

Research Article

Primordial Dark Energy from a Condensate of Spinors in a 5D Vacuum

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Received 28 October 2013; Accepted 9 December 2013

Academic Editor: Kishor Adhav

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We explore the possibility that the expansion of the universe can be driven by a condensate of spinors which are free of interactions in a 5D relativistic vacuum defined in an extended de Sitter spacetime which is Riemann flat. The extra coordinate is considered as noncompact. After making a static foliation on the extra coordinate, we obtain an effective 4D (inflationary) de Sitter expansion which describes an inflationary universe. We found that the condensate of spinors studied here could be an interesting candidate to explain the presence of dark energy in the early universe. The dark energy density which we are talking about is poured into smaller subhorizon scales with the evolution of the inflationary expansion.

1. Introduction

Modern versions of 5D general relativity abandon the cylinder and compactification conditions used in original Kaluza-Klein (KK) theories, which caused problems with the cosmological constant and the masses of particles, and consider a large extra dimension. The main question that these approaches address is whether the four-dimensional properties of matter can be viewed as being purely geometrical in origin. In particular, the induced matter theory (IMT) [1, 2] is based on the assumption that ordinary matter and physical fields that we can observe in our 4D universe can be geometrically induced from a 5D Ricci-flat metric with a space-like noncompact extra dimension in which we define a physical vacuum. The Campbell-Magaard theorem (CMT) [3] serves as a ladder to go between manifolds whose dimensionality differs by one. Due to this theorem one can say that every solution of the 4D Einstein equations with arbitrary energy-momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in a relativistic vacuum: $G_{AB} = 0$. (We shall consider that capital letters A and B run from 0 to 4 in a 5D extended de

Sitter spacetime (where the 3D Euclidean space is in Cartesian coordinates), small letters a and b run from 0 to 5 in a 5D Minkowsky spacetime (in cartesian coordinates), Greek letters α and β run from 0 to 3, and latin letters i and j run from 1 to 3.). Due to this fact the stress-energy may be a 4D manifestation of the embedding geometry and, therefore, by making a static foliation on the space-like extra coordinate of an extended 5D de Sitter spacetime, it is possible to obtain an effective 4D universe that suffered an exponential accelerated expansion driven by an effective scalar field with an equation of state typically dominated by vacuum [4–7]. An interesting problem in modern cosmology relies on explaining the physical origin of the cosmological constant, which is responsible for the exponential expansion of the early inflationary universe. The standard explanation for the early universe expansion is that it is driven by the inflaton field [8]. Many cosmologists mean that such acceleration (as well as the present day accelerated expansion of the universe) could be driven by some exotic energy called *dark energy*. Most versions of inflationary cosmology require one scalar inflaton field which drives the accelerated expansion of the early universe with an equation of state governed by

the vacuum [9]. The parameters of this scalar field must be rather finely tuned in order to allow adequate inflation and an acceptable magnitude for density perturbations. The need for this field is one of the less satisfactory features of inflationary models. Consequently, we believe that it is of interest to explore variations of inflation in which the role of the scalar field is played by some other field [10, 11]. Recently the possibility that such expansion can be explained by a condensate of dark spinors has been explored [12]. This interesting idea was recently revived in the framework of the induced matter theory (IMT) [13]. In this work we shall extend this idea.

2. The Effective Lagrangian in a 5D Riemann-Flat Spacetime

We are concerned with a 5D Riemann-flat spacetime with a line element given by

$$dS^2 = \left(\frac{\Psi}{\psi_0} \right)^2 \left[dt^2 - e^{2t/\psi_0} (dx^2 + dy^2 + dz^2) \right] - d\psi^2, \quad (1)$$

where t, x, y and z are the usual local spacetime coordinate systems and ψ is the noncompact space-like extra dimension.

We start from an effective Lagrangian density for non massive fermions in 5D:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2} (\nabla_A \bar{\Psi}) (\nabla^A \Psi). \quad (2)$$

At this point it is easy to obtain the equations of motion from a variational principle. The Euler-Lagrange equations for both Ψ and $\bar{\Psi}$ can be obtained making the following functional derivatives:

$$\frac{\delta \mathcal{L}_{\text{eff}}}{\delta \bar{\Psi}} = 0, \quad (3)$$

$$\nabla_A \frac{\delta \mathcal{L}_{\text{eff}}}{\delta (\nabla_A \bar{\Psi})} = \frac{1}{2} \nabla_A \nabla^A \Psi, \quad (4)$$

$$\frac{\delta \mathcal{L}_{\text{eff}}}{\delta g_{MN}} = \frac{1}{2} \nabla^M \bar{\Psi} \nabla^N \Psi + \nabla_P J^{MNP}, \quad (5)$$

where the effective current J^{MNP} is symmetric with respect to permutations of M and N ,

$$J^{MNP} = \frac{1}{8} (\nabla^M \bar{\Psi} f^{NP} \Psi + \bar{\Psi} \nabla^M \Psi), \quad (6)$$

and $f^{NP} = [\gamma^N, \gamma^P]$ is antisymmetric [14]. At this point we are in conditions of introducing the stress tensor $T^{MN} = 2(\delta \mathcal{L}_{\text{eff}} / \delta g_{MN}) - g^{MN} \mathcal{L}_{\text{eff}}$:

$$T^{MN} = \nabla^M \bar{\Psi} \nabla^N \Psi + 2 \nabla_P J^{MNP} + \frac{1}{2} g^{MN} (g_{AB} \nabla^A \bar{\Psi} \nabla^B \Psi). \quad (7)$$

Applying the compatibility condition on the metric $\nabla_C g^{AB} = 0$, we obtain

$$\nabla_A \nabla^A \Psi = \nabla_A (g^{AB} \nabla_B \Psi) = (\nabla_A g^{AB}) \nabla_B \Psi + g^{AB} \nabla_A \nabla_B \Psi = 0, \quad (8)$$

and we obtain that the equation for the spinor Ψ takes the form

$$g^{AB} \nabla_A \nabla_B \Psi = 0. \quad (9)$$

The same procedure yields an identical equation for the field $\bar{\Psi}$. On the other hand, the 5D Einstein equation for the Riemann-flat metric (1) is

$$\langle 0 | T_{AB} | 0 \rangle = 0, \quad (10)$$

where $\langle 0 | T_{AB} | 0 \rangle$ denotes the expectation value of T_{AB} in the vacuum state $|0\rangle$.

2.1. Tensorial Formulation of the Equation of Motion. Using the formalism previously introduced, the double Nabla can be expressed explicitly:

$$\begin{aligned} \nabla_A \nabla_B \Psi &= \partial_A \nabla_B \Psi + \Gamma_A \nabla_B \Psi - \omega_{AB}{}^C \nabla_C \Psi \\ &= \partial_A \partial_B \Psi - \frac{1}{4} \partial_A \omega_B^{ab} \gamma_a \gamma_b \Psi - \frac{1}{4} \omega_B^{ab} \gamma_a \gamma_b \partial_A \Psi \\ &\quad - \frac{1}{4} \omega_A^{ab} \gamma_a \gamma_b \partial_B \Psi \\ &\quad + \frac{1}{16} \omega_A^{ab} \omega_B^{cd} \gamma_a \gamma_b \gamma_c \gamma_d \Psi - \omega_{AB}{}^C \partial_C \Psi \\ &\quad + \frac{1}{4} \omega_{AB}{}^C \omega_C^{ab} \gamma_a \gamma_b \Psi, \end{aligned} \quad (11)$$

where the spin connection is $\omega_M^{ab} = -e_N^a [\partial_M e_a^b g^{AN} + e_B^b g^{AB} \Gamma_{AM}^N]$ and $\Gamma_M = -(1/8) \omega_M^{ab} [\gamma_a, \gamma_b]$.

(The tensors can be written using the vielbein e_a^A and its inverse e_A^a , such that, if $e_a^A e_A^b = \delta_a^b$, then

$$\eta_{ab} = e_a^A e_b^B g_{AB}, \quad (12)$$

where η_{ab} is the 5D Minkowsky tensor metric with signature $(+, -, -, -, -)$.)

Thus, after replacing the last expression in the equation of motion (9) we obtain

$$\begin{aligned} g^{AB} \partial_A \partial_B \Psi - \frac{1}{2} g^{AB} \omega_{(A}^{ab} \sigma_{ab} \partial_{B)} \Psi \\ - g^{AB} \omega_{AB}{}^C \partial_C \Psi + \frac{1}{4} g^{AB} \partial_A \omega_B^{ab} \sigma_{ab} \Psi \\ + \frac{1}{16} g^{AB} \omega_A^{ab} \omega_B^{cd} \sigma_{ab} \sigma_{cd} \Psi - \frac{1}{4} g^{AB} \omega_{AB}{}^C \omega_C^{ab} \sigma_{ab} \Psi = 0. \end{aligned} \quad (13)$$

Here, we have made use of the fact that $\gamma_a \gamma_b = (1/2) \{\gamma_a, \gamma_b\} + (1/2) [\gamma_a, \gamma_b] = g_{ab} \mathbb{1} + \sigma_{ab}$, $\omega_M^{ab} = -\omega_M^{ba}$ and $\omega_M^{ab} \gamma_a \gamma_b = \omega_M^{ab} \sigma_{ab}$. Once we simplify some terms, we obtain

$$\begin{aligned} \frac{1}{2} g^{AB} \omega_{(A}^{ab} \sigma_{ab} \partial_{B)} \Psi = \frac{1}{2} g^{AB} \omega_A^{ab} \sigma_{ab} \partial_B \Psi, \\ \omega_{AB}{}^C = \omega_A^{DC} g_{DB} = \omega_A^{dc} g_{DB} e_d^D e_c^C, \end{aligned} \quad (14)$$

$$g^{AB} \omega_{AB}{}^C = g^{AB} \omega_A^{dc} g_{DB} e_d^D e_c^C = \omega_A^{dc} \delta_D^A e_d^D e_c^C = \omega_A^{dc} e_d^A e_c^C.$$

Finally, the equation of motion for the spinors assumes its final form

$$\begin{aligned}
& g^{AB} \partial_A \partial_B \Psi - \frac{1}{2} g^{AB} \omega_A^{ab} \sigma_{ab} \partial_B \Psi \\
& - \omega_A^{ab} e_a^A e_b^C \partial_C \Psi + \frac{1}{4} g^{AB} \partial_A \omega_B^{ab} \sigma_{ab} \Psi \\
& + \frac{1}{16} g^{AB} \omega_A^{ab} \omega_B^{cd} \sigma_{ab} \sigma_{cd} \Psi \\
& - \frac{1}{4} \omega_A^{ab} e_a^A e_b^C \omega_C^{cd} \sigma_{cd} \Psi = 0,
\end{aligned} \tag{15}$$

which is very difficult to be resolved because the fields are coupled.

2.2. Conformal Mapping Based Solution. In order to simplify the structure of (15), we shall introduce the following transformation on the spinor components:

$$\Psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{16}$$

where components are grouped as

$$\varphi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{17}$$

With this representation we obtain the equation of motion for φ_1 and φ_2 :

$$\begin{aligned}
& \widehat{\mathcal{O}}\varphi_1 + \frac{3\psi_0}{\psi^2} \frac{\partial \varphi_1}{\partial t} - \frac{4}{\psi} \frac{\partial \varphi_1}{\partial \psi} + \frac{1}{4\psi^2} \varphi_1 - \frac{i\psi_0}{\psi^2} e^{-t/\psi_0} \vec{\sigma} \cdot \vec{\nabla} \varphi_1 \\
& = -\frac{i\psi_0}{\psi^2} \frac{\partial \varphi_2}{\partial t} + \frac{3i}{2\psi^2} \varphi_2 - \frac{\psi_0}{\psi^2} e^{-t/\psi_0} \vec{\sigma} \cdot \vec{\nabla} \varphi_2, \\
& \widehat{\mathcal{O}}\varphi_2 + \frac{3\psi_0}{\psi^2} \frac{\partial \varphi_2}{\partial t} - \frac{4}{\psi} \frac{\partial \varphi_2}{\partial \psi} + \frac{1}{4\psi^2} \varphi_2 + \frac{i\psi_0}{\psi^2} e^{-t/\psi_0} \vec{\sigma} \cdot \vec{\nabla} \varphi_2 \\
& = \frac{i\psi_0}{\psi^2} \frac{\partial \varphi_1}{\partial t} - \frac{3i}{2\psi^2} \varphi_1 - \frac{\psi_0}{\psi^2} e^{-t/\psi_0} \vec{\sigma} \cdot \vec{\nabla} \varphi_1.
\end{aligned} \tag{18}$$

Here, we have adopted the following conventions:

$$\begin{aligned}
& \widehat{\mathcal{O}}\varphi = \left(\frac{\psi_0}{\psi} \right)^2 \frac{\partial^2 \varphi}{\partial t^2} - \left(\frac{\psi_0}{\psi} \right)^2 e^{-2t/\psi_0} \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial \psi^2}, \\
& \vec{\sigma} = \sigma_1 \hat{i} + \sigma_2 \hat{j} + \sigma_3 \hat{k}, \\
& \vec{\sigma} \cdot \vec{\nabla} \varphi = \sigma_1 \frac{\partial \varphi}{\partial x} + \sigma_2 \frac{\partial \varphi}{\partial y} + \sigma_3 \frac{\partial \varphi}{\partial z}.
\end{aligned} \tag{19}$$

Now we can use the conformal mapping defining new complex fields $\Phi_+ = \varphi_1 + i\varphi_2$ and $\Phi_- = \varphi_1 - i\varphi_2$. Rewriting (18) in terms of these new fields, it is possible to decouple

the first equation, while the other coupling becomes a source for the second equation:

$$\widehat{\mathcal{O}}\Phi_+ + \frac{4\psi_0}{\psi^2} \frac{\partial \Phi_+}{\partial t} - \frac{4}{\psi} \frac{\partial \Phi_+}{\partial \psi} - \frac{5}{4\psi^2} \Phi_+ = 0, \tag{20}$$

$$\begin{aligned}
& \widehat{\mathcal{O}}\Phi_- + \frac{2\psi_0}{\psi^2} \frac{\partial \Phi_-}{\partial t} - \frac{4}{\psi} \frac{\partial \Phi_-}{\partial \psi} + \frac{7}{4\psi^2} \Phi_- \\
& = \frac{i2\psi_0}{\psi^2} e^{-t/\psi_0} \vec{\sigma} \cdot \vec{\nabla} \Phi_+.
\end{aligned} \tag{21}$$

Then, after few calculations, the Lagrangian density written in terms of the new fields takes the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2} \left(\nabla_A \bar{\varphi}_1 \nabla^A \varphi_1 + \nabla_A \bar{\varphi}_2 \nabla^A \varphi_2 \right) \tag{22}$$

or, alternatively, can be written as a function of the pair $\varphi_1 = (1/2)(\Phi_+ + \Phi_-)$, $\varphi_2 = (1/2i)(\Phi_+ - \Phi_-)$:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} \left(\nabla_A \bar{\Phi}_+ \nabla^A \Phi_+ + \nabla_A \bar{\Phi}_- \nabla^A \Phi_- \right). \tag{23}$$

On the other hand the 5D energy-momentum (EM) tensor is represented by ${}^{(5)}T_{AB} = 2(\delta \mathcal{L}_{\text{eff}} / \delta g^{AB}) - g_{AB} \mathcal{L}_{\text{eff}}$. This procedure takes place in a 5D vacuum. Therefore, the effective Lagrangian and the EM tensor are involved directly with the cosmological observables we wish to evaluate. The observables to which we refer are energy density and pressure. Both come from the diagonal part of the EM tensor.

2.3. Extra Dimensional Solution for Φ_+ . We shall use the variable separation method in the homogeneous PDE (20); we obtain the following set of ODEs:

$$\nabla^2 R + \kappa^2 R = 0, \tag{24}$$

$$\frac{\partial^2 T_\kappa^{(+)}}{\partial t^2} + \frac{2}{\psi_0} \frac{\partial T_\kappa^{(+)}}{\partial t} + \left(\kappa^2 e^{-2t/\psi_0} - M_1^2 \right) T_\kappa^{(+)} = 0, \tag{25}$$

$$\psi^2 \frac{\partial^2 \Lambda}{\partial \psi^2} + 4\psi \frac{\partial \Lambda}{\partial \psi} + \left(\frac{5}{4} - M_1^2 \psi_0^2 \right) \Lambda = 0. \tag{26}$$

Equation (24) has a solution that can be written in terms of plane wavefront

$$R(\vec{r}) \sim e^{\pm i\vec{\kappa} \cdot \vec{r}}. \tag{27}$$

The second equation (25) has a general solution:

$$\begin{aligned}
\Lambda^{(+)}(\psi) &= C_1 \left(\frac{\psi}{\psi_0} \right)^{-((3/2) + \sqrt{1 + M_1^2 \psi_0^2})} \\
&+ C_2 \left(\frac{\psi}{\psi_0} \right)^{-((3/2) - \sqrt{1 + M_1^2 \psi_0^2})}.
\end{aligned} \tag{28}$$

Since we are interested in ‘‘localized’’ static solutions, that is, those that decay to zero when ψ tends to infinity, we must choose $C_2 = 0$, so that $n \equiv ((3/2) + \sqrt{1 + M_1^2 \psi_0^2}) > 0$. This choice makes $M_1^2 = (((n - (3/2))^2 - 1) / \psi_0^2) \geq 0$, with $n \geq 3$ and $n \in \mathbb{R}$, in order to get $\sqrt{1 + M_1^2 \psi_0^2} \geq 0$.

2.4. *Extra Dimensional Solution for Φ_- .* Now we are able to calculate the coupling term of the inhomogeneous equation (21) for each mode (see (37)):

$$\begin{aligned} & \frac{2i\psi_0}{\psi^2} e^{-t/\psi_0} \vec{\sigma} \cdot \vec{\nabla} \Phi_{+,\kappa} \\ &= -\frac{2^\nu \Gamma(\nu)}{\sqrt{\pi} \kappa^{\nu-1} \psi_0^{(1/2)+\nu}} \\ & \quad \times e^{i\vec{\kappa} \cdot \vec{x}} e^{(\nu-3)(t/\psi_0)} \left(\frac{\psi}{\psi_0} \right)^{-((7/2)+\sqrt{\nu^2-3})}. \end{aligned} \quad (29)$$

Using the last expression in (21), we obtain a degenerate two-component system; for the spinor $\Phi_{-,\kappa}$. (Henceforth we are concerned with asymptotic solutions; that is, only the infrared limit has cosmological significance.) Again, a plane wavefront satisfies the spatial part. By inserting $\Phi_{-,\kappa} = G_\kappa(t, \psi) e^{i\vec{\kappa} \cdot \vec{r}}$ and multiplying by $(\psi/\psi_0)^2$, we obtain

$$\begin{aligned} & \frac{\partial^2 G_\kappa^{(-)}}{\partial t^2} + \frac{2}{\psi_0} \frac{\partial G_\kappa^{(-)}}{\partial t} + \left(\kappa^2 e^{-2t/\psi_0} + \frac{7}{4\psi_0^2} \right) G_\kappa \\ & - \left[\left(\frac{\psi}{\psi_0} \right)^2 \frac{\partial^2 G_\kappa^{(-)}}{\partial \psi^2} + \frac{4\psi}{\psi_0^2} \frac{\partial G_\kappa^{(-)}}{\partial \psi} \right] \\ & \simeq -\frac{2^\nu \Gamma(\nu) \kappa^{1-\nu}}{\sqrt{\pi} \psi_0^{(1/2)+\nu}} e^{((\nu-3)/\psi_0)t} \left(\frac{\psi}{\psi_0} \right)^{-[(3/2)+\sqrt{\nu^2-3}]}. \end{aligned} \quad (30)$$

This inhomogeneous PDE can be converted to one with a constant coupling. In order to make the right side of (30) constant, we shall propose

$$G_\kappa^{(-)}(t, \psi) = \psi_0^{-2} e^{((\nu-3)/\psi_0)t} \left(\frac{\psi}{\psi_0} \right)^{-[(3/2)+\sqrt{\nu^2-3}]} K_\kappa^{(-)}(t, \psi). \quad (31)$$

Finally, we must solve the following equation:

$$\begin{aligned} & \frac{\partial^2 K_\kappa^{(-)}}{\partial t^2} + \frac{2(\nu-2)}{\psi_0} \frac{\partial K_\kappa^{(-)}}{\partial t} - \left(\frac{\psi}{\psi_0} \right)^2 \frac{\partial K_\kappa^{(-)}}{\partial \psi^2} \\ & - \frac{2}{\psi_0^2} \left[\frac{1}{2} - \sqrt{\nu^2-3} \right] \psi \frac{\partial K_\kappa^{(-)}}{\partial \psi} \\ & + \left[\kappa^2 e^{-2t/\psi_0} + \frac{10-4\nu}{\psi_0^2} \right] K_\kappa^{(-)} \\ & \simeq -\frac{2^\nu \Gamma(\nu)}{\sqrt{\pi} \kappa^{\nu-1} \psi_0^{-(3/2)+\nu}}. \end{aligned} \quad (32)$$

3. Effective Dynamics on the 4D Hypersurface $\psi=1/H_0$

In order to describe the effective 4D dynamics of the physical system in the early inflationary universe with an effective

4D de Sitter expansion, we shall consider a static foliation on the 5D metric (1). The resulting 4D hypersurface after making the static foliation $\psi = \psi_0 = 1/H_0$ describes an effective 3D spatially flat, isotropic, and homogeneous de Sitter four-dimensional expanding universe with a constant Hubble parameter H_0 , with a line element

$$dS^2 \longrightarrow ds^2 = dt^2 - e^{2H_0 t} dr^2. \quad (33)$$

From the relativistic point of view, an observer moves in a comoving frame with the five-velocity $U^\Psi = 0$ on a 4D hypersurface with a scalar curvature ${}^{(4)}R = 12/\psi_0^2 = 12H_0^2$, such that the Hubble parameter H_0 and thus also the cosmological constant $\Lambda_0 = 3H_0^2/(8\pi G)$ are defined by the foliation $H_0 = \psi_0^{-1}$.

3.1. *Time-Dependent Modes of Φ_+ .* The solution for the time-dependent equation (26) can be expanded in terms of first- and second-kind Hankel functions:

$$\begin{aligned} & T_\kappa^{(+)}(t) \\ &= e^{-2H_0 t} \left[C_3 \mathcal{H}_\nu^{(1)} \left(\frac{\kappa}{H_0} e^{-H_0 t} \right) + C_4 \mathcal{H}_\nu^{(2)} \left(\frac{\kappa}{H_0} e^{-H_0 t} \right) \right], \end{aligned} \quad (34)$$

where $\nu = \sqrt{4 + M_1^2 \psi_0^2} \geq 2$. After making a Bunch-Davies normalization of the modes [15] we obtain the solution

$$T_\kappa^{(+)}(t) = \frac{i}{2} \sqrt{\frac{\pi}{H_0}} e^{-2H_0 t} \mathcal{H}_\nu^{(2)} \left(\frac{\kappa}{H_0} e^{-H_0 t} \right). \quad (35)$$

Since we are interested to describe the universe on super-Hubble cosmological scales, we must acquire $\kappa\psi_0 e^{-t/\psi_0} \ll 1$; we reject solutions that go to zero at late times. The asymptotic behavior of $T_\kappa^{(+)}(t)$ on cosmological scales will be

$$T_\kappa^{(+)}(t) \simeq \frac{i}{2} \sqrt{\frac{\pi}{H_0}} \Gamma(\nu) e^{-2H_0 t} \left(\frac{\kappa}{2H_0} e^{-H_0 t} \right)^{-\nu}. \quad (36)$$

Finally, the degenerate two-component spinor Φ_+ can be expanded as a function of the modes

$$\begin{aligned} & \Phi_{+,\kappa} \left(t, \vec{r}, \psi_0 = \frac{1}{H_0} \right) \\ & \simeq iC_1 \frac{2^{\nu-1} \Gamma(\nu)}{\sqrt{\pi}} H_0^{\nu-(1/2)} \kappa^{-\nu} e^{i\vec{\kappa} \cdot \vec{r}} e^{(\nu-2)H_0 t} \end{aligned} \quad (37)$$

and their conjugated complex.

3.2. *The Time-Dependent Modes for Φ_- .* The homogeneous solution $K_\kappa^{(-)}(t, \psi)|_{\text{hom}}$, of (32), is

$$\begin{aligned} K_\kappa^{(-)}(t, \psi)|_{\text{hom}} &= e^{-(\nu-2)t/\psi_0} \left[\bar{C}_3 \mathcal{K}_\mu^{(1)}(\kappa \psi_0 e^{-(t/\psi_0)}) \right. \\ &\quad \left. + \bar{C}_4 \mathcal{K}_\mu^{(2)}(\kappa \psi_0 e^{-(t/\psi_0)}) \right] \\ &\times \left[\bar{C}_1 \left(\frac{\psi}{\psi_0} \right)^{(\sqrt{\nu^2-3}-\sqrt{\nu^2-4\nu+7+M_2^2\psi_0^2})} \right. \\ &\quad \left. + \bar{C}_2 \left(\frac{\psi}{\psi_0} \right)^{(\sqrt{\nu^2-3}+\sqrt{\nu^2-4\nu+7+M_2^2\psi_0^2})} \right], \end{aligned} \quad (38)$$

where $\mu = \sqrt{\nu^2 - 4\nu + 4 + M_2^2\psi_0^2} \geq 0$ and the squared mass of $\Phi_{(-)}$ is $[M_2(m, n)]^2 = ((m - 3/2)^2 - (n - 3/2)^2 + (4\sqrt{(n - 3/2)^2 + 3 - 10})/\psi_0^2)$, which is definitely positive for $m \geq 1$ (with $n \geq 3$). In order get to the ψ -dependent solution of $\lim_{\psi \rightarrow \infty} K_\kappa^{(-)}(t, \psi)|_{\text{hom}} \rightarrow 0$, we shall require that $\bar{C}_2 = 0$ and $m > 3/2 + \sqrt{3 + (\sqrt{(n - 3/2)^2 + 3 - 2})^2}$, for $n \geq 3$, such that $(m, n) \in \mathbb{Z}$. After taking the asymptotic limit on cosmological scales we obtain that the modes $\Phi_{-\kappa}(t, \vec{r}, \psi_0 = 1/H_0)$, for $\mu = 1$, are

$$\Phi_{-\kappa}(t, \vec{r}, \psi_0) \simeq A_2 \frac{H_0^{1/2}}{\sqrt{\pi\kappa}} e^{i\vec{\kappa} \cdot \vec{r}}, \quad (39)$$

where $A_2 = \bar{C}_4 \bar{C}_1$. Notice that we have neglected the inhomogeneous part of its solution because it is negligible on these large super-Hubble scales at the end of inflation.

As can be demonstrated the solution $K_\kappa^{(-)}(t, \psi_0) \simeq K_\kappa^{(-)}(t, \psi_0)|_{\text{hom}}$ on cosmological scales, once we consider $H_0 = 1/\psi_0 = 1 \times 10^{-9} M_p$. Hence, the homogeneous solution $K_\kappa^{(-)}(t, \psi_0)|_{\text{hom}}$ is a very acceptable solution at the end of inflation for the time-dependent modes of Φ_- . In other words, at the end of inflation the effective 4D bosons Φ_\pm can be decoupled on cosmological scales.

3.3. *4D Einstein Equations.* The effective 4D Lagrangian density (23) is expressed in terms of the fields $\Phi_\pm(x^\mu, \psi_0)$, which can be thought of as two minimally coupled bosons;

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} \left[\nabla_\mu \bar{\Phi}_+ \nabla^\mu \Phi_+ + \nabla_\mu \bar{\Phi}_- \nabla^\mu \Phi_- \right] + V(\Phi_+, \Phi_-). \quad (40)$$

Since

$$\begin{aligned} \nabla_4 \bar{\Phi}_+ &= \frac{\partial \bar{\Phi}_+}{\partial \psi} (1 \ 1) = -\left(\frac{n}{\psi} \right) \bar{\Phi}_+ (1 \ 1), \\ \nabla_4 \bar{\Phi}_- &= \frac{\partial \bar{\Phi}_-}{\partial \psi} (1 \ 1) = -\left(\frac{m}{\psi} \right) \bar{\Phi}_- (1 \ 1), \end{aligned}$$

$$\begin{aligned} \nabla_4 \Phi_+ &= \frac{\partial \Phi_+}{\partial \psi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\left(\frac{n}{\psi} \right) \Phi_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \nabla_4 \Phi_- &= \frac{\partial \Phi_-}{\partial \psi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\left(\frac{m}{\psi} \right) \Phi_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned} \quad (41)$$

the effective 4D potential resulting is

$$\begin{aligned} V(\Phi_+, \Phi_-) &= -\left(\frac{1}{4} \right) \left[\nabla_4 \bar{\Phi}_+ \nabla^4 \Phi_+ + \nabla_4 \bar{\Phi}_- \nabla^4 \Phi_- \right] \Big|_{\psi=1/H_0} \\ &= \frac{H_0^2}{4} (n^2 \|\Phi_+\|^2 + m^2 \|\Phi_-\|^2) \Big|_{\psi=1/H_0}, \end{aligned} \quad (42)$$

which is induced by the static foliation on the fifth coordinate $\psi = \psi_0 = 1/H_0$. This effective 4D potential is responsible to provide us with the dynamics of the fields $\Phi_\pm(x^\mu, \psi_0)$ on the effective 4D hypersurface on which the equation of state is $\omega = P/\rho = -1$. The energy density and pressure related to these fields are obtained from the diagonal part of the energy-momentum tensor written in a mixed manner:

$$\begin{aligned} \rho &= \left\langle E \left| \frac{1}{4} \left[\|\nabla_0 \Phi_+\|^2 + \|\nabla_0 \Phi_-\|^2 \right] \right. \right. \\ &\quad \left. \left. - \frac{e^{-2H_0 t}}{4} \left[\vec{\nabla} \Phi_- \cdot \vec{\nabla} \bar{\Phi}_- + \vec{\nabla} \Phi_+ \cdot \vec{\nabla} \bar{\Phi}_+ \right] \right. \right. \\ &\quad \left. \left. + V(\Phi_+, \Phi_-) + F_0^0 \right| E \right\rangle \Big|_{\psi=1/H_0}, \end{aligned} \quad (43)$$

$$\begin{aligned} P &= \left\langle E \left| \frac{1}{4} \left[\|\nabla_0 \Phi_+\|^2 + \|\nabla_0 \Phi_-\|^2 \right] \right. \right. \\ &\quad \left. \left. - \frac{e^{-2H_0 t}}{12} \left[\vec{\nabla} \Phi_- \cdot \vec{\nabla} \bar{\Phi}_- + \vec{\nabla} \Phi_+ \cdot \vec{\nabla} \bar{\Phi}_+ \right] \right. \right. \\ &\quad \left. \left. - V(\Phi_+, \Phi_-) + F_j^j \delta_j^j \right| E \right\rangle \Big|_{\psi=1/H_0}, \end{aligned}$$

where $|E\rangle$ is some quantum state, $F_0^0 = C_3/\pi[(H_0^7/8\kappa^2) + (H_0^9/\kappa^4)]$, $F_j^j = A_3/\pi[(15H_0^7/32\kappa^2) + (H_0^9/2\kappa^4)]\delta_j^j$, and

$$\begin{aligned} \nabla_0 \Phi_\pm &= \left[\partial_0 \mp \frac{1}{4\psi_0} \right] \Phi_\pm \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \nabla_j \Phi_+ &= \partial_j \Phi_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \nabla_1 \Phi_- &= \left[\partial_1 \Phi_- + i \frac{H_0 e^{H_0 t}}{2} \Phi_- \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \nabla_2 \Phi_- &= \partial_2 \Phi_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \frac{H_0 e^{H_0 t}}{2} \Phi_- \begin{pmatrix} -i \\ i \end{pmatrix}, \\ \nabla_3 \Phi_- &= \partial_3 \Phi_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \frac{H_0 e^{H_0 t}}{2} \Phi_- \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
\nabla_1 \bar{\Phi}_- &= \left[\partial_1 \Phi_- - i \frac{H_0 e^{H_0 t}}{2} \Phi_- \right] (1 \ 1), \\
\nabla_2 \bar{\Phi}_- &= \partial_2 \Phi_-^* (1 \ 1) - i \frac{H_0 e^{H_0 t}}{2} \Phi_-^* (i \ -i), \\
\nabla_3 \bar{\Phi}_- &= \partial_3 \Phi_-^* (1 \ 1) - i \frac{H_0 e^{H_0 t}}{2} \Phi_-^* (1 \ -1).
\end{aligned} \tag{44}$$

An interesting asymptotic solution can be obtained by considering the expectation values of, for instance, some quadratic scalar $\Sigma^2(\vec{x}, t)$, as

$$\begin{aligned}
\Sigma^2(t) &= \langle E | \Sigma^2(\vec{x}, t) | E \rangle \\
&= \frac{1}{(2\pi)^3} \int_{\kappa_*}^{\epsilon \kappa_0^+(t)} d^3 \kappa \Sigma_\kappa(\vec{x}, t) \Sigma_\kappa^*(t)(\vec{x}, t),
\end{aligned} \tag{45}$$

where $\kappa_* > 0$ is some minimum cut for the wavenumber to be determined and $\kappa_0^+(t) = H_0 e^{H_0 t}$ and $\kappa_0^-(t) = 2H_0 e^{H_0 t}$ are the maximum wavenumbers to the modes of Φ_+ and Φ_- , respectively. The expectation values for the radiation energy density ρ and the pressure P are given by the following expressions:

$$\begin{aligned}
\rho &= \left[A_2^2 \left(\frac{173H_0^4 \epsilon}{128\pi^3} - \frac{\epsilon^3 H_0^4}{24\pi^3} \right) + C_3 \frac{H_0^8 \epsilon}{16\pi^3} \right] e^{H_0 t} \\
&\quad - A_2^2 \frac{173k_* H_0^3}{128\pi^3} + C_1^2 \frac{101H_0^5}{32k_* \pi^3} \\
&\quad + C_3 \left(-\frac{k_* H_0^7}{16\pi^3} + \frac{H_0^9}{2k_* \pi^3} \right) \\
&\quad - \left[C_1^2 \left(\frac{101H_0^4}{64\pi^3 \epsilon} + \frac{H_0^4 \epsilon}{\pi^3} \right) + C_3 \frac{H_0^8}{4\pi^3 \epsilon} \right] e^{-H_0 t} \\
&\quad + \left[C_1^2 \frac{k_* H_0^3}{2\pi^3} + A_2^2 \frac{k_*^3 H_0}{24\pi^3} \right] e^{-2H_0 t}, \\
P &= \left[A_3 \frac{15H_0^8 \epsilon}{64\pi^3} - A_2^2 \left(\frac{219H_0^4 \epsilon}{128\pi^3} + \frac{H_0^4 \epsilon^3}{24\pi^3} \right) \right] e^{H_0 t} \\
&\quad - C_1^2 \frac{99H_0^5}{32k_* \pi^3} + A_2^2 \frac{219k_* H_0^3}{128\pi^3} \\
&\quad + A_3 \left(-\frac{15k_* H_0^7}{64\pi^3} + \frac{H_0^9}{4k_* \pi^3} \right) \\
&\quad + \left[C_1^2 \left(\frac{99H_0^4}{64\pi^3 \epsilon} - \frac{H_0^4 \epsilon}{\pi^3} \right) - A_3 \frac{H_0^8}{8\pi^3 \epsilon} \right] e^{-H_0 t} \\
&\quad + \left[A_2^2 \frac{k_*^3 H_0}{24\pi^3} + C_1^2 \frac{k_* H_0^3}{2\pi^3} \right] e^{-2H_0 t}.
\end{aligned} \tag{46}$$

Since we are interested to find solutions with $\mu = 1$ and $\nu = 2$ that correspond to $\partial_0 \Phi_\pm = 0$, we must consider the values

$n = 5/2$ and $m = 7/2$. In order to calculate the coefficients corresponding to the factors $e^{\pm H_0 t}$ and if we require that $\rho = -P = 3H_0^2/(8\pi G)$, we obtain that

$$\begin{aligned}
C_1^2 &= \frac{6k_* \pi^2 (-519 + 16\epsilon^2)}{H_0 \epsilon^2 [32H_0^2 (-519 + 16\epsilon^2) - k_*^2 (101 + 64\epsilon^2)]} \\
&= \frac{12k_* \pi^2 (657 + 16\epsilon^2)}{H_0 \epsilon^2 [64H_0^2 (657 + 16\epsilon^2) - 15k_*^2 (-99 + 64\epsilon^2)]},
\end{aligned} \tag{47}$$

$$A_2^2 = -\frac{3C_1^2 (101 + 64\epsilon^2)}{2(-519 + 16\epsilon^2)} = -\frac{45C_1^2 (-99 + 64\epsilon^2)}{4(657 + 16\epsilon^2)}, \tag{48}$$

such that from (47) we obtain that $\epsilon = 6.68586$. Furthermore, due to the fact that the equation of state is $\rho = -P = 3H_0^2/(8\pi G)$, we must require that

$$\begin{aligned}
&\frac{32H_0^2 (519 - 16\epsilon^2) + k_*^2 (101 + 64\epsilon^2)}{k_* (-519 + 16\epsilon^2)} \\
&= \frac{64H_0^2 (657 + 16\epsilon^2) - 15k_*^2 (-99 + 64\epsilon^2)}{2k_* (657 + 16\epsilon^2)},
\end{aligned} \tag{49}$$

from which we obtain that $\kappa_* = 1.45598H_0 = 1.45598 \times 10^{-9} M_p$. Notice that we have neglected in P and ρ terms which are very small with respect to $3H_0^2/(8\pi G)$ and decrease as $e^{-2H_0 t}$. With the values earlier mentioned for κ_* , H_0 , and M_p , we arrive at the numerical values $(C_1)^2 = -2.63042 \times 10^{32}$, $(A_2)^2 = 5.95601 \times 10^{33}$, $C_3 = 4.8693 \times 10^{70} M_p^{-4}$, and $A_3 = 9.081 \times 10^{70} M_p^{-4}$, which correspond to $\rho = -P = 1.19366 \times 10^{-19} M_p^4$. These values perfectly accord with that one expects during an inflationary vacuum dominated expansion of the early universe. A very important fact is that the dark energy is outside the horizon at the beginning of inflation but during the inflationary epoch enters to causally connected regions. In other words the dark energy is concentrated in the range of scales (physical scales) $2\pi/[\epsilon \kappa_0^\pm(t)] \simeq (\pi/H_0) e^{-H_0 t} < \lambda_{\text{phys}} < 2\pi/\kappa_*$. Hence, the effective 4D scalar (massive) field Φ_- should be an interesting candidate to explain dark energy in the early inflationary universe.

4. Final Remarks

We have explored the possibility that the expansion of the universe during the primordial inflationary phase of the universe can be driven by a condensate of spinor fields. In our picture ϕ_\pm are effective fields which from a condensate of two entangled spinors. The fields ϕ_\pm are decoupled at the end of inflation. In all our analysis we have neglected the role of the inflaton field, which (in a de Sitter expansion) is freezed in amplitude and nearly scale invariant but decays at the end of inflation into other fields. The point here is how we explain the existence of dark energy once the inflaton field energy density goes to zero. Our proposal is consistent to prove that the dark energy could be physically explained

though the entanglement of spinor fields that behave as effective 1-spin and 0-spin bosons on a 4D hypersurface on which the universe suffers a vacuum dominated expansion. The equation of state of the universe is determined by the static foliation $\psi = 1/H_0$. Our calculations show that the vector boson ϕ_+ is massless and with spin 1, and therefore compatible with the properties of a massless vector boson. On the other hand the field ϕ_- is a scalar boson which could be (jointly with the inflaton) responsible for the expansion of the universe and would be a good candidate to explain the existence of the dark energy. (Other fields such as the curvaton field [16] have been proposed in the literature to explain it.) A very interesting fact is that the (dark) energy density which we are talking about is poured into smaller subhorizon scales with the evolution of the inflationary expansion.

Acknowledgments

The authors acknowledge UNMdP and CONICET, Argentina, for financial support.

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