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Research Article

Quantum Tunnelling for Hawking Radiation from Both Static and Dynamic Black Holes

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The paper deals with Hawking radiation from both a general static black hole and a nonstatic spherically symmetric black hole. In case of static black hole, tunnelling of nonzero mass particles is considered and due to complicated calculations, quantum corrections are calculated only up to the first order. The results are compared with those for massless particles near the horizon. On the other hand, for dynamical black hole, quantum corrections are incorporated using the Hamilton-Jacobi method beyond semiclassical approximation. It is found that different order correction terms satisfy identical differential equation and are solved by a typical technique. Finally, using the law of black hole mechanics, a general modified form of the black hole entropy is obtained considering modified Hawking temperature.

1. Introduction

Hawking radiation is one of the most important effects in black hole (BH) physics. Classically, nothing can escape from the BH across its event horizon. But in 1974, there was a dramatic change in view when Hawking and Hartle [1, 2] showed that BHs are not totally black; they radiate analogous to thermal black body radiation. Since then, there has been lots of attraction to this issue and various approaches have been developed to derive Hawking radiation and its corresponding temperature [3–7]. However, in the last decade, two distinct semiclassical methods have been developed which enhanced the study of Hawking radiation to a great extent. The first approach developed by Parikh and Wilczek [8, 9] is based on the heuristic pictures of visualisation of the source of radiation as tunnelling and is known as radial null geodesic method. The essence of this method is to calculate the imaginary part of the action for the *s-wave* emission (across the horizon) using the radial null geodesic equation and is then related to the Boltzmann factor to obtain Hawking radiation by the relation:

$$\Gamma \propto \exp\left\{-\frac{2}{h}\left(\operatorname{Im} S^{\text{out}} - \operatorname{Im} S^{\text{in}}\right)\right\} = \exp\left\{-\frac{E}{T_H}\right\}, \quad (1)$$

where E is the energy associated with the tunnelling particle and T_H is the usual Hawking temperature.

The alternative way of looking into this aspect is known as complex paths method developed by Srinivasan et al. [10, 11]. In this approach, the differential equation of the action $S(r,t)$ of a classical scalar particle can be obtained by plugging the scalar field wave function $\phi(r,t) = \exp\{-(i/\hbar)S(r,t)\}$ into the Klein-Gordon (KG) equation in a gravitational background. Then, the Hamilton-Jacobi (HJ) method is employed to solve the differential equation for S. Finally, Hawking temperature is obtained using the "principle of detailed balance" [10–12] (time-reversal invariant). It should be noted that the first method is limited to massless particles only. Also, this method is applicable to such coordinate system only in which there is no singularity across the horizon. On the other hand, in complex paths method, the emitted particles are considered without self-gravitation and the action is assumed to satisfy the relativistic HJ equation. Here tunnelling of both massless and massive particles is possible and it is applicable to any coordinate system to describe the BH.

Most of the studies [13–18] dealing with the Hawking radiation are connected to semiclassical analysis. Recently, Banerjee and Majhi [19] and Corda et al. [20, 21] initiated the calculation of Hawking temperature beyond the semiclassical limit. Mostly, both groups have considered tunnelling of massless particle and evaluated the modified Hawking temperature with quantum corrections.

In the present work, at first we consider a general nonstatic metric for dynamical BH. HJ method is extended beyond semiclassical approximation to consider all the terms in the expansion of the one particle action. It is found that the higher order terms (quantum corrections) satisfy identical differential equations as the semiclassical action and the complicated terms are eliminated considering BH horizon as one way barrier. We derive the modified Hawking temperature using both the above approaches which are found to be identical at the semiclassical level. Also, modified form of the BH entropy with quantum correction has been evaluated.

Subsequently, in the next section, we consider tunnelling of particles having nonzero mass beyond semiclassical approximation. Due to nonzero mass, the imaginary part of the action cannot be evaluated using first approach; only HJ method will be applicable. Further, the complicated form of the equations involved restricted us to only first order quantum correction.

2. Method of Radial Null Geodesic: A Survey of Earlier Works

This section deals with a brief survey of the method of radial null geodesics method [8] considering the picture of Hawking radiation as quantum tunnelling. In a word, the method correlates the imaginary part of the action for the classically forbidden process of *s-wave* emission across the horizon with the Boltzmann factor for the black body radiation at the Hawking temperature. We start with a general class of nonstatic spherically symmetric BH metric of the form

$$ds^2 = -A(r,t)\,dt^2 + \frac{dr^2}{B(r,t)} + r^2 d\Omega_2^2, \tag{2}$$

where the horizon r_h is located at $A(r_h,t) = 0 = B(r_h,t)$ and the metric has a coordinate singularity at the horizon. To remove the coordinate singularity, we make the following Painleve-type transformation of coordinates:

$$dt \longrightarrow dt - \sqrt{\frac{1-B}{AB}}dr \tag{3}$$

and as a result metric (2) transforms to

$$ds^2 = -Adt^2 + 2\sqrt{A\left(\frac{1}{B}-1\right)}dtdr + dr^2 + r^2 d{\Omega_2}^2. \tag{4}$$

This metric (i.e., the choice of coordinates) has some distinct features over the former one, as follows.

(i) The metric is singularity free across the horizon.

(ii) At any fixed time, we have a flat spatial geometry.

(iii) Both the metric will have the same boundary geometry at any fixed radius.

The radial null geodesic (characterized by $ds^2 = 0 = d{\Omega_2}^2$) has the differential equation (using (3)):

$$\frac{dr}{dt} = \sqrt{\frac{A}{B}}\left[\pm 1 - \sqrt{1-B(r,t)}\right], \tag{5}$$

where outgoing or ingoing geodesic is identified by the + or − sign within the square bracket in (4). In the present case, we deal with the absorption of particles through the horizon (i.e., + sign only) and according to Parikh and Wilczek [8], the imaginary part of the action is obtained as

$$\begin{aligned}\operatorname{Im} S &= \operatorname{Im}\int_{r_{\text{in}}}^{r_{\text{out}}} p_r\,dr = \operatorname{Im}\int_{r_{\text{in}}}^{r_{\text{out}}}\int_0^{p_r} dp_r'\,dr \\ &= \operatorname{Im}\int_{r_{\text{in}}}^{r_{\text{out}}}\left\{\int_0^H \frac{dH'}{dr/dt}\right\}dr.\end{aligned} \tag{6}$$

Note that in the last step of the above derivation we have used the Hamilton's equation $\dot r = (dH/dp_r)|_r$, where (r,p_r) are canonical pair. Further, it is to be mentioned that in quantum mechanics, the action of a tunnelled particle in a potential barrier having energy larger than the energy of the particle will be imaginary as $p_r = \sqrt{2m(E-V)}$. For the present nonstatic BH, the mass of the BH is not constant and hence the dH' integration extends over all the values of energy of outgoing particle, from zero to $E(t)$ [22] (say). As we are dealing with tunnelling across the BH horizon, so using Taylor series expansion about the horizon r_h we write

$$\begin{aligned}A(r,t)|_t &= \left.\frac{\partial A(r,t)}{\partial r}\right|_t (r-r_h) + O(r-r_h)^2\Big|_t, \\ B(r,t)|_t &= \left.\frac{\partial B(r,t)}{\partial r}\right|_t (r-r_h) + O(r-r_h)^2\Big|_t.\end{aligned} \tag{7}$$

So, in the neighbourhood of the horizon, the geodesic equation (4) can be approximated as

$$\frac{dr}{dt} \approx \frac{1}{2}\sqrt{A'(r_h,t)\,B'(r_h,t)}\,(r-r_h). \tag{8}$$

Substituting this value of dr/dt in the last step of (5) we have

$$\operatorname{Im} S = \frac{2\pi E(t)}{\sqrt{A'(r_h,t)\,B'(r_h,t)}}, \tag{9}$$

where the choice of contour for r-integration is on the upper half complex plane to avoid the coordinate singularity at r_h. Thus, the tunnelling probability is given by

$$\Gamma \sim \exp\left\{-\frac{2}{\hbar}\operatorname{Im} S\right\} = \exp\left\{-\frac{4\pi E(t)}{\hbar\sqrt{A'B'}}\right\}, \tag{10}$$

which in turn equates with the Boltzmann factor $\exp\{E(t)/T\}$; the expression for the Hawking temperature is

$$T_H = \frac{\hbar\sqrt{A'(r_h,t)\,B'(r_h,t)}}{4\pi}. \tag{11}$$

From the above expression, it is to be noted that T_H is time dependent.

Recently, a drawback of the above approach has been noted [23–25]. It has been shown that $\Gamma \sim \exp\{-(2/\hbar)\,\mathrm{Im}\,S\} = \exp\{-(2/\hbar)\,\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r dr\}$ is not canonically invariant and hence is not a proper observable; it should be modified as $\exp\{-\mathrm{Im}\oint p_r dr/\hbar\}$. The closed path goes across the horizon and back. For tunnelling across the ordinary barrier, it is immaterial whether the particle goes from the left to the right or the reverse path. So in that case

$$\oint p_r\, dr = 2\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r\, dr \tag{12}$$

and there is no problem of canonical invariance. But difficulty arises for BH horizon which behaves as a barrier for particles going from inside of the BH to outside but it does not act as a barrier for particles going from outside to the inside. So relation (12) is no longer valid. Also, using tunnelling the probability is $\Gamma \sim \exp\{-\mathrm{Im}\oint p_r dr/\hbar\}$, so there will be a problem of factor two in Hawking temperature [24, 26, 27].

Further, the above analysis of tunnelling approach remains incomplete unless effects of self-gravitation and back reaction are taken into account. But unfortunately, no general approaches to account for the above effects are there in the literature; only few results are available for some known BH solutions [26–32].

Finally, it is worth mentioning that so far the above tunnelling approach is purely semiclassical in nature and quantum corrections are not included. Also, this method is applicable for Painleve-type coordinates only; one cannot use the original metric coordinates to avoid horizon singularity. Lastly, the tunnelling approach is not applicable for massive particles [19].

3. Hamilton-Jacobi Method: Quantum Corrections

We will now follow the alternative approach as mentioned in the introduction, that is, the HJ method to evaluate the imaginary part of the action and hence the Hawking temperature. We will analyze the beyond semiclassical approximation by incorporating possible quantum corrections. As this method is not affected by the coordinate singularity at the horizon so we will use the general BH metric (2) for convenience.

In the background of the gravitational field described by the metric (2), massless scalar particles obey the Klein-Gordon equation

$$\frac{\hbar^2}{\sqrt{-g}}\partial\left[g^{\mu\nu}\sqrt{-g}\partial_\nu\right]\psi = 0. \tag{13}$$

For spherically symmetric BH, as we are only considering radial trajectories, so we will consider (t,r)-sector in the spacetime given by (2); that is, we concentrate on two-dimensional BH problems. Using (2), the above Klein-Gordon equation becomes

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial\psi}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial\psi}{\partial r} - AB\frac{\partial^2\psi}{\partial r^2} = 0. \tag{14}$$

Using the standard ansatz for the semiclassical wave function, namely,

$$\psi(r,t) = \exp\left\{-\frac{i}{\hbar}S(r,t)\right\}, \tag{15}$$

the differential equation for the action S is

$$\left(\frac{\partial S}{\partial t}\right)^2 - AB\left(\frac{\partial S}{\partial r}\right)^2 + i\hbar\left[\frac{\partial^2 S}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial S}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial S}{\partial r} - AB\frac{\partial^2 S}{\partial r^2}\right]. \tag{16}$$

To solve this partial differential equation we expand the action S in powers of Planck's constant $\hbar$ as

$$S(r,t) = S_0(r,t) + \Sigma\hbar^k S_k(r,t), \tag{17}$$

with k being a positive integer. Note that, in the above expansion, terms of the order of Planck's constant and its higher powers are considered as quantum corrections over the semiclassical action S_0. Now substituting ansatz (17) for S into (16) and equating different powers of $\hbar$ on both sides, we obtain the following set of partial differential equations:

$$\hbar^0 : \left(\frac{\partial S}{\partial t}\right)^2 - AB\left(\frac{\partial S}{\partial r}\right)^2 = 0, \tag{18}$$

$$\hbar^1 : \frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - AB\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r} + \frac{i}{2}\left[\frac{\partial^2 S_0}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial S_0}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial S_0}{\partial r} - AB\frac{\partial^2 S_0}{\partial r^2}\right] = 0, \tag{19}$$

$$\hbar^2 : \left(\frac{\partial S_1}{\partial t}\right)^2 + 2\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - AB\left(\frac{\partial S_1}{\partial r}\right)^2 - 2AB\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r} + i\left[\frac{\partial^2 S_1}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial S_1}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial S_1}{\partial r} - AB\frac{\partial^2 S_1}{\partial r^2}\right] = 0, \tag{20}$$

and so on.

Apparently, different order partial differential equations are very complicated but fortunately there will be lot of simplifications if, in the partial differential equation corresponding to $\hbar^k$, all previous partial differential equations are used and finally we obtain identical partial differential equation, namely,

$$\hbar^k : \frac{\partial S_k}{\partial t} = \pm\sqrt{A(r,t)B(r,t)}\frac{\partial S_k}{\partial r}, \tag{21}$$

for $k = 0, 1, 2, \ldots$.

Thus, quantum corrections satisfy the same differential equation as the semiclassical action S_0. Hence, the solutions will be very similar. To solve S_0, it is to be noted that due to nonstatic BHs the metric coefficients are functions of r and t and hence standard HJ method cannot be applied; some generalization is needed. We start with a general metric [22]

$$S_0(r,t) = \int_0^t \omega_0(t')\, dt + D_0(r,t). \tag{22}$$

Here $\omega_0(t)$ behaves as the energy of the emitted particle and the justification of the choice of the integral is that the outgoing particle should have time-dependent continuum energy.

Now substituting the above ansatz for $S_0(r,t)$ into (18) and using the radial null geodesic in the usual metric from (2), namely,

$$\frac{dr}{dt} = \pm\sqrt{AB}, \tag{23}$$

we have

$$\frac{\partial D_0}{\partial r} + \frac{\partial D_0}{\partial t}\frac{dt}{dr} = \mp\omega_0(t)\frac{dt}{dr}. \tag{24}$$

that is,

$$\frac{dD_0}{dr} = \mp\frac{\omega_0(t)}{\sqrt{AB}}, \tag{25}$$

which gives

$$D_0 = \mp\omega_0(t)\int_0^r \frac{dr}{\sqrt{AB}}. \tag{26}$$

Hence, the complete semiclassical action takes the form

$$S_0(r,t) = \int_0^t \omega_0(t')\, dt' \mp \omega_0(t)\int_0^r \frac{dr}{\sqrt{AB}}. \tag{27}$$

Here the − (or +) sign corresponds to absorption (or emission) particle. As solution (27) contains an arbitrary time-dependent function $\omega_0(t)$, so a general solution for S_k can be written as

$$S_k(r,t) = \int_0^t \omega_k(t')\, dt' \mp \omega_0(t)\int_0^r \frac{dr}{\sqrt{AB}}, \quad k = 1, 2, 3, \ldots. \tag{28}$$

Thus, from (15), using solutions (27) and (28) into (17), the wave functions for absorption and emission of scalar particle can be expressed as

$$\begin{aligned}\psi_{\text{emm.}}(r,t) = \exp\Bigg\{-\frac{i}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' &+ \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &- \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\}, \\ \psi_{\text{abs.}}(r,t) = \exp\Bigg\{-\frac{i}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' &+ \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &+ \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\},\end{aligned} \tag{29}$$

respectively. Due to tunnelling across the horizon, there will be a change of sign of the metric coefficients in the (r,t)-part of the metric and as a result, function of t coordinate has an imaginary part which will contribute to the probabilities. So we write

$$\begin{aligned}P_{\text{abs.}} &= \left|\psi_{\text{abs.}}(r,t)\right|^2 \\ &= \exp\Bigg\{\frac{2\,\text{Im}}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' + \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &\quad + \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\},\end{aligned} \tag{30}$$

$$\begin{aligned}P_{\text{emm.}} &= \left|\psi_{\text{emm.}}(r,t)\right|^2 \\ &= \exp\Bigg\{\frac{2\,\text{Im}}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' + \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &\quad - \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\}.\end{aligned} \tag{31}$$

To have some simplification, we will now use the physical fact that all incoming particles certainly cross the horizon; that is, $P_{\text{abs.}} = 1$. So from (30),

$$\begin{aligned}&\text{Im}\left(\int_0^t \omega_0(t')\, dt' + \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\right) \\ &\quad = -\,\text{Im}\left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\end{aligned} \tag{32}$$

and hence $P_{\text{emm.}}$ simplifies to

$$P_{\text{emm.}} = \exp\left\{-\frac{4}{\hbar}\left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\text{Im}\int_0^r \frac{dr}{\sqrt{AB}}\right\}. \tag{33}$$

Then from the principle of "detailed balance" [10–12] (which states that transitions between any two states take place with equal frequency in either direction at equilibrium), we write

$$P_{\text{emm.}} = \exp\left\{-\frac{\omega_0(t)}{T_h}\right\} P_{\text{in}} = \exp\left\{-\frac{\omega_0(t)}{T_h}\right\}. \quad (34)$$

So, comparing (33) and (34), the temperature of the BH is given by

$$T_h = \frac{\hbar}{4}\left[1 + \Sigma_k \hbar^k \frac{\omega_k(t)}{\omega_0(t)}\right]^{-1} \left[\text{Im} \int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1}, \quad (35)$$

where

$$T_h = \frac{\hbar}{4}\left[\text{Im} \int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1} \quad (36)$$

is the usual Hawking temperature of the BH. Thus, due to quantum corrections, the temperature of the BH is modified from the Hawking temperature and both temperatures are functions of t and r. Note that (36) is the standard expression for semiclassical Hawking temperature and it is valid for nonspherical metric also. However, for spherical metric, one can use the Taylor series expansions (7) near the horizon and obtain T_H as given in (11) by performing the contour integration. The ambiguity of factor of two (as mentioned earlier) in the Hawking temperature does not arise here.

Further, one may note that solutions (27) or (28) are the unique solutions to (18) or (21) except for a premultiplication factor. This arbitrary multiplicative factor does not appear in the expression for Hawking temperature; only the particle energy (ω_0) or ω_k is rescaled. As quantum correction term contains ω_k/ω_0, so it does not involve the arbitrary multiplicative factor and hence it is unique.

To have some interpretation about the arbitrary functions $\omega_k(t)$ appearing in the quantum correction terms, we make use of dimensional analysis. As S_0 has the dimension $\hbar$, so the arbitrary function $\omega_k(t)$ has the dimension $\hbar^{-k}$. In standard choice of units, namely, $G = c = K_B = 1$, $\hbar \sim M_p^2$ and so $\omega_k \sim M^{-2k}$, where M is the mass of the BH.

Similar to the Hawking temperature, the surface gravity of the BH is modified due to quantum corrections. If κ_c is the semiclassical surface gravity corresponding to Hawking temperature, that is, $\kappa_c = 2\pi T_H$, then the quantum corrected surface gravity $\kappa = 2\pi T_H$ is related to the semiclassical value by the relation:

$$\kappa = \kappa_c \left[1 + \Sigma_k \hbar^k \frac{\omega_k(t)}{\omega_0(t)}\right]^{-1}. \quad (37)$$

Moreover, based on the dimensional analysis, if we choose, for simplicity,

$$\omega_k(t) = \frac{a^k \omega_0(t)}{M^{2k}}, \qquad \text{"}a\text{" is a dimensionless parameter,} \quad (38)$$

then expression (37) is simplified to

$$\kappa = \kappa_0 \left(1 - \frac{\hbar a}{M^2}\right)^{-1}. \quad (39)$$

This is related to the one loop back reaction effects in the spacetime [6, 33] with the parameter a corresponding to trace anomaly. Higher order loop corrections to the surface gravity can be obtained similarly by suitable choice of the functions $\omega_k(t)$. For static BHs, Banerjee and Majhi [19] have studied these corrections in detail. Lastly, it is worth mentioning that identical result for BH temperature may be obtained if we use the Painleve coordinate system as in the previous section.

4. Entropy Function and Quantum Correction

We will now examine how the semiclassical Bekenstein-Hawking area law, namely, $S_{\text{BH}} = (A/4\hbar)$ (A is the area of the horizon) is modified due to quantum corrections described in the previous section. The first law of the BH mechanics, which is essentially the energy conservation relation, related the change of BH mass (M) to the change of its entropy (S_{BH}), electric charge (Q), and angular momentum (J) as

$$dM = T_h dS_{\text{BH}} + \Phi dQ + \Omega dJ. \quad (40)$$

Here, Ω is the angular velocity and Φ is the electrostatic potential. So, for nonrotating uncharged BHs, the entropy has the simple form

$$S_{\text{BH}} = \int \frac{dM}{T_h}, \quad (41)$$

or using (35) for T_h, we get

$$S_{\text{BH}} = \int \left[1 + \Sigma_k \hbar^k \frac{\omega_k(t)}{\omega_0(t)}\right] \frac{dM}{T_H}. \quad (42)$$

For choice (38) corresponding to one loop back reaction effects, we have from (42) the quantum corrected BH entropy as

$$S_{\text{BH}} = \int \left[1 + \frac{a\hbar}{M} + \frac{a^2\hbar^2}{M^2} + \cdots\right] \frac{dM}{T_H}. \quad (43)$$

The first term is the usual semiclassical Bekenstein-Hawking entropy and the subsequent terms are the quantum corrections of different order. For static BHs, Banerjee and Majhi [19] have shown the correction terms of which the leading one gives the standard logarithmic correction. On the other hand, for nonstatic BHs, as the proportionality factors are time-dependent and arbitrary (see (42)) so the leading order correction term may not be logarithmic. For future work, we will attempt to determine physical interpretation of the arbitrary time-dependent proportionality factors so that quantum corrections may be evaluated.

5. Hamilton-Jacobi Method for Massive Particles: Quantum Corrections

The KG equation for a scalar field ψ describing a scalar particle of mass m_0 has the form [10]

$$\left(\Box + \frac{m_0^2}{\hbar^2}\right)\psi = 0, \quad (44)$$

where the box operator "□" is evaluated in the background of a general static BH metric of the form

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_2{}^2. \tag{45}$$

The explicit form of the KG equation for the metric (45) is

$$\begin{aligned} &-\frac{1}{A}\frac{\partial^2\psi}{\partial t^2} + B\frac{\partial^2\psi}{\partial r^2} + \frac{1}{2A}\frac{\partial(AB)}{\partial r}\frac{\partial\psi}{\partial r} + \frac{2B}{r}\frac{\partial\psi}{\partial r} \\ &+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) \\ &+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{m_0^2}{\hbar^2}\psi(t,r,\theta,\phi). \end{aligned} \tag{46}$$

Due to spherical symmetry, we can decompose ϕ in the form

$$\psi(t,r,\theta,\phi) = \Phi(t,r)\,Y_l^m(\theta,\phi), \tag{47}$$

where ϕ satisfies [10]

$$\begin{aligned} &\frac{1}{A}\frac{\partial^2\psi}{\partial t^2} - B\frac{\partial^2\psi}{\partial r^2} - \frac{1}{2A}\frac{\partial(AB)}{\partial r}\frac{\partial\psi}{\partial r} - \frac{2B}{r}\frac{\partial\psi}{\partial r} \\ &+ \left\{\frac{l(l+1)}{r^2} + \frac{m_0^2}{\hbar^2}\right\}\Phi(t,r) = 0. \end{aligned} \tag{48}$$

If we substitute the standard ansatz for the semiclassical wave function, namely,

$$\phi(t,r) = \exp\left\{-\frac{i}{\hbar}S(r,t)\right\}, \tag{49}$$

then the action S will satisfy the following differential equation:

$$\begin{aligned} &\left[\frac{1}{A}\left(\frac{\partial S}{\partial t}\right)^2 - B\left(\frac{\partial S}{\partial r}\right)^2 - E_0^2(r)\right] \\ &\quad -\frac{\hbar}{i}\left[\frac{1}{A}\frac{\partial^2 S}{\partial t^2} - B^2\frac{\partial^2 S}{\partial r^2} - \left\{\frac{1}{2A}\frac{\partial(AB)}{\partial r} + \frac{2B}{r}\right\}\frac{\partial S}{\partial r}\right] = 0, \end{aligned} \tag{50}$$

where $E_0^2 = m_0^2 + (L^2/r^2)$ and $L^2 = l(l+1)\hbar^2$ is the angular momentum. To incorporate quantum corrections over the semiclassical action, we expand the actions in powers of Planck constant $\hbar$ as

$$S(r,t) = S_0(r,t) + \Sigma_k \hbar^k S_k(r,t), \tag{51}$$

where S_0 is the semiclassical action and k is a positive integer. Now substituting this ansatz for S in the differential equation (50) and equating different powers of $\hbar$ on both sides, we obtain the following set of partial differential equations:

$$\hbar^0 : \frac{1}{A}\left(\frac{\partial S}{\partial t}\right)^2 - B\left(\frac{\partial S}{\partial r}\right)^2 - E_0^2(r) = 0, \tag{52}$$

$$\begin{aligned} \hbar^1 : &\frac{2}{A}\frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - 2B\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r} \\ &- \frac{1}{i}\left[\frac{1}{A}\frac{\partial^2 S_0}{\partial t^2} - B^2\frac{\partial^2 S_0}{\partial r^2}\right. \\ &\left. - \left\{\frac{1}{2A}\frac{\partial(AB)}{\partial r} + \frac{2B}{r}\right\}\frac{\partial S_0}{\partial r}\right] = 0, \end{aligned} \tag{53}$$

$$\begin{aligned} \hbar^2 : &\frac{1}{A}\left(\frac{\partial S_1}{\partial t}\right)^2 + \frac{2}{A}\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - B\left(\frac{\partial S_1}{\partial r}\right)^2 - 2B\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r} \\ &- \frac{1}{i}\left[\frac{1}{A}\frac{\partial^2 S_1}{\partial t^2} - B^2\frac{\partial^2 S_1}{\partial r^2}\right. \\ &\left. - \left\{\frac{1}{2A}\frac{\partial(AB)}{\partial r} + \frac{2B}{r}\right\}\frac{\partial S_1}{\partial r}\right] = 0, \end{aligned} \tag{54}$$

and so on.

To solve the semiclassical action S_0, we start with the standard separable choice [10]

$$S_0(r,t) = \omega_0 t + D_0(r). \tag{55}$$

Substituting this choice in (52), we obtain

$$D_0 = \pm\int_0^r \sqrt{\frac{\omega_0^2 - AE_0^2}{AB}}\,dr = \pm I_0 \quad (\text{say}), \tag{56}$$

where + or − sign corresponds to absorption or emission of scalar particle. Now substituting this choice for S_0 in (53), we have the differential equation for first order corrections S_1 as

$$\begin{aligned} &\frac{\partial S_1}{\partial t} \mp \sqrt{AB}\sqrt{1 - \frac{AE_0^2}{\omega_0^2}}\frac{\partial S_1}{\partial r} \\ &\mp \frac{\sqrt{AB}}{i}\left[-\frac{1}{r}\sqrt{1 - \frac{AE_0^2}{\omega^2}}\right. \\ &\left. + \frac{(\partial A/\partial r)\left(E_0^2/\omega^2\right) - \left(2AL^2/\omega_0^2 r^3\right)}{4\sqrt{1 - (AE_0^2/\omega^2)}}\right] = 0. \end{aligned} \tag{57}$$

As before, S_1 can be written in separable form as

$$S_1 = \omega_1 t + D_1(r), \tag{58}$$

where

$$D_1 = \int_0^r \frac{dr}{\sqrt{AB}\sqrt{1-(AE_0^2/\omega_0^2)}} \times \left[\pm\omega_1 - \frac{\sqrt{AB}}{i} \times \left\{ -\frac{1}{r}\sqrt{1-\frac{AE_0^2}{\omega^2}} + \frac{(\partial A/\partial r)\left(E_0^2/\omega^2\right) - \left(2AL^2/\omega_0^2 r^3\right)}{4\sqrt{1-(AE_0^2/\omega^2)}} \right\} \right] = \pm I_1 - I_2. \tag{59}$$

Now due to complicated form, if we retain terms up to first order quantum corrections, that is,

$$S = S_0 + \hbar S_1 = (\omega_0 + \hbar\omega_1)\,t + \{D_0 + \hbar D_1(r)\}, \tag{60}$$

then the wave function denoting absorption and emission solutions of the KG equation (48) using (49) are of the form

$$\begin{aligned} \phi_{\text{abs.}} &= \exp\left\{-\frac{i}{\hbar}\left(\overline{\omega_0+\hbar\omega_1}t + \overline{I_0+\hbar I_1-\hbar I_2}\right)\right\}, \\ \phi_{\text{emm.}} &= \exp\left\{-\frac{i}{\hbar}\left(\overline{\omega_0+\hbar\omega_1}t - \overline{I_0+\hbar I_1-\hbar I_2}\right)\right\}. \end{aligned} \tag{61}$$

It is to be noted that in course of tunnelling across the horizon, the coordinate nature changes; that is, more precisely the signs of the metric coefficients in the (r,t)-hyperplane are altered. Thus, we can interpret this as that the time coordinate has an imaginary part in crossing the horizon and accordingly the temporal part has contribution to the probabilities [19, 33]. Thus, absorption and emission probabilities are given by

$$P_{\text{abs.}} = \left|\phi_{\text{in}}\right|^2 = \exp\left\{\frac{2}{\hbar}\left(\text{Im}\ \overline{\omega_0+\hbar\omega_1 t}\right) + \text{Im}\ \overline{I_0+\hbar I_1} - \text{Im}\ \hbar I_2\right\}, \tag{62}$$

$$P_{\text{emm.}} = \left|\phi_{\text{out}}\right|^2 = \exp\left\{-\frac{i}{\hbar}\left(\text{Im}\ \overline{\omega_0+\hbar\omega_1 t}\right) - \text{Im}\ \overline{I_0+\hbar I_1} - \text{Im}\ \hbar I_2\right\}. \tag{63}$$

In the classical limit $\hbar \to 0$, there is no reflection, so all ingoing particles should be absorbed and hence [33]

$$\lim_{\hbar\to 0} P_{\text{abs.}} = 1. \tag{64}$$

So, from (62), we must have

$$\text{Im}\ \omega_0 t = \text{Im}\ I_0, \qquad \text{Im}\left(\omega_1 t - I_2\right) = \text{Im}\ I_1 \tag{65}$$

and as a result $P_{\text{emm.}}$ simplifies to

$$P_{\text{emm.}} = \exp\left[-\frac{4\omega_0}{\hbar} \times \text{Im}\left\{\int_0^r \frac{dr}{\sqrt{AB}}\left(\sqrt{1-\frac{AE_0^2}{\omega_0^2}} + \frac{\hbar\left(\omega_1/\omega_0\right)}{\sqrt{1-\left(AE_0^2/\omega_0^2\right)}}\right)\right\}\right]. \tag{66}$$

Using the principle of "detailed balance" [10, 11, 20, 21], namely,

$$P_{\text{emm.}} = \exp\left\{-\frac{E}{T_h}\right\} P_{\text{in}} = \exp\left\{-\frac{E}{T_h}\right\}, \tag{67}$$

the temperature of the BH is given by

$$T_h = \frac{\hbar E}{4\omega_0}\left[\text{Im}\left\{\int_0^r \frac{dr}{\sqrt{AB}} \times \left(\sqrt{1-\frac{AE_0^2}{\omega_0^2}} + \frac{\hbar(\omega_1/\omega_0)}{\sqrt{1-\left(AE_0^2/\omega_0^2\right)}}\right)\right\}\right]^{-1}, \tag{68}$$

where the semiclassical Hawking temperature of the BH has the expression

$$T_H = \frac{\hbar E}{4\omega_0}\left[\text{Im}\int_0^r \frac{dr}{\sqrt{AB}}\sqrt{1-\frac{AE_0^2}{\omega_0^2}}\right]^{-1}. \tag{69}$$

Now, to obtain the modified form of the surface gravity of the BH, we start with the usual relation between surface gravity and Hawking temperature, namely,

$$\kappa_H = 2\pi T_H, \tag{70}$$

where T_H is given by (69).

So the quantum corrected surface gravity is given by

$$\kappa_{\text{QC}} = 2\pi T_h. \tag{71}$$

Further, for the present nonrotating, uncharged, static BHs, using the law of BH thermodynamics $dM = T_h dS$, we have the expression for the entropy of the BH as

$$S_{\text{BH}} = \int \frac{4\omega_0}{\hbar E}\left(1+\frac{\hbar\omega_1}{\omega_0}\right) dM \int_0^r \frac{dr}{\sqrt{AB}}. \tag{72}$$

Finally, it is easy to see from (68) that near the horizon the presence of E_0^2 term can be neglected as it is multiplied by the metric coefficient A. Therefore, the quantum corrected (up to first order) temperature of the BH (in (68)) reduces to

$$T_h = \frac{\hbar E}{4\omega_0}\left(1 + \frac{\hbar\omega_1}{\omega_0}\right)^{-1}\left[\int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1} \tag{73}$$

and the Hawking temperature (given in (69)) becomes

$$T_H = \frac{\hbar E}{4\omega_0}\left[\int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1}. \tag{74}$$

So we have

$$T_h = \left(1 + \frac{\hbar\omega_1}{\omega_0}\right)^{-1} T_H. \tag{75}$$

We see that if the energy of the tunnelling particle is chosen as ω_0 (i.e., $E = \omega_0$) and $\omega_1 = \beta_1/M$ (for notations see Banerjee and Majhi [19]) then the Hawking temperature given by (74) is the usual one derived for massless particles and the quantum corrected temperature T_h given in (75) agrees with that of Banerjee and Majhi [19] for massless particle. Therefore, Hawking temperature near the horizon remains the same for both massless and nonzero mass tunnelling particles and it agrees with the claim of Srinivasan and Padmanabhan [10] and Banerjee and Majhi [19]. For future work, it will be interesting to calculate the temperature of the BH for tunnelling nonzero mass particle with full quantum corrections and examine whether the result agrees with that of Banerjee and Majhi [19] near the horizon. Finally, it will be nice to determine quantum corrected entropy of the BH in a convenient form.

6. Summary of the Work

This work is an attempt to study quantum corrections to Hawking radiation of massless particle from a dynamical BH as well as for massive particle from a static BH. At first, radial null geodesic tunnelling approach has been used with Painleve-type choice of coordinate system to derive semiclassical Hawking temperature. Then full quantum mechanical calculations have performed writing action in a power series of the Planck constant $\hbar$ to evaluate the quantum corrections to the Hawking temperature. Subsequently, quantum corrected surface gravity has been calculated and it is found that one loop back reaction effects in the spacetime can be obtained by suitable choice of the arbitrary functions and parameters. Finally, an expression for the quantum corrected entropy of the BH has been evaluated. It is found that, due to the presence of the arbitrary functions in the expression for entropy, the leading order quantum correction may not be logarithmic in nature. On the other hand, in the case of Hawking radiation of massive particle from static BH, it is found that Hawking temperature near the horizon does not depend on the mass term as predicted by Srinivasan and Padmanabhan [10] and Banerjee et al. [16–18]. For future work, we will try to find a solution for the partial differential equation (18) in a more simple form so that more physical interpretations can be done from the BH parameters. Also, it will be interesting to calculate temperature of the BH for tunnelling nonzero mass particle with full quantum correction and examine whether the result agrees with that of Banerjee and Majhi [19] near the horizon. Finally, it will be nice to determine quantum corrected entropy of the BH in a convenient form.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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