. We have shown that the first higher derivative corrections to the low-energy effective action around superstring vacua with $\mathcal{N} = 5, 6$ supersymmetry are $\mathcal{R}^4$ terms as in $\mathcal{N} = 8$. Contrary to $\mathcal{N} \leq 4$, no $\mathcal{R}^2$ terms appear. In this respect $\mathcal{N} = 5, 6$ supersymmetric models in $D = 4$, having no massless matter multiplets to add, behave similarly to their common $\mathcal{N} = 8$ supersymmetric parent. It is worth stressing again that such nonvanishing threshold corrections confirm that, as in the $\mathcal{N} = 8$ case, superstring calculations do not reproduce field theory results. As in $\mathcal{N} = 8$ supergravity, it is known that $\mathcal{R}^4$ corrections are absent in $\mathcal{N} = 5, 6$ supergravity due to the anomaly free continuous duality symmetry [1].

Relying on previous results on vector boson scattering at one loop in unoriented D-brane worlds [38], we have studied four graviton scattering amplitudes and derived explicit formulae for the one-loop threshold corrections in asymmetric orbifolds that realize the above vacua. In addition to a term $1/n \times f^{\mathcal{N}=8} \times c\mathcal{R}^4$, coming from the $(0, 0)$ sector, contributions from nontrivial sectors of the orbifold to $f^{\mathcal{N}=5,6} \times c\mathcal{R}^4$ display a close similarity with heterotic threshold corrections in the presence of Wilson lines [41, 42, 44]. For illustrative purposes, we have computed the relevant integrals for $\mathcal{N} = 6$ in $D = 5$ exposing the expected $SO(1, 5)$ T-duality symmetry. The analysis in $D = 4$ is technically more involved and will be performed elsewhere. We have also identified the relevant $1/3$ or $1/5$ BPS bound states of Euclidean D-branes that contribute to the nonperturbative dependence of the thresholds on R-R scalars and on the axio-dilaton. By analogy with $\mathcal{N} = 8$ it is natural to conjecture the possible structure of the automorphic form of the relevant U-duality group. A more detailed analysis of this issue is however necessary. Finally, in view of the potential UV finiteness of $\mathcal{N} = 5, 6$ supergravities, we have discussed how to compute tree-level MHV amplitudes using generating function and orbifolds techniques [28]. All other tree-level amplitudes should follow from factorization and in fact should coincide with $\mathcal{N} = 8$ amplitudes involving only $\mathcal{N} = 5$ or $\mathcal{N} = 6$ supergravity states in the external legs. Loop amplitudes require a separate investigation. In particular no generalization of the KLT relations is known beyond tree level [58].

Acknowledgments

Discussions with C. Bachas, M. Cardella, S. Ferrara, F. Fucito, M. Green, E. Kiritsis, S. Kovacs, L. Lopez, J. F. Morales, N. Obers, R. Poghossyan, R. Richter, M. Samsonyan, and A. V. Santini are kindly acknowledged. This work was partially supported by the ERC Advanced Grant no. 226455 *"Superfields"* and by the Italian MIUR-PRIN Contract 2007-5ATT78 *"Symmetries of the Universe and of the Fundamental Interactions"*.

References

[1] N. Marcus, "Composite anomalies in supergravity," *Physics Letters B*, vol. 157, no. 5-6, pp. 383–388, 1985.

[2] A. Ceresole and S. Ferrara, "Black holes and attractors in supergravity," http://arxiv.org/abs/1009.4175.
[3] Z. Bern, L. J. Dixon, and R. Roiban, "Is $\mathcal{N} = 8$ supergravity ultraviolet finite?" *Physics Letters B*, vol. 644, no. 4, pp. 265–271, 2007.
[4] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, "Ultraviolet behavior of $\mathcal{N} = 8$ supergravity at four loops," *Physical Review Letters*, vol. 103, no. 8, Article ID 081301, 4 pages, 2009.
[5] K. S. Stelle, "Is $\mathcal{N} = 8$ supergravity a finite field theory?" *Fortschritte der Physik*, vol. 57, no. 5–7, pp. 446–450, 2009.
[6] N. Beisert, H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales, and S. Stieberger, "$E_{7(7)}$ constraints on counterterms in $\mathcal{N} = 8$ supergravity," *Physics Letters B*, vol. 694, no. 3, pp. 265–271, 2010.
[7] R. Kallosh, "The ultraviolet finiteness $\mathcal{N} = 8$ supergravity," *Journal of High Energy Physics*, vol. 2010, no. 12, 2010.
[8] M. B. Green, H. Ooguri, and J. H. Schwarz, "Nondecoupling of maximal supergravity from the superstring," *Physical Review Letters*, vol. 99, no. 4, Article ID 041601, 4 pages, 2007.
[9] M. Bianchi, S. Ferrara, and R. Kallosh, "Perturbative and non-perturbative $\mathcal{N} = 8$ supergravity," *Physics Letters B*, vol. 690, no. 3, pp. 328–331, 2010.
[10] M. Bianchi, S. Ferrara, and R. Kallosh, "Observations on arithmetic invariants and U-duality orbits in $\mathcal{N} = 8$ supergravity," *Journal of High Energy Physics*, vol. 2010, no. 3, 2010.
[11] S. Ferrara and C. Kounnas, "Extended supersymmetry in four-dimensional type II strings," *Nuclear Physics B*, vol. 328, no. 2, pp. 406–438, 1989.
[12] M. Bianchi, "Bound-states of D-branes in L-R asymmetric superstring vacua," *Nuclear Physics B*, vol. 805, no. 1-2, pp. 168–181, 2008.
[13] K. S. Narain, M. H. Sarmadi, and C. Vafa, "Asymmetric orbifolds," *Nuclear Physics B*, vol. 288, no. 3-4, pp. 551–577, 1987.
[14] A. Dabholkar and J. A. Harvey, "String islands," *Journal of High Energy Physics*, vol. 3, no. 2, 16 pages, 1999.
[15] I. Antoniadis, C. P. Bachas, and C. Kounnas, "Four-dimensional superstrings," *Nuclear Physics B*, vol. 289, no. 1, pp. 87–108, 1987.
[16] I. Antoniadis and C. Bachas, "4d fermionic superstrings with arbitrary twists," *Nuclear Physics B*, vol. 298, no. 3, pp. 586–612, 1988.
[17] H. Kawai, D. C. Lewellen, and S. H. H. Tye, "Construction of fermionic string models in four dimensions," *Nuclear Physics B*, vol. 288, no. 1, pp. 1–76, 1987.
[18] H. Kawai, D. C. Lewellen, and S. H. H. Tye, "Classification of closed-fermionic-string models," *Physical Review D*, vol. 34, no. 12, pp. 3794–3804, 1986.
[19] H. Kawai, D. C. Lewellen, and S. H. H. Tye, "Construction of four-dimensional fermionic string models," *Physical Review Letters*, vol. 57, no. 15, pp. 1832–1835, 1986.
[20] H. Kawai and D. C. Lewellen, "Erratum: construction of four-dimensional fermionic string models," *Physical Review Letters*, vol. 58, no. 4, p. 429, 1987.
[21] J. Broedel and L. J. Dixon, "$\mathcal{R}^4$ counterterm and *E7(7)* symmetry in maximal supergravity," *Journal of High Energy Physics*, vol. 2010, no. 5, 2010.
[22] M. B. Green and M. Gutperle, "Effects of D-instantons," *Nuclear Physics B*, vol. 498, no. 1-2, pp. 195–227, 1997.
[23] K. Becker, M. Becker, and A. Strominger, "Fivebranes, membranes and non-perturbative string theory," *Nuclear Physics B*, vol. 456, no. 1-2, pp. 130–152, 1995.
[24] M. B. Green, J. G. Russo, and P. Vanhove, "Automorphic properties of low energy string amplitudes in various dimensions," *Physical Review D*, vol. 81, no. 8, Article ID 086008, 49 pages, 2010.
[25] M. B. Green, S. D. Miller, J. G. Russo, and P. Vanhove, "Eisenstein series for higher-rank groups and string theory amplitudes," http://arxiv.org/abs/1004.0163.
[26] B. Pioline, "$\mathcal{R}^4$ couplings and automorphic unipotent representations," *Journal of High Energy Physics*, vol. 2010, no. 3, Article ID 116, 2010.
[27] J. M. Drummond, "Hidden simplicity of gauge theory amplitudes," *Classical and Quantum Gravity*, vol. 27, no. 21, 2010.
[28] M. Bianchi, H. Elvang, and D. Z. Freedman, "Generating the amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG," *Journal of High Energy Physics*, vol. 2008, no. 9, 2008.
[29] J. Bagger and N. Lambert, "Modeling multiple M2-branes," *Physical Review D*, vol. 75, no. 4, Article ID 045020, 7 pages, 2007.
[30] A. Gustavsson, "Algebraic structures on parallel M2 branes," *Nuclear Physics B*, vol. 811, no. 1-2, pp. 66–76, 2009.

[31] O. Aharony, O. Bergman, and D. L. Jafferis, "Fractional M2-branes," *Journal of High Energy Physics*, vol. 2008, no. 11, Article ID 043, 27 pages, 2008.
[32] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *Journal of High Energy Physics*, vol. 2008, no. 10, Article ID 091, 38 pages, 2008.
[33] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park, and P. Yi, "A nonperturbative test of M2-brane theory," *Journal of High Energy Physics*, vol. 2008, no. 11, 2008.
[34] A. Cagnazzo, D. Sorokin, and L. Wulff, "String instanton in $AdS_4 \times CP^3$," *Journal of High Energy Physics*, vol. 2010, no. 5, 29 pages, 2010.
[35] M. Bianchi, R. Poghossian, and M. Samsonyan, "Precision spectroscopy and higher spin symmetry in the ABJM model," *Journal of High Energy Physics*, vol. 2010, no. 10, 2010.
[36] N. Drukker, M. Marino, and P. Putrov, "From weak to strong coupling in ABJM theory," *Communications in Mathematical Physics*, 2011, http://arxiv.org/abs/1007.3837.
[37] R. D'Auria, S. Ferrara, and C. Kounnas, "$\mathcal{N} = (4,2)$ chiral supergravity in six dimensions and solvable Lie algebras," *Physics Letters B*, vol. 420, no. 3-4, pp. 289–299, 1998.
[38] M. Bianchi and A. V. Santini, "String predictions for near future colliders from one-loop scattering amplitudes around D-brane worlds," *Journal of High Energy Physics*, vol. 2006, no. 12, Article ID 010, 2006.
[39] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory, vol 2*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, UK, 1987.
[40] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory, vol. 1*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, UK, 1987.
[41] C. Bachas, C. Fabre, E. Kiritsis, N. A. Obers, and P. Vanhove, "Heterotic/type I duality and D-brane instantons," *Nuclear Physics B*, vol. 509, no. 1-2, pp. 33–52, 1998.
[42] E. Kiritsis and N. A. Obers, "Heterotic/type-I duality in $D < 10$ dimensions, threshold corrections and D-instantons," *Journal of High Energy Physics*, vol. 1, no. 10, pp. 13–70, 1997.
[43] M. Bianchi, E. Gava, F. Morales, and K. S. Narain, "D-strings in unconventional type I vacuum configurations," *Nuclear Physics B*, vol. 547, no. 1-2, pp. 96–126, 1999.
[44] N. A. Obers and B. Pioline, "Exact thresholds and instanton effects in string theory," *Fortschritte der Physik*, vol. 49, no. 4–6, pp. 359–375, 2001.
[45] M. Bianchi and J. F. Morales, "Unoriented D-brane instantons vs heterotic worldsheet instantons," *Journal of High Energy Physics*, vol. 2008, no. 2, Article ID 073, 27 pages, 2008.
[46] L. J. Dixon, V. Kaplunovsky, and J. Louis, "Moduli dependence of string loop corrections to gauge coupling constants," *Nuclear Physics B*, vol. 355, no. 3, pp. 649–688, 1991.
[47] P. Mayr and H. P. Nilles, "String unification and threshold corrections," *Physics Letters B*, vol. 317, no. 1-2, pp. 53–59, 1993.
[48] M. Trapletti, "On the unfolding of the fundamental region in integrals of modular invariant amplitudes," *Journal of High Energy Physics*, vol. 2003, no. 2, Article ID 012, 20 pages, 2003.
[49] M. Cardella, "A novel method for computing torus amplitudes for $\mathbb{Z}_N$ orbifolds without the unfolding technique," *Journal of High Energy Physics*, vol. 2009, no. 5, Article ID 010, 16 pages, 2009.
[50] W. Lerche, S. Stieberger, and N. P. Warner, "Quartic gauge couplings from K3 geometry," *Advances in Theoretical and Mathematical Physics*, vol. 3, no. 5, pp. 1575–1611, 1999.
[51] V. P. Nair, "A current algebra for some gauge theory amplitudes," *Physics Letters B*, vol. 214, no. 2, pp. 215–218, 1988.
[52] P. Katsaroumpas, B. Spence, and G. Travaglini, "One-loop $\mathcal{N} = 8$ supergravity coefficients from $\mathcal{N} = 4$ super Yang-Mills," *Journal of High Energy Physics*, vol. 2009, no. 8, Article ID 096, 33 pages, 2009.
[53] Z. Bern, J. J. M. Carrasco, and H. Johansson, "The structure of multiloop amplitudes in gauge and gravity theories," *Nuclear Physics B*, vol. 205-206, pp. 54–60, 2010.
[54] J. M. Drummond, P. J. Heslop, and P. S. Howe, " A note on $\mathcal{N} = 8$ counterterms," http://arxiv.org/abs/1008.4939.
[55] H. Tye and Y. Zhang, "Comment on the identities of the gluon tree amplitudes," http://arxiv.org/abs/1007.0597.
[56] S. H. Henry Tye and Y. Zhang, "Dual identities inside the gluon and the graviton scattering amplitudes," *Journal of High Energy Physics*, vol. 2010, no. 6, Article ID 071, 45 pages, 2010.
[57] Z. Bern, T. Dennen, Y. T. Huang, and M. Kiermaier, "Gravity as the square of gauge theory," *Physical Review D*, vol. 82, no. 6, 2010.
[58] H. Kawai, D. C. Lewellen, and S. H. H. Tye, "A relation between tree amplitudes of closed and open strings," *Nuclear Physics B*, vol. 269, no. 1, pp. 1–23, 1986.


Journal of Gravity
Journal of Photonics
The Scientific World Journal
Journal of Soft Matter
Advances in Condensed Matter Physics
Journal of Fluids
Journal of Aerodynamics
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Statistical Mechanics
International Journal of Optics
Journal of Computational Methods in Physics
Advances in High Energy Physics
Journal of Thermodynamics
Physics Research International
Advances in Astronomy
Journal of Solid State Physics
Journal of Astrophysics
International Journal of Superconductivity
Journal of Biophysics
Journal of Atomic and Molecular Physics