

## Research Article

# The K-Theoretic Formulation of D-Brane Aharonov-Bohm Phases

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The topological calculation of Aharonov-Bohm phases associated with D-branes in the absence of a Neveu-Schwarz  $B$ -field is explored. The K-theoretic classification of Ramond-Ramond fields in Type II and Type I theories is used to produce formulae for the Aharonov-Bohm phase associated with a torsion flux. A topological construction shows that K-theoretic pairings to calculate such phases exist and are well defined. An analytic perspective is then taken, obtaining a means for determining Aharonov-Bohm phases by way of the reduced eta-invariant. This perspective is used to calculate the phase for an experiment involving the  $(-1) -8$  system in Type I theory and compared with previous calculations performed using different methods.

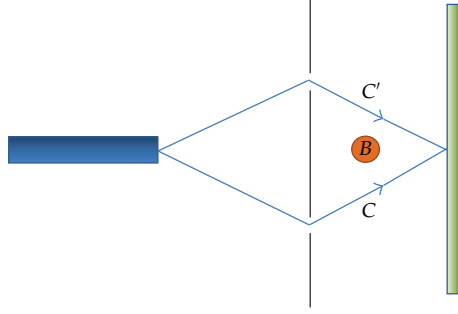
## 1. Introduction

It is well known that the existence of a magnetic field will affect the phase of electrically charged particles, even when the particles do not pass through the region containing the magnetic field. The canonical example was formulated by Aharonov and Bohm [1], and is shown in Figure 1.

First, note that the existence of the  $B$ -field induces a connection  $\omega = (ie/\hbar)A$ , where  $A$  is the Lie-algebra valued one form determined by the field. Next, we let  $\gamma = C - C' \in H_1(X)$ . Then it is found that the phase acquired by a particle traveling along  $\gamma$  is

$$\Phi[A, \gamma] = \frac{ie}{\hbar} \oint_{\gamma} A. \quad (1.1)$$

Imposing the reality condition  $\Phi[A_1, \gamma] \sim \Phi[A_2, \gamma]$  if  $A_1 - A_2 \in \mathbb{Z}$ , the set of equivalence classes  $[\Phi[A, \gamma]]$  are  $\mathbb{R}/\mathbb{Z} \cong U(1)$ -valued. So we may view  $[\Phi[A, \cdot]]$  as an element of



**Figure 1:** The canonical Aharonov-Bohm setup, with electrically charged particles moving along each of two paths,  $C$  and  $C'$ , before interfering on a screen.

$H^1(X; U(1))$ . Then we see that the Aharonov-Bohm phase is given by a pairing  $H_1(X) \times H^1(X; U(1)) \rightarrow U(1)$  defined as  $[\Phi[A, \gamma]]$ .

When we consider D-branes, however, things are not so simple. In [2, 3] it was shown that D-brane charges and Ramond-Ramond (RR) fields in Types IIA, IIB, and I theories are classified by K-theory. Therefore, the calculation of Aharonov-Bohm phases for D-branes will necessarily involve some sort of K-theoretic pairing.

In this paper, Sections 2.1–2.4 produce a number of details concerning the topological formulation of D-brane Aharonov-Bohm phases in Type IIA theory, building off of a brief speculative discussion in [4]. It is shown that the pairing outlined in [4] exists and is well-defined. Section 2.5 provides adaptations of this pairing to the Type IIB and Type I settings. In Section 3, the focus shifts to the use of the reduced eta-invariant as a means for calculating the K-theoretic pairing. A brief overview of relevant mathematical technology is presented in Section 3.1 and then utilized in Section 3.2 for the  $(-1) - 8$  system in Type I theory. It is shown that our result agrees with a calculation performed in [5] using different methods.

## 2. The Topological Formulation

Let us begin by considering Type IIA theory on  $\mathbb{R}_t \times X_9$ , and suppose we have a brane producing a torsion flux. This flux defines an element of  $K_{\text{tors}}^0(X)$ , the torsion subgroup of  $K^0(X)$ , where  $X \equiv X_8 = \partial X_9$ .

### 2.1. The Long Exact Sequence

We next wish to lift our element of  $K_{\text{tors}}^0(X)$  to an element of  $K^{-1}(X; U(1))$ . Before doing this, it is useful to consider the analogous cohomological situation. From the exact coefficient sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow U(1), \quad (2.1)$$

we may obtain the long exact cohomological sequence

$$\dots \xrightarrow{\delta^{k-1}} H^k(X; \mathbb{Z}) \xrightarrow{i^k} H^k(X; \mathbb{R}) \xrightarrow{j^k} H^k(X; U(1)) \xrightarrow{\delta^k} H^{k+1}(X; \mathbb{Z}) \xrightarrow{i^{k+1}} H^{k+1}(X; \mathbb{R}) \longrightarrow \dots, \quad (2.2)$$

where the maps  $\delta^k : H^k(X; U(1)) \rightarrow H^{k+1}(X; \mathbb{Z})$  are called the Bockstein homomorphisms.

Since  $\mathbb{R}/n\mathbb{R} \cong 0$  for any  $n \in \mathbb{Z} - \{0\}$ , the kernel of  $i^{k+1}$  is the set of torsion elements of  $H^{k+1}(X; \mathbb{Z})$ , denoted  $H_{\text{tors}}^{k+1}(X; \mathbb{Z})$ . Therefore, we may write the following exact sequence,

$$H^k(X; U(1)) \xrightarrow{\delta^k} H_{\text{tors}}^{k+1}(X; \mathbb{Z}) \longrightarrow 0. \quad (2.3)$$

Thus for any torsion class there is a lift in  $H^k(X; U(1))$ . This lift is an integral cochain, that is, closed in  $U(1)$  but not in  $\mathbb{Z}$ . Simple diagram-chasing shows that this lift is well defined [6].

A completely analogous argument goes through in K-theory. Indeed, one may write a long exact sequence similar to that above, and all subsequent statements and actions carry over. Specifically, the long exact sequence

$$\dots \longrightarrow K^{-1}(X) \xrightarrow{\text{ch}} K^{-1}(X; \mathbb{R}) \xrightarrow{\alpha} K^{-1}(X; U(1)) \xrightarrow{\beta} K^0(X) \xrightarrow{\text{ch}} K^0(X; \mathbb{R}) \longrightarrow \dots \quad (2.4)$$

gives the exact sequence

$$K^{-1}(X; U(1)) \xrightarrow{\beta} K_{\text{tors}}^0(X) \xrightarrow{\text{ch}} 0. \quad (2.5)$$

Here,  $\text{ch}$  is the Chern character and  $\beta$  is the forgetful map. Lifting an element of  $K_{\text{tors}}^0(X)$  via the Bockstein  $\beta$  then gives an element of  $K^{-1}(X; U(1))$ . As in the cohomological case, diagram-chasing shows that this lift is well defined.

## 2.2. The K-Cup Product

Next, note that the test brane defines an element of  $K^0(X)$ . We will pair this with our element of  $K^{-1}(X; U(1))$  via the K-cup product [7–9]. Again we begin by considering the analogous cohomological case. There, one starts with a mapping

$$S^p(X) \times S^q(X; G) \xrightarrow{\cup} S^{p+q}(X; G), \quad (2.6)$$

which assigns to each  $p$ -cochain  $c^p$  and  $q$ -cochain  $c^q$  a  $(p+q)$ -cochain  $c^{p+q}$  by letting  $c^p$  act on the front  $p$ -face and  $c^q$  act on the  $q$ -back face, then multiplying the results by the usual product operation sending  $(n, g)$  to  $ng$ . It follows that  $\cup$  gives a well-defined product operation [6]

$$H^p(X) \times H^q(X; G) \xrightarrow{\cup} H^{p+q}(X; G). \quad (2.7)$$

In K-theory, we define a similar product operation via tensor products of bundles. The external K-cup product is a group morphism  $K(X) \otimes K(Y) \rightarrow K(X \times Y)$  which assigns to each  $a \otimes b \in K(X) \otimes K(Y)$  the element  $(K(p_x)(a))(K(p_y)(b)) \in K(X \times Y)$ , where  $p_x : X \times Y \rightarrow X$  and  $p_y : X \times Y \rightarrow Y$  are the projection operators.

When extended to higher K-groups, this product becomes  $K^{-i}(X) \times K^{-j}(Y) \rightarrow K^{-i-j}(X \times Y)$ . To see this, we start with a pairing

$$\tilde{K}^{-i}(X) \times \tilde{K}^{-j}(Y) \rightarrow \tilde{K}^{-i-j}(X \wedge Y) \quad (2.8)$$

given by tensor product. Next we use the defined relationship  $K^{-i}(X) \equiv \tilde{K}^{-i}(X^+) \equiv \tilde{K}(\Sigma^i(X^+))$ , where  $X^+ \equiv X \cup \{pt.\}$  is  $X$  with a disjoint basepoint and  $\Sigma^i(X) = S^i \wedge X$  is the smash product of  $S^i$  with  $X$ . Then we obtain a pairing  $K^{-i}(X) \times K^{-j}(Y) \rightarrow K^{-i-j}(X \times Y)$ .

Letting  $X = Y$  and then composing with the map from  $K^{-i}(X \times X)$  to  $K^{-i}(X)$  induced by the diagonal map  $X \rightarrow X \times X$  gives the product

$$K^{-i}(X) \times K^{-j}(X) \longrightarrow K^{-i-j}(X). \quad (2.9)$$

It follows that there is a pairing

$$K^{-i}(X) \times K^{-j}(X; G) \longrightarrow K^{-i-j}(X; G), \quad (2.10)$$

with  $(n, g) \rightarrow ng$  as in the cohomological case.

We see that we may therefore pair

$$K^0(X) \times K^{-1}(X; U(1)) \longrightarrow K^{-1}(X; U(1)) \quad (2.11)$$

via the K-cup product with generalized coefficients. Thus, given a brane producing a torsion flux and a charged test brane, we obtain an element of  $K^{-1}(X; U(1))$ .

### 2.3. K-Homology

To measure the Aharonov-Bohm phase at infinity, we must move the test brane on a closed path in  $X$ . This path defines an element of  $H_1(X)$ . In order to pair our path with  $K^{-1}(X; U(1))$ , we must lift the path to an element of  $K_1(X)$ , the K-homology of  $X$  [10].

We may parametrize our path by a function  $f : S^1 \rightarrow X$ . Note that  $S^1$  is a compact  $Spin^C$ -manifold without boundary, and  $f$  is by definition a continuous map. To put a complex vector bundle on  $S^1$  is easy, since every complex vector bundle on  $S^1$  is trivial. It is natural, then, to let a K-cycle associated with our path be given by  $(S^1, e^n, f)$  where  $e^n$  is the trivial complex vector bundle with fibre  $\mathbb{C}^n$ . Since  $S^1$  is odd-dimensional, we have defined an element of  $K_1(X)$ .

To show that this lift is unique, we use the bordism and direct sum relations. Consider the K-cycle  $(S^1, e^2, f_2)$ . Since  $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$  the direct sum relation gives  $(S^1, e^2, f_2) \sim (S^1, e^1, f_2) \cup (S^1, e^1, f_2)$ . Let  $W$  be the compact 2-dimensional  $Spin^C$ -manifold shown in Figure 2, and put the trivial rank 1 complex vector bundle on it.

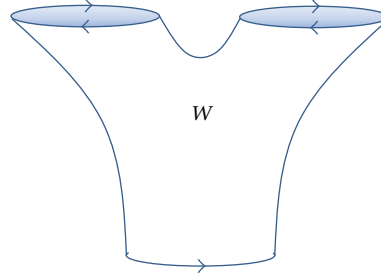
Then for appropriate choice of continuous  $\phi : W \rightarrow X$ , we have

$$(\partial W, E|_{\partial W}, \phi|_{\partial W}) \cong (S^1, e^1, f_2) \cup (S^1, e^1, f_2) \cup (-S^1, e^1, f_1) \sim (S^1, e^2, f_2) \cup (-S^1, e^1, f_1). \quad (2.12)$$

Hence

$$(S^1, e^2, f_2) \sim (S^1, e^1, f_1). \quad (2.13)$$

Clearly this generalizes to any K-cycle  $(S^1, e^N, f_N)$ . Thus our lift from  $H_1(X)$  to  $K_1(X)$  is unique.



**Figure 2:** Surface  $W$  carries the trivial rank 1 complex vector bundle on it and serves as a bordism between  $(-S^1, e^1, f_1)$  and two copies of  $(S^1, e^1, f_2)$ .

#### 2.4. The Intersection Form

In the cohomological case, there is an intersection pairing on a compact oriented  $n$ -dimensional manifold  $X$

$$H^k(X; \mathbb{Z}) \times H^{n-k}(X; U(1)) \longrightarrow U(1) \quad (2.14)$$

defined by

$$\alpha \times \beta \longmapsto \alpha \cdot \beta \equiv \langle \alpha \cup \beta, [X] \rangle, \quad (2.15)$$

that is, cup product followed by integration over an orientation class  $[X] \in H_n(X; \mathbb{Z})$ .

We use this to define another pairing

$$H^k(X; \mathbb{Z}) \times H_{\text{tors}}^{n-k+1}(X; \mathbb{Z}) \longrightarrow U(1) \quad (2.16)$$

in the following way. Since any torsion class  $[\alpha] \in H_{\text{tors}}^{n-k+1}(X; \mathbb{Z})$  has a well-defined lift  $[\alpha'] \in H^{n-k}(X; U(1))$ , we may define the desired pairing as

$$\langle \beta \cup \alpha', [X] \rangle \in U(1). \quad (2.17)$$

Returning to the K-theoretic case, recall that from our torsion flux, test brane, and path of the test brane we have defined elements of  $K^1(X; U(1))$  and  $K_1(X)$ . We would now like to pair these elements and get an element of  $U(1)$ , the Aharonov-Bohm phase. This is achieved with the use of the so-called intersection form [11].

The intersection form is the nondegenerate pairing

$$K_{\text{cpt}}^i(T^*X) \times K^i(X; U(1)) \longrightarrow K_{\text{cpt}}^0(T^*X; U(1)) \xrightarrow{p_t} U(1), \quad (2.18)$$

(with cpt denoting compact support) which is induced by the K-cup product and the direct image mapping  $p_t : K^0(T^*X, U(1)) \rightarrow K^0(pt, U(1)) = U(1)$  corresponding to the map  $p : X \rightarrow pt$ .

Poincaré duality and the Thom isomorphism give [10, 12]

$$K_{\text{cpt}}^1(T^*X) \cong K_1(X). \quad (2.19)$$

To see this isomorphism topologically, first let  $S(T^*X)$  denote the unit sphere bundle of  $T^*X$ . Also let  $\pi : S(T^*X) \rightarrow X$  be the projection. It was shown in [13] that elements of  $K^1(T^*X)$  are in one-to-one correspondence with stable homotopy classes of self-adjoint symbols on  $X$ . Then an element of  $K^1(T^*X)$  is a pair  $(E, \sigma)$  where  $E$  is a Hermitian vector bundle on  $X$  and  $\sigma : \pi^*(E) \rightarrow \pi^*(E)$  is a self-adjoint automorphism of  $\pi^*(E)$ . Then  $\sigma$  gives the decomposition  $\pi^*(E) = E_+ \oplus E_-$  where  $E_{\pm}$  is spanned by the eigenvectors with  $\pm$  eigenvalues of  $\sigma$ .

Setting  $\widehat{X} = S(T^*X)$ , note that  $\dim(\widehat{X}) = 2(\dim(X)) - 1$  so that it is odd-dimensional. Furthermore,  $\widehat{X}$  is a  $Spin^C$ -manifold since  $T\widehat{X}$  has a  $Spin^C$  structure from  $TX \oplus T^*X \cong \mathbb{C} \otimes_{\mathbb{R}} TX$ . Then the triple  $(\widehat{X}, E_+, \pi)$  is an element of  $K_1(X)$ . If we define  $c(E, \sigma) = (\widehat{X}, E_+, \pi)$ , then

$$c : K^1(T^*X) \longrightarrow K_1(X) \quad (2.20)$$

is an isomorphism.

Thus the intersection form is a nondegenerate pairing between K-homology and K-theory,

$$K_1(X) \times K^1(X; U(1)) \longrightarrow U(1). \quad (2.21)$$

This is precisely what we need to give the Aharonov-Bohm phase.

To summarize our formulation for the Type IIA case, the torsion flux defined an element of  $K_{\text{tors}}^0(X)$  which we lifted to  $K^{-1}(X; U(1))$  by the long exact sequence of K-groups associated with the exact coefficient sequence. The test brane defined an element of  $K^0(X)$  which we paired with our element of  $K^{-1}(X; U(1))$  via the K-cup product to again get an element of  $K^{-1}(X; U(1))$ . The path of our test brane defined an element of  $H_1(X; \mathbb{Z})$  which we lifted to an element of  $K_1(X; \mathbb{Z})$ . The intersection form then took  $K_1(X; \mathbb{Z}) \times K^1(X; U(1)) \rightarrow U(1)$ , which we call the Aharonov-Bohm phase.

## 2.5. The Type IIB and Type I Cases

Now that we have given the topological details of the K-theoretic formula for Aharonov-Bohm phase in the Type IIA case, we would like to develop similar statements for the Type IIB and Type I cases.

In the Type IIB situation, the torsion flux takes values in  $K_{\text{tors}}^1(X)$ , with  $X = \partial X_9$  as before. Then we may again use the exact coefficient sequence to give a long exact sequence of K-groups and lift our element of  $K_{\text{tors}}^1(X)$  to  $K^0(X; U(1))$ . Now our test brane defines an element of  $K^1(X)$ , and we again use the K-cup product to pair these elements as

$$K^1(X) \times K^0(X; U(1)) \longrightarrow K^1(X; U(1)). \quad (2.22)$$

Again we lift the path of the test brane from  $H_1(X)$  to  $K_1(X)$  and use the intersection form to pair

$$K_1(X) \times K^1(X; U(1)) \longrightarrow U(1). \quad (2.23)$$

Finally, we may make a similar proposal in the Type I scenario. Here, the torsion flux is valued in  $KO_{\text{tors}}^{-1}(X)$ , and the test brane defines an element of  $KO^{-1}(X)$ . All the properties of

the complex K-theory that we employed carry over to the KO-groups. The only real difference between these theories is the form of Bott Periodicity, but that does not seriously affect our discussion. So we lift the torsion flux to  $KO(X;U(1))$ , then pair the test brane charge to it,

$$KO^{-1}(X) \times KO(X;U(1)) \longrightarrow KO^{-1}(X;U(1)), \quad (2.24)$$

by a KO-cup product which is completely analogous to the K-cup product. We can complexify to obtain an element of  $K^{-1}(X;U(1))$ . Now we lift the path of the test brane from  $H_1(X)$  to  $K_1(X)$ . Then we use the intersection form to pair

$$K_1(X) \times K^{-1}(X;U(1)) \longrightarrow U(1), \quad (2.25)$$

giving us the Aharonov-Bohm phase.

Some explanation is required to justify labelling this  $U(1)$  as an Aharonov-Bohm phase. The question is whether this phase occurs within the partition function for a D-brane that participates in an Aharonov-Bohm experiment. The interaction between the pair of D-branes involved in such an experiment will be mediated by open strings connecting the two branes, and to produce an Aharonov-Bohm phase they must be sensitive to their relative orientations. Only fermions which become massless when the branes coincide are capable of detecting their relative orientations. While the Neveu-Schwarz sector open string zero point energy is sometimes greater than zero [14], Ramond-sector open strings always have a zero point energy equal to zero. Therefore, there will be massless fermions whose sensitivity to relative orientation will affect the partition function, generating an Aharonov-Bohm phase.

In the Type I case, these open string interactions can be viewed from the perspective of the effective gauge theory defined on the worldvolume of 9-branes used to construct the D-brane system. The two D-branes correspond to topological defects in the gauge bundle defined on the 9-brane system, and the K-theoretic pairing specified above measures the topological phase induced by the relative motion of the defects. We will return to this gauge bundle perspective later, in Section 3.2.

### 3. Analytical Aspects

Here we describe the pairing

$$K_1(X; \mathbb{Z}) \times K^1(X;U(1)) \longrightarrow U(1) \quad (3.1)$$

from an analytic point of view. We begin by reviewing relevant material from [15] to define the reduced eta-invariant, and relate it to the topological pairing from Section 2. We then use the eta-invariant to calculate the phase for an Aharonov-Bohm experiment involving a  $(-1)-8$  brane system in Type I theory.

#### 3.1. The Analytic Formulation

First we define a  $\mathbb{Z}_2$ -graded cocycle in  $K^1(X;U(1))$  to be a quadruple  $\mathcal{U} = (V_{\pm}, h^{V_{\pm}}, \nabla^{E_{\pm}}, \omega)$ , where

- (i)  $V = V_+ \oplus V_-$  is a  $\mathbb{Z}_2$ -graded vector bundle on  $X$ ,

- (ii)  $h^V = h^{V_+} \oplus h^{V_-}$  is a Hermitian metric on  $V$ ,
- (iii)  $\nabla^V = \nabla^{V_+} \oplus \nabla^{V_-}$  is a Hermitian connection on  $V$ ,
- (iv)  $\omega \in \Omega^{\text{odd}}/im(d)$  satisfies  $d\omega = \text{ch}_{\mathbb{Q}}(\nabla^V)$ .

We may define a  $\mathbb{Z}_2$ -graded cocycle in  $KO^1(X;U(1))$  analogously, by replacing the adjectives ‘‘complex’’ and ‘‘Hermitian’’ with ‘‘real’’ and ‘‘symmetric,’’ respectively.

Next, recall that a K-cycle in  $K_1(X)$  is a triple  $(M, E, f)$  with  $M$  a closed odd-dimensional  $Spin^{\mathbb{C}}$ -manifold,  $E$  a complex vector bundle on  $M$ , and  $f : M \rightarrow X$  a continuous map. We will in fact let  $f$  be smooth here. Again note that there is an analogous real formulation. For the rest of this subsection, however, we will restrict our attention to the complex case.

Since  $M$  is  $Spin^{\mathbb{C}}$ , the principle  $GL(\dim(M))$ -bundle on  $M$  may be reduced to a principle  $Spin^{\mathbb{C}}$ -bundle, call it  $P$ . We may associate to  $P$  a Hermitian line bundle  $L$  on  $M$  [16]. Choose a Hermitian connection  $\nabla^L$  on  $L$ , a Hermitian metric  $h^E$  on  $E$ , and a Hermitian connection  $\nabla^E$  on  $E$ .

Let  $\hat{A}(\nabla^{TM}) \in \Omega^{\text{even}}(M)$  be the closed form representing  $\hat{A}(TM) \in H^{\text{even}}(M; \mathbb{Q})$ . Also, let  $\exp[c_1(\nabla^L)/2] \in \Omega^{\text{even}}(M)$  be the closed form representing  $\exp[c_1(L)/2] \in H^{\text{even}}(M; \mathbb{Q})$ . Finally, let the spinor bundle of  $M$  be  $S_M$ .

Then given a  $\mathbb{Z}_2$ -graded cocycle  $\mathcal{U} \in K^1(X;U(1))$ , we let  $D_{f^*\nabla^{V_{\pm}}}$  denote the Dirac-type operator acting on sections of  $S_M \otimes E \otimes f^*V_{\pm}$ . Its reduced eta-invariant is [17]

$$\bar{\eta}(D_{f^*\nabla^{V_{\pm}}}) = \frac{1}{2} [\bar{\eta}(D_{f^*\nabla^{V_+}}) + \dim(\text{Ker}(D_{f^*\nabla^{V_+}}))] \text{ mod } \mathbb{Z}. \quad (3.2)$$

Then the reduced eta-invariant of  $f^*\mathcal{U}$  is the  $\mathbb{R}/\mathbb{Z}$ -valued function

$$\bar{\eta}(f^*\mathcal{U}) = \bar{\eta}(D_{f^*\nabla^{V_+}}) - \bar{\eta}(D_{f^*\nabla^{V_-}}) - \int_M \hat{A}(\nabla^{TM}) \wedge \exp\left[\frac{c_1(\nabla^L)}{2}\right] \wedge \text{ch}_{\mathbb{Q}}(\nabla^E) \wedge f^*\omega. \quad (3.3)$$

Finally, given a cycle  $\mathcal{K} = (M, E, f)$  in  $K_1(X)$  and a  $\mathbb{Z}_2$ -graded cocycle  $\mathcal{U}$  for  $K^1(X;U(1))$ , their  $\mathbb{R}/\mathbb{Z}$ -valued pairing is

$$\langle [\mathcal{K}], [\mathcal{U}] \rangle = \bar{\eta}(f^*\mathcal{U}). \quad (3.4)$$

The proof that this is in fact the correct pairing is given in [15], and is based largely on the corresponding proof in [17]. We claim that this yields the Aharonov-Bohm phase as

$$2\pi i \bar{\eta}(f^*\mathcal{U}). \quad (3.5)$$

Note that we can use the D-brane charges instead of the RR fields in our prescriptions. The K-theory class associated with an RR-field may be mapped in a well-defined way to the K-theory class associated with the D-brane charge via the isomorphism

$$K_{\text{cpt}}^i(M) \cong \frac{K^{i-1}(\partial M)}{j(K^{i-1}(M))}, \quad (3.6)$$



where  $j$  restricts a K-theory class from  $M$  to  $\partial M$ . The analogous isomorphism holds for the KO-theory as well. Therefore, if desired we may change our prescriptions to begin with the K-theory classes associated with the charges of the D-branes.

### 3.2. Calculation in the Type I Case

We now consider an Aharonov-Bohm experiment for a  $(-1) - 8$  system of Type I D-branes. The path of the instanton defines the K-cycle  $(M, e^1, f)$  as discussed above. We use the 32 nine-branes required for tadpole cancellation to construct our system without adding extra branes/antibranes. It will be convenient to work from the K-theory classes of the D-brane charges instead of those of the RR fields.

The 8-brane determines the nontrivial element of  $KO_{\text{tors}}^0(S^1) = \mathbb{Z}_2$ , which we view as  $KO_{\text{tors}}^0(\mathbb{R}^1)$  with compact support. Such an element is given by the pair  $(E_8, pt)$  where  $E_8$  is a rank 1 bundle and  $pt$  is the trivial rank 0 bundle. We lift this via the Bockstein to an element  $(E'_8, pt) \in KO^{-1}(\mathbb{R}; U(1))$ . Note that  $E'_8$  is also a rank 1 bundle.

Next, the  $(-1)$ -brane determines an element of  $KO^0(S^{10}) = \mathbb{Z}_2$ . This is the pair  $(E_{-1}, pt)$ . Taking the KO-cup product we get  $(E_{-1} \otimes E'_8, pt) \in KO^{-1}(\mathbb{R}^{11}; U(1))$ . After complexifying these bundles, we then obtain a  $\mathbb{Z}_2$ -graded cocycle  $V = (E_{-1} \otimes E'_8 \otimes \mathbb{C}) \oplus pt$  in  $K^{-1}(\mathbb{R}^{11}; U(1))$ .

We also have the associated Dirac operator along  $M$  for the  $E_{-1} \otimes E'_8 \otimes \mathbb{C}$  component of the cocycle,

$$\mathcal{D}_a^+ = \mathcal{D}(A'_{-1}) + \mathcal{D}(A_8) + \Gamma^9 a, \quad (3.7)$$

where  $a$  parametrizes  $M$  as the distance between the 8-brane and the  $(-1)$ -brane as above, and the  $+$  superscript indicates that it corresponds to the  $V_+ \equiv E_{-1} \otimes E'_8 \otimes \mathbb{C}$  component. Since  $E'_8$  is a rank 1 bundle, the index theorem [17, 18] says that  $\mathcal{D}(A'_{-1}) + \mathcal{D}(A_8)$  has one zero mode of definite chirality with respect to  $\prod_{\mu=0}^9 \Gamma^\mu$  and  $\Gamma^9$  (in the nontrivial instanton number sector).

Thus  $\mathcal{D}_a^+$  has eigenvalue  $a$ .

Our analytic pairing can be evaluated by using the pullback via  $f$  of  $V = (E_{-1} \otimes E'_8 \otimes \mathbb{C}) \oplus pt$  from  $X$  to  $M$ . Since the eigenvalue of  $\mathcal{D}_a^+$  is equal to  $a$ , we get  $\bar{\eta}(D_{f^* \nabla V_+}) = 1/2$ . Also, since  $E_{-1}$  and  $E'_8$  are each  $SO(n)$ -bundles, their connection and curvature forms are  $so(n)$ -valued, hence have zero trace. Complexification does not change this, so that the Chern character of  $V$  is zero. Then since  $d\omega = \text{ch}_{\mathbb{Q}}(\nabla V) = 0$  and  $\omega \in \Omega^{\text{odd}}/im(d)$ , we get  $\omega = 0$  and  $f^* \omega = 0$ . Finally, note that  $\bar{\eta}(D_{f^* \nabla V_-}) = 0$  since  $V_- = pt$ .

Then we find that our pairing gives

$$\bar{\eta}(f^* \mathcal{U}) = \bar{\eta}(D_{f^* \nabla V_+}) - \bar{\eta}(D_{f^* \nabla V_-}) - \int_M \hat{A}(\nabla^{TM}) \wedge e^{(c_1(L)/2)} \wedge \text{ch}_{\mathbb{Q}}(\nabla^{e^1}) \wedge f^* \omega = \frac{1}{2}. \quad (3.8)$$

Consequently, we get an Aharonov-Bohm phase of  $2\pi i(1/2) = i\pi$ , and the monodromy is  $\exp(i\pi) = -1$ .

This result agrees with the calculation performed in [5] for the  $(-1) - 8$  system, which was performed by examining changes in massless fermionic contributions to the amplitude as the instanton is moved.

## 4. Summary

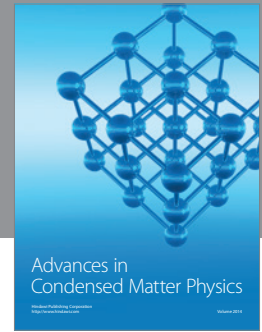
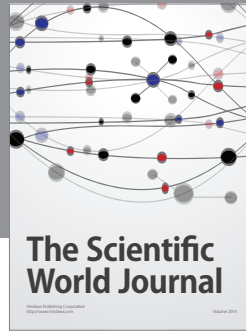
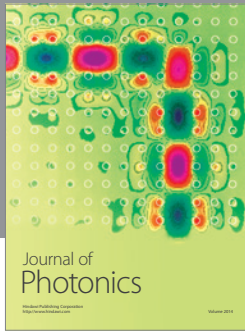
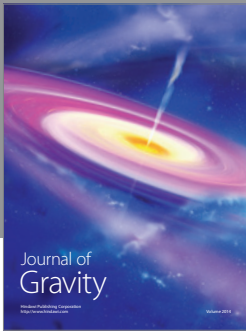
In this paper we have developed formulae to calculate the Aharonov-Bohm phase of torsion Ramond-Ramond fluxes in the Type II and Type I string theories based upon the K-theoretic classification of Ramond-Ramond fields and D-brane charges. These formulae were constructed in two different but equivalent fashions, one being purely topological and the other employing the reduced eta-invariant. The topological pairing was shown to exist and be well defined. The analytic perspective was used to calculate the phase for the  $(-1) - 8$  system in Type I theory, allowing us to test our formulae by comparison with independent calculations.

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