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Research Article

Discreteness of Curved Spacetime from GUP

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Diverse theories of quantum gravity expect modifications of the Heisenberg's uncertainty principle near the Planck scale to a so-called Generalized uncertainty principle (GUP). It was shown by some authors that the GUP gives rise to corrections to the Schrodinger , Klein-Gordon, and Dirac equations. By solving the GUP corrected equations, the authors arrived at quantization not only of energy but also of box length, area, and volume. In this paper, we extend the above results to the case of curved spacetime (Schwarzschild metric). We showed that we arrived at the quantization of space by solving Dirac equation with GUP in this metric.

1. Introduction

Diverse approaches to quantum gravity expect a minimum measurable length and a modification of the Heisenberg uncertainty principle to a so-called generalized uncertainty principle or GUP. This implies a modification of the commutation relations between position coordinates and momentum. In [1], the following proposed GUP is consistent with Doubly special relativity or DSR theories and black hole physics which ensure that $[x_i, x_j] = 0 = [p_i, p_j]$:

$$\begin{aligned} \left[x_i \cdot p_j\right] &= ih\left[\delta_{ij} - a\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + a^2\left(p^2\delta_{ij} + 3p_i p_j\right)\right], \\ \Delta x \Delta p &\geq \frac{h}{2}\left[1 - 2a\langle p\rangle + 4a^2\langle p^2\rangle\right] \\ &\geq \frac{h}{2}\left[1 + \left(\frac{a}{\sqrt{\langle p^2\rangle}} + 4a^2\right)\Delta p^2 + 4a^2\langle p\rangle^2 - 2a\sqrt{\langle p^2\rangle}\right], \end{aligned} \tag{1}$$

where $a = a_0/M_{\rm pl}c = a_0 L_{\rm pl}/h$, $M_{\rm pl}$ = Planck mass, $L_{\rm pl} \approx 10^{-35}$ m = Planck length, and $M_{\rm pl}c^2$ = Planck energy $\approx 10^{19}$ GeV. GUP-induced terms become important near the Planck scale (for earlier version of GUP motivated by string theory, black hole physics, and DSR, see [2–14], and for some phenomenological implications, see [1, 15, 16]).

Equation (1) implies the following minimum measurable length and maximum measurable momentum [1, 17]:

$$\begin{aligned} \Delta x &\geq (\Delta x)_{\min} \approx a_0 L_{\rm pl}, \\ \Delta p &\leq (\Delta p)_{\max} \approx \frac{M_{\rm pl}c}{a_0}. \end{aligned} \tag{2}$$

It is natural to take $a_0 = 1$; for more details see [17].

The following definitions are proposed in [1] and used in [1, 17]:

$$x_i = x_{0i}, \qquad p_i = p_{0i}\left(1 - ap_0 + 2a^2 p_0^2\right) \tag{3}$$

with x_{0i} and p_{0j} satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = ih\delta_{ij}$, such that $p_{0i} = -ih(\partial/\partial x_{0i})$, $p_0^2 = \sum_{j=1}^{3} p_{0j}p_{0j}$.

In [1], it was shown that any nonrelativistic Hamiltonian of the form $H = p^2/2m + V(\overrightarrow{r})$ can be written as $H = p_0^2/2m - (a/m)p_0^3 + V(\overrightarrow{r}) + O(a^2)$ using (3). This corrected Hamiltonian implies not only the usual quantization of energy, but also that the box length is quantized. In [17], the above results were extended to a relativistic particle in two and three dimensions. In this paper we study Dirac equation in Schwarzschild metric using GUP and show that we arrive at the quantization of space.

2. GUP Dirac Equations in Schwarzschild Metric

Dirac equation in Schwarzschild metric without (GUP) can be written as follows [18]:

$$\left(c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p}\right)+\beta mc^2\right)\psi=\frac{E}{\sqrt{\zeta}}\psi, \quad (4)$$

where m is the rest mass of the particle, ψ is the Dirac spinor, $\overrightarrow{\alpha}$ and β are Dirac matrices, $\overrightarrow{p}\equiv(p_r,p_\theta,p_\varphi)$ are momentum operators, $\zeta=1-r_s/r$, r_s is the Schwarzschild radius of massive body, related to its mass M by $r_s=2GM/C^2$, G is the gravitational constant, and c is the speed of light in free space. Using $\psi=\binom{\chi_1}{\chi_2}$, (4) can be written as

$$\begin{aligned} c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p}\right)\chi_2+mc^2\chi_1&=\frac{E}{\sqrt{\zeta}}\chi_1,\\ c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p}\right)\chi_1-mc^2\chi_2&=\frac{E}{\sqrt{\zeta}}\chi_2. \end{aligned} \quad (5)$$

Now, using GUP correction (3) and (5) take the form

$$\begin{aligned} c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)\chi_2+mc^2\chi_1-cap_0^2\chi_1&=\frac{E}{\sqrt{\zeta}}\chi_1,\\ c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)\chi_1-mc^2\chi_2-cap_0^2\chi_2&=\frac{E}{\sqrt{\zeta}}\chi_2, \end{aligned} \quad (6)$$

where

$$p_0^2=-h^2\left[\frac{\sqrt{\zeta}}{r^2}\frac{\partial}{\partial r}\left(r^2\sqrt{\zeta}\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\times\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \quad (7)$$

We study Dirac equations in (6) in Schwarzschild metric with spherical cavity with radius R defined by the potential

$$\begin{aligned} U(r)&=0, \quad r\le R,\\ U(r)&=U_0\longrightarrow\infty, \quad r>R, \end{aligned} \quad (8)$$

so, we can write the corrected GUP Dirac equations with spherical cavity defined by (8) in Schwarzschild metric as

$$\begin{aligned} c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)\chi_2+\left(mc^2+U\right)\chi_1-cap_0^2\chi_1&=\frac{E}{\sqrt{\zeta}}\chi_1,\\ c\left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)\chi_1-\left(mc^2+U\right)\chi_2-cap_0^2\chi_2&=\frac{E}{\sqrt{\zeta}}\chi_2. \end{aligned} \quad (9)$$

Notice that, when $a=0$, $\zeta=1$, equations in (9) are usual Dirac equations in flat spacetime. When $a\neq 0$, $\zeta=1$, (9) are Dirac equations with GUP in flat spacetime proposed in [17]. When $a=0$, $\zeta\neq 1$, equations in (9) are Dirac equations in Schwarzschild metric without GUP defined in [18]. We follow the analysis of [17, 19] and related references [20, 21].

We assume the form of Dirac spinor as

$$\psi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}=\begin{pmatrix} g_k(r)\,y_{jl}^{j_3}(\hat{r})\\ if_k(r)\,y_{jl^-}^{j_3}(\hat{r})\end{pmatrix}, \quad (10)$$

$$\begin{aligned} y_{jl}^{j_3}(\hat{r})=&\left(l\ \ \frac{1}{2}\ \ j_3-\frac{1}{2}\ \ \frac{1}{2}\mid j\ \ j_3\right)\times Y_l^{j_3-(1/2)}(\hat{r})\begin{pmatrix}1\\0\end{pmatrix}\\ &+\left(l\ \ \frac{1}{2}\ \ j_3+\frac{1}{2}\ \ \frac{-1}{2}\mid j\ \ j_3\right)\times Y_l^{j_3+(1/2)}(\hat{r})\begin{pmatrix}0\\1\end{pmatrix}, \end{aligned} \quad (11)$$

where $Y_l^{j_3-(1/2)}(\hat{r})$ and $Y_l^{j_3+(1/2)}(\hat{r})$ are spherical harmonics and $(j_1\ \ j_2\ \ m_1\ \ m_2\mid j\ \ j_3)$ are Clebsh-Gordon coefficients, χ_1,χ_2 are eigenstates of L^2 ($\overrightarrow{L}$ is the angular momentum operator) with eigenvalues $h^2l(l+1)$ and $h^2l^-(l^-+1)$, respectively, such that the following hold:

if

$$k=j+\frac{1}{2}>0, \quad (12)$$

then

$$l=k=j+\frac{1}{2}, \qquad l^-=k-1=j-\frac{1}{2}, \quad (13)$$

and if

$$k=-\left(j+\frac{1}{2}\right)<0, \quad (14)$$

then

$$l=-(k+1)=j-\frac{1}{2}, \qquad l^-=-k=j+\frac{1}{2}. \quad (15)$$

We use $(\overrightarrow{\alpha}\cdot\overrightarrow{A})(\overrightarrow{\alpha}\cdot\overrightarrow{B})=\overrightarrow{A}\cdot\overrightarrow{B}+i\overrightarrow{\alpha}\cdot(\overrightarrow{A}\times\overrightarrow{B})$ and the related identity $(\overrightarrow{\alpha}\cdot\overrightarrow{r})(\overrightarrow{\alpha}\cdot\overrightarrow{r})=r^2$; so we have

$$\begin{aligned} \left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)&=\frac{\left(\overrightarrow{\alpha}\cdot\overrightarrow{r}\right)\left(\overrightarrow{\alpha}\cdot\overrightarrow{r}\right)\left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)}{r^2}\\ &=\frac{\overrightarrow{\alpha}\cdot\overrightarrow{r}}{r^2}\left(\overrightarrow{r}\cdot\overrightarrow{p_0}+i\overrightarrow{\alpha}\cdot\left(\overrightarrow{r}\times\overrightarrow{p_0}\right)\right)\\ &=\frac{\overrightarrow{\alpha}\cdot\overrightarrow{r}}{r^2}\left(\overrightarrow{r}\cdot\overrightarrow{p_0}+i\overrightarrow{\alpha}\cdot\overrightarrow{L}\right). \end{aligned} \quad (16)$$

But from the definition of the momentum operators in Schwarzschild metric [18], we can write (16) as

$$\left(\overrightarrow{\alpha}\cdot\overrightarrow{p_0}\right)=\left(\overrightarrow{\alpha}\cdot\hat{r}\right)\left(-ih\sqrt{\zeta}\frac{\partial}{\partial r}+\frac{i}{r}\overrightarrow{\alpha}\cdot\overrightarrow{L}\right). \quad (17)$$

Also, we have

$$\left(\overrightarrow{\alpha}\cdot\overrightarrow{L}+1\right)\chi_{1,2}=\mp\chi_{1,2},$$

$$\left(\overrightarrow{\alpha}\cdot\widehat{r}\right)\gamma^{j_3}_{jl}\left(\widehat{r}\right)=-\gamma^{j_3}_{jl^-}\left(\widehat{r}\right),\qquad\left(\overrightarrow{\alpha}\cdot\widehat{r}\right)\gamma^{j_3}_{jl^-}\left(\widehat{r}\right)=-\gamma^{j_3}_{jl}\left(\widehat{r}\right).\tag{18}$$

Next, from the definition of p_0^2 in Schwarzschild metric [18] we can write

$$p_0^2F(r)Y_l^m=h^2\left[-\frac{\sqrt{\zeta}}{r^2}\frac{d}{dr}\left(r^2\sqrt{\zeta}\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}\right]F(r)Y_l^m.\tag{19}$$

So, using (17), (18), and (19), we can obtain from (9) the following equations:

$$\begin{aligned}&-ch\sqrt{\zeta}\frac{df_k}{dr}+\frac{c(k-1)}{r}f_k+\left(mc^2+U\right)g_k\\&\quad+cah^2\left[\frac{\sqrt{\zeta}}{r^2}\frac{d}{dr}\left(\sqrt{\zeta}r^2\frac{dg_k}{dr}\right)-\frac{l(l+1)}{r^2}g_k\right]=\frac{E}{\sqrt{\zeta}}g_k,\\&ch\sqrt{\zeta}\frac{dg_k}{dr}+\frac{c(k+1)}{r}g_k-\left(mc^2+U\right)f_k\\&\quad+cah^2\left[\frac{\sqrt{\zeta}}{r^2}\frac{d}{dr}\left(\sqrt{\zeta}r^2\frac{df_k}{dr}\right)-\frac{l^-(l^-+1)}{r^2}f_k\right]=\frac{E}{\sqrt{\zeta}}f_k.\end{aligned}\tag{20}$$

It can be shown that MIT bag boundary condition (at $r=R$) is equivalent to [19, 20]

$$\widetilde{\psi}\psi=0.\tag{21}$$

As in [17], we can expect new nonperturbative solutions of the forms $f_k=F_K(r)e^{i\epsilon r/ah}$ and $g_k=G_K(r)e^{i\epsilon r/ah}$ (where $\epsilon=O(1)$) for which (20) simplifies to

$$\begin{aligned}ah\frac{d^2g_k}{dr^2}&=\sqrt{\zeta}\frac{df_k}{dr},\\ah\frac{d^2f_k}{dr^2}&=-\sqrt{\zeta}\frac{dg_k}{dr},\end{aligned}\tag{22}$$

where we have dropped terms which are ignorable for small a.

When $\zeta=1$, equations in (22) are identical to the (60)-(61) in reference [17], and in this case we have the following solutions: $f_k^N=iNe^{ir/ah}$, $g_k^N=Ne^{ir/ah}$, N is constant; so we can assume the solutions of (22) as

$$\begin{aligned}f_k^N&=iNe^{ir/ah},\\g_k^N&=NB(r)e^{ir/ah}.\end{aligned}\tag{23}$$

By applying (23) on (22), we find that

$$a^2h^2\frac{d^2B(r)}{dr^2}+B(r)+\sqrt{\zeta}-\frac{2}{\sqrt{\zeta}}=0.\tag{24}$$

Consider that r_s is very small, so we can approximate (24) to

$$a^2h^2\frac{d^2B(r)}{dr^2}+B(r)-1-\frac{3r_s}{2r}=0.\tag{25}$$

The solution of (25) takes the form

$$\begin{aligned}B(r)=&c_1\sin\left(\frac{r}{ah}\right)+c_2\cos\left(\frac{r}{ah}\right)\\&+\frac{1.5r_s}{ah}\left[C_i\left(\frac{r}{ah}\right)\sin\left(\frac{r}{ah}\right)\right.\\&\left.-S_i\left(\frac{r}{ah}\right)\cos\left(\frac{r}{ah}\right)\right]+1,\end{aligned}\tag{26}$$

where c_1 and c_2 are constants, $S_i(r/ah)$ and $C_i(r/ah)$ are the sine integral function and cosine integral function defined as $S_i(y)=\int_0^y(\sin t/t)dt$ and $C_i(y)=-\int_y^\infty(\cos t/t)dt$. For more details about this functions see [22, 23].

Therefore, the solutions of (22) take the form

$$\begin{aligned}f_k^N&=iNe^{ir/ah},\\g_k^N&=N\left[1+\sigma(r)\right]e^{ir/ah},\end{aligned}\tag{27}$$

where

$$\begin{aligned}\sigma(r)=&c_1\sin\left(\frac{r}{ah}\right)+c_2\cos\left(\frac{r}{ah}\right)\\&+\frac{1.5r_s}{ah}\left[C_i\left(\frac{r}{ah}\right)\sin\left(\frac{r}{ah}\right)\right.\\&\left.-S_i\left(\frac{r}{ah}\right)\cos\left(\frac{r}{ah}\right)\right].\end{aligned}\tag{28}$$

Here, one must have $\lim_{r_s\to o}c_1=\lim_{r_s\to o}c_2=0$, and, in this case ($\sigma(r)=0$, $r_s=0$), the results are the same of [17].

Now, the boundary condition (21) gives

$$\left|g_k(R)+g_k^N(R)\right|^2=\left|f_k(R)+f_k^N(R)\right|^2,\tag{29}$$

which to $O(a)$ translates to

$$\begin{aligned}&\left(g_k^2-f_k^2\right)+2N\left[g_k(1+\sigma(R))\cos\left(\frac{R}{ah}\right)-f_k\sin\left(\frac{R}{ah}\right)\right]\\&\quad+N^2\left[(1+\sigma(R))^2-1\right]=0.\end{aligned}\tag{30}$$

From (30), we have

$$f_k=g_k,\tag{31}$$

$$\begin{aligned}\tan\left(\frac{R}{ah}\right)&=1+\sigma(R),\\\sigma(R)=0,&\quad\text{or}\quad\sigma(R)=-2.\end{aligned}\tag{32}$$

Observing (28), we can write

$$\begin{aligned}\frac{\sigma(R)}{\cos(R/ah)}=&c_1\tan\left(\frac{R}{ah}\right)+c_2\\&+\frac{1.5r_s}{ah}\left[C_i\left(\frac{R}{ah}\right)\tan\left(\frac{R}{ah}\right)-S_i\left(\frac{R}{ah}\right)\right].\end{aligned}\tag{33}$$

From (32) we have

$$\text{if} \quad \sigma(R) = 0 \quad \text{then} \quad \tan\left(\frac{R}{ah}\right) = 1,$$
$$\text{if} \quad \sigma(R) = -2 \quad \text{then} \quad \tan\left(\frac{R}{ah}\right) = -1. \tag{34}$$

For the case of $\sigma(R) = 0$ and $r_s = 0$ we have the same result of discreteness of space in flat spacetime [17].

For the second case $\sigma(R) = -2$, we have, from (33),

$$\frac{4ah}{3r_s \cos(R/ah)} + \frac{2Cah}{3r_s} = C_i\left(\frac{R}{ah}\right) + S_i\left(\frac{R}{ah}\right), \quad C \text{ is constant.} \tag{35}$$

If we take the constant $C = 0$, then we have

$$\frac{4ah}{3r_s \cos(R/ah)} = C_i\left(\frac{R}{ah}\right) + S_i\left(\frac{R}{ah}\right). \tag{36}$$

Suppose that we choose $r_s = (4/3)ah$ (very small as assumption); then we have

$$\frac{1}{\cos(R/ah)} = C_i\left(\frac{R}{ah}\right) + S_i\left(\frac{R}{ah}\right). \tag{37}$$

Equation (37) has infinite number of solutions; we write some numerical values of it, $(R/ah) = 5, 565$, $(R/ah) = 7.159$, $(R/ah) = 11$, and 755, $(R/ah) = 13.448$. So, the radius of cavity R has been quantized in terms of ah and we again arrived at the quantization of space in Schwarzschild-like metric.

3. Conclusion

Dirac equations with GUP in Schwarzschild metric have been studied. We showed that the assumption of existence of a minimum measurable length and a corresponding modification of uncertainty principle yields discreteness of space in this metric. But the question now arises, what is the guarantee that this result will continue to hold for more generic curved spacetimes? We expect that this discreteness will always appear provided that generalized uncertainty principle enters into the theory, but in fact we have no mathematical proof of existence of such discreteness of space if we work on the general metric of the general relativity theory.

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