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Research Article

Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity

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It is demonstrated that provided a theory involves a minimal length, this theory must be free from such infinitesimal quantities as infinitely small variations in surface of the holographic screen, its volume, and entropy. The corresponding infinitesimal quantities in this case must be replaced by the "minimal variations possible"—finite quantities dependent on the existent energies. As a result, the initial low-energy theory (quantum theory or general relativity) inevitably must be replaced by a minimal length theory that gives very close results but operates with absolutely other mathematical apparatus.

At the present time all high-energy generalizations (limits) of the basic components in fundamental physics (quantum theory [1] and gravity [2]) of necessity lead to a minimal length on the order of the Planck length $l_{\min} \propto l_P$. This follows from string theory [3–6], loop quantum gravity [7], and other approaches [8–22].

But it is clear that provided a minimal length exists, it is existent at all the energy scales and not at high (Planck's) scales only.

What is inferred on this basis for real physics? At least, it is suggested that the use of infinitesimal quantities dx_μ in a mathematical apparatus of both quantum theory and gravity is incorrect, despite the fact that both these theories give the results correlating well with the experiment (e.g., [23]).

Indeed, in all cases, the infinitesimal quantities dx_μ bring about an infinitely small length ds [2]

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \tag{1}$$

that is inexistent because of $l_{\min}$.

The same is true for any function Υ dependent only on different parameters L_i whose dimensions of length of the exponents are equal to or greater than 1 $\nu_i \geq 1$:

$$\Upsilon \equiv \Upsilon\left(L_i^{\nu_i}\right). \tag{2}$$

Obviously, the infinitely small variation $d\Upsilon$ of Υ is senseless as, according to (2), we have

$$d\Upsilon \equiv d\Upsilon\left(\nu_i L_i^{\nu_i - 1} dL_i\right). \tag{3}$$

But, because of $l_{\min}$, the infinitesimal quantities dL_i make no sense and hence $d\Upsilon$ makes no sense too.

Instead of these infinitesimal quantities, it seems reasonable to denote them as "minimal variations possible" $\Delta_{\min}$ of the quantity L having the dimension of length, that is, the quantity

$$\Delta_{\min} L = l_{\min}. \tag{4}$$

And then

$$\Delta_{\min}\Upsilon \equiv \Delta_{\min}\Upsilon\left(\nu_i L_i^{\nu_i - 1} \Delta_{\min} L_i\right) = \Delta_{\min}\Upsilon\left(\nu_i L_i^{\nu_i - 1} l_{\min}\right). \tag{5}$$

However, the "minimal variations possible" of any quantity having the dimensions of length (4) which are equal to $l_{\min} \propto l_P$ require, according to the Heisenberg Uncertainty Principle (HUP) [24], maximal momentum $p_{\max} \propto P_{Pl}$ and energy $E_{\max} \propto E_P$. Here l_P, P_{Pl}, E_P are Planck's length, momentum, and energy, respectively.

But at low energies (far from the Planck energy) there are no such quantities and hence in essence $\Delta_{\min} L = l_{\min} \propto l_P$ (4) corresponds to the high-energy (Planck's) case only.

For the energies lower than Planck's energy, the "minimal variations possible" $\Delta_{\min}L$ of the quantity L having the dimensions of length must be greater than $l_{\min}$ and dependent on the present E:

$$\Delta_{\min} \equiv \Delta_{\min,E}, \Delta_{\min,E}L > l_{\min}. \quad (6)$$

Besides, as we have a minimal length unit $l_{\min}$, it is clear that any quantity having the dimensions of length is "quantized"; that is, its value measured in the units $l_{\min}$ equals an integer number and we have

$$L = N_L l_{\min}, \quad (7)$$

where N_L is positive integer number.

The problem is that how the "minimal variations possible" $\Delta_{\min,E}$ (6) are dependent on the energy or, similarly, on the scales of the measured lengths?

To solve the above-mentioned problem, initially we can use the space-time quantum fluctuations (STQF) with regard to quantum theory and gravity.

The definition (STQF) is closely associated with the notion of "space-time foam." The notion "space-time foam," introduced by Wheeler about 60 years ago for the description and investigation of physics at Planck's scales (Early Universe) [25, 26], is fairly settled. Despite the fact that in the last decade numerous works have been devoted to physics at Planck's scales within the scope of this notion, for example, [27–46], by this time still their is no clear understanding of the "space-time foam" as it is.

On the other hand, it is undoubtful that a quantum theory of the Early Universe should be a deformation of the well-known quantum theory.

In my works with colleagues [47–56], I have put forward one of the possible approaches to resolution of a quantum theory at Planck's scales on the basis of the density matrix deformation.

In accordance with the modern concepts, the space-time foam [26] notion forms the basis for space-time at Planck's scales (Big Bang). This object is associated with the quantum fluctuations generated by uncertainties in measurements of the fundamental quantities, inducing uncertainties in any distance measurement. A precise description of the space-time foam is still lacking along with an adequate quantum gravity theory. But for the description of quantum fluctuations we have a number of interesting methods (e.g., [36–46, 57, 58]).

In what follows, we use the terms and symbols from [38]. Then for the fluctuations $\widetilde{\delta} l$ of the distance l we have the following estimate:

$$\left(\widetilde{\delta} l\right)_\gamma \gtrsim l_P^\gamma l^{1-\gamma} = l_P \left(\frac{l}{l_P}\right)^{1-\gamma} = l \left(\frac{l_P}{l}\right)^\gamma = l\lambda_l^\gamma, \quad (8)$$

or that same one:

$$\left|\left(\widetilde{\delta} l\right)_\gamma\right|_{\min} = \beta l_P^\gamma l^{1-\gamma} = \beta l_P \left(\frac{l}{l_P}\right)^{1-\gamma} = \beta l \lambda_l^\gamma, \quad (9)$$

where $0 < \gamma \le 1$, coefficient β is of order 1, and $\lambda_l \equiv l_P/l$.

From (8) and (9), we can derive the quantum fluctuations for all the primary characteristics, specifically for the time $(\widetilde{\delta} t)_\gamma$, energy $(\widetilde{\delta} E)_\gamma$, and metrics $(\widetilde{\delta} g_{\mu\nu})_\gamma$. In particular, for $(\widetilde{\delta} g_{\mu\nu})_\gamma$, we can use formula (10) in [38]:

$$\left(\widetilde{\delta} g_{\mu\nu}\right)_\gamma \gtrsim \lambda^\gamma. \quad (10)$$

Further in the text it is assumed that the theory involves a minimal length on the order of Planck's length

$$l_{\min} \propto l_P \quad (11)$$

or similarly

$$l_{\min} = \xi l_P, \quad (12)$$

where the coefficient ξ is on the order of unity too.

In this case the origin of the minimal length is not important. For simplicity, we assume that it comes from the generalized uncertainty principle (GUP) that is an extension of HUP for Planck's energies, where gravity must be taken into consideration [3–22]:

$$\Delta x \ge \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \quad (13)$$

Here α' is the model-dependent dimensionless numerical factor.

Inequality (13) leads to the minimal length $l_{\min} = \xi l_P = 2\sqrt{\alpha'} l_P$.

Therefore, in this case, replacement of Planck's length by the minimal length in all the above formulae is absolutely correct and is used without detriment to the generality [59]:

$$l_P \longrightarrow l_{\min}. \quad (14)$$

Thus, $\lambda_l \equiv l_{\min}/l$ and then (8)–(10) upon the replacement of (14) remain unchanged.

As noted in the overview [38], the value $\gamma = 2/3$ derived in [57, 58] is totally consistent with the Holographic Principle [60–63].

The following points of importance should be noted [59].

(1.1) It is clear that at Planck's scales, that is, at the minimal length scales,

$$l \longrightarrow l_{\min}, \quad (15)$$

models for different values of the parameter γ are coincident.

(1.2) As noted, specifically in (7), provided some quantity has a minimal measuring unit, values of this quantity are multiples of this unit.

Naturally, any quantity having a minimal measuring unit is uniformly discrete.

The latter property is not met, in particular, by the energy E.

As $E \sim 1/l$, where l is measurable scale, the energy E is a discrete but nonuniform quantity. It is clear that the difference between the adjacent values of E is the less the lower E. In other words, for $l \gg l_{\min}$, that is,

$$E \ll E_P, \quad (16)$$

E becomes a practically continuous quantity.

(1.3) In fact, the parameter λ_l was introduced earlier in [47–56] as a deformation parameter on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (Early Universe) that are considered to be the quantum mechanics with the fundamental length (QMFL):

$$0 < \alpha_x = \frac{l_{\min}^2}{x^2} \leq \frac{1}{4}, \tag{17}$$

where x is the measuring scale, $l_{\min} \sim l_p$.

The deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [64].

Obviously, everywhere, apart from the limiting point $\lambda_x = 1$ or $x = l_{\min}$, we have

$$\lambda_x = \sqrt{\alpha_x}. \tag{18}$$

From (17) it is seen that at the limiting point $x = l_{\min}$ the parameter α_x is not defined due to the appearance of singularity [47–56]. But at this point its definition may be extended (regularized).

The parameter α_l has the following clear physical meaning:

$$\alpha_l^{-1} \sim S^{BH}, \tag{19}$$

where

$$S^{BH} = \frac{A}{4l_p^2} \tag{20}$$

is the well-known Bekenstein-Hawking formula for the black hole entropy in the semiclassical approximation [65, 66] for the black hole event horizon surface A, with the characteristics linear dimension ("radius") $R = l$. This is especially obvious in the spherically symmetric case.

In what follows we use both parameters λ_x and α_x.

Turning back to the introductory section of this work and to the definition $\Delta_{\min,E}L$, we assume the following:

$$\left|\Delta_{\min,E}L\right| = \left|\left(\widetilde{\delta}L\right)_\gamma\right|_{\min}, \tag{21}$$

where $|(\widetilde{\delta}L)_\gamma|_{\min}$ is from formula (9), γ is fixed parameter from formulae (8) and (9), and $E = c\hbar/L$.

In physics, and in thermodynamics in particular, the extensive quantities or parameters are those proportional to the mass of a system or to its volume. Proceeding from definition (2) of the function $\Upsilon(L_i^{\nu_i})$, one can generalize this notion, taking, as a generalized extensive quantity (GEQ) of some spatial system Ω, the function dependent only on the linear dimensions of this system, with the exponents no less than 1.

The function $\Upsilon(L_i^{\nu_i})$, $\nu_i \geq 1$, (2) is GEQ of the system Ω with the characteristic linear dimensions L_i, $i = 1, \ldots, n$, or, identically, a sum of the systems Ω_i, $i = 1, \ldots, n$, each of which has its individual characteristic linear dimension L_i.

Then from the initial formulae (2)–(6) it directly follows that provided the minimal length $l_{\min}$ is existent, there are no infinitesimal variations of GEQ.

In the first place, this is true for such simplest objects as the n-dimensional sphere B_n, $n \geq 2$, whose surface area (area of the corresponding hypersphere S_n) and volume V_n represent GEQs and are equal to the following:

$$S_n = nC_nR^{n-1}; \qquad V_n = C_nR^n, \tag{22}$$

where R—radius of a sphere the length of which is a characteristic linear size, $C_n = \pi^{n/2}/\Gamma((n/2)+1)$, and $\Gamma(x)$ is a gamma-function.

Of course, the same is true for the n-dimensional cube (or hypercube) A_n; its surface area and its volume are GEQs, and a length of its edge is a characteristic linear dimension.

Provided $l_{\min}$ exists, there are no infinitesimal increments for both the surface area and volume of A_n or B_n; only minimal variations possible for these quantities are the case.

In what follows we consider only the spatial systems whose surface areas and volumes are GEQs.

Let us consider a simple but very important example of gravity in horizon spaces.

Gravity and thermodynamics of horizon spaces and their interrelations are currently most actively studied [67–79]. Let us consider a relatively simple illustration—the case of a static spherically symmetric horizon in space-time, the horizon being described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2. \tag{23}$$

The horizon location will be given by a simple zero of the function $f(r)$, at the radius $r = a$.

This case is studied in detail by Padmanabhan in his works [67, 78] and by the author of this paper in [80]. We use the notation system of [78]. Let, for simplicity, the space be denoted by $\mathscr{H}$.

It is known that for horizon spaces one can introduce the temperature that can be identified with an analytic continuation to imaginary time. In the case under consideration ([78], Equation (116)),

$$k_BT = \frac{\hbar cf'(a)}{4\pi}. \tag{24}$$

Therewith, the conditions $f(a) = 0$ and $f'(a) \neq 0$ must be fulfilled.

Then at the horizon $r = a$ Einstein's equations have the form

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a - \frac{1}{2}\right] = 4\pi Pa^2, \tag{25}$$

where $P = T_r^r$ is the trace of the momentum-energy tensor and radial pressure.

Now we proceed to the variables "α" from formula (17) to consider (25) in a new notation, expressing a in terms of the corresponding deformation parameter α. In what follows we omit the subscript in formula (17) of α_x, where the context implies which index is the case. In particular, here we use α instead of α_a. Then we have

$$a = l_{\min}\alpha^{-1/2}. \tag{26}$$

Therefore,

$$f'(a) = -2l_{\min}^{-1}\alpha^{3/2} f'(\alpha). \tag{27}$$

Substituting this into (25), we obtain in the considered case of Einstein's equations in the "α-representation" as follows [80]:

$$\frac{c^4}{G}\left(-\alpha f'(\alpha) - \frac{1}{2}\right) = 4\pi P \alpha^{-1} l_{\min}^2. \tag{28}$$

Multiplying the left- and right-hand sides of the last equation by α, we get

$$\frac{c^4}{G}\left(-f'(\alpha)\alpha^2 - \frac{1}{2}\alpha\right) = 4\pi P l_{\min}^2. \tag{29}$$

L.h.s. of (29) is dependent on α. Because of this, r.h.s. of (29) must be dependent on α as well; that is, $P = P(\alpha)$; that is,

$$\frac{c^4}{G}\left(-f'(\alpha)\alpha^2 - \frac{1}{2}\alpha\right) = 4\pi P(\alpha) l_{\min}^2. \tag{30}$$

Note that in this specific case the parameter α within constant factors is coincident with the Gaussian curvature K_a [81] corresponding to a:

$$\frac{l_{\min}^2}{a^2} = l_{\min}^2 K_a. \tag{31}$$

Substituting r.h.s of (31) into (30), we obtain the Einstein equation on horizon, in this case in terms of the Gaussian curvature,

$$\frac{c^4}{G}\left(-f'(K_a)K_a^2 - \frac{1}{2}K_a\right) = 4\pi P(K_a). \tag{32}$$

This means that up to the constants

$$-f'(K_a)K_a^2 - \frac{1}{2}K_a = P(K_a), \tag{33}$$

that is, the Gaussian curvature, K_a is a solution of Einstein equations in this case. Then we examine different cases of the solution (33) with due regard for considerations of formula (21).

(2.1) First, let us assume that $a \gg l_{\min}$. As, according to (7), the radius a is quantized, we have $a = N_a l_{\min}$ with the natural number $N_a \gg 1$. Then it is clear that the Gaussian curvature $K_a = 1/a^2 \approx 0$ takes a (nonuniform) discrete series of values close to zero, and, within the factor $1/l_{\min}^2$, this series represents inverse squares of natural numbers:

$$(K_a) = \left(\frac{1}{N_a^2}, \frac{1}{(N_a \pm 1)^2}, \frac{1}{(N_a \pm 2)^2}, \ldots\right). \tag{34}$$

Let us return to formulas (9) and (21) for $l = a$; consider

$$\left|\left(\left(\widetilde{\delta}a\right)_\gamma\right)_{\min}\right| = \beta N_a l_{\min} N_a^{-\gamma} = \beta N_a^{1-\gamma} l_{\min}, \tag{35}$$

where β in this case contains the proportionality factor that relates $l_{\min}$ and l_P.

Then, according to (21), $a_{\pm 1}$ is a measurable value of the radius r following after a, and we have

$$\begin{aligned}(a_{\pm 1})_\gamma &\equiv a \pm \left(\left(\widetilde{\delta}a\right)_\gamma\right)_{\min} = a \pm \beta N_a^{1-\gamma} l_{\min} \\ &= N_a\left(1 \pm \beta N_a^{-\gamma}\right) l_{\min}.\end{aligned} \tag{36}$$

But, as $N_a \gg 1$, for sufficiently large N_a and fixed γ, the bracketed expression in r.h.s. (36) is close to 1:

$$1 \pm \beta N_a^{-\gamma} \approx 1. \tag{37}$$

Obviously, we get

$$\lim_{N_a \to \infty}\left(1 \pm \beta N_a^{-\gamma}\right) \longrightarrow 1. \tag{38}$$

As a result, the Gaussian curvature $K_{a_{\pm 1}}$ corresponding to $r = a_{\pm 1}$

$$K_{a_{\pm 1}} = \frac{1}{a_{\pm 1}^2} \propto \frac{1}{N_a^2\left(1 \pm \beta N_a^{-\gamma}\right)^2} \tag{39}$$

in the case under study is only slightly different from K_a.

And this is the case for sufficiently large values of N_a, for any value of the parameter γ, and for $\gamma = 1$ as well, corresponding to the absolute minimum of fluctuations $\approx l_{\min}$, or more precisely, to $\beta l_{\min}$. However, as all the quantities of the length dimension are quantized and the factor β is on the order of 1, actually we have $\beta = 1$.

Because of this, provided the minimal length is involved, $l_{\min}$ (9) is read as

$$\left|\left(\widetilde{\delta}l\right)_1\right|_{\min} = l_{\min}. \tag{40}$$

But, according to (12), $l_{\min} = \xi l_P$ is on the order of Planck's length, and it is clear that the fluctuation $|(\widetilde{\delta}l)_1|_{\min}$ corresponds to Planck's energies and Planck's scales. The Gaussian curvature K_a, due to its smallness ($K_a \ll 1$ up to the constant factor $l_{\min}^{-2}$) and smooth variations independent of γ (formulas (36)–(39)), is insensitive to the differences between various values of γ.

Consequently, for sufficiently small Gaussian curvature K_a, we can take any parameter from the interval $0 < \gamma \leq 1$ as γ.

It is obvious that the case $\gamma = 1$, that is, $|(\widetilde{\delta}l)_1|_{\min} = l_{\min}$, is associated with infinitely small variations da of the radius $r = a$ in the Riemannian geometry.

Since then K_a is varying practically continuously, in terms of K_a up to the constant factor we can obtain the following expression:

$$d\left[L(K_a)\right] = d\left[P(K_a)\right], \tag{41}$$

where we have

$$L(K_a) = -f'(K_a)K_a^2 - \frac{1}{2}K_a, \tag{42}$$

that is, l.h.s of (32) (or (33)).

But in fact, as in this case the energies are low, it is more correct to consider

$$L\left(\left(K_{a_{\pm1}}\right)_{\gamma}\right) - L\left(K_a\right) = \left[P\left(K_{a_{\pm1}}\right)_{\gamma}\right] - \left[P\left(K_a\right)\right] \equiv F_{\gamma}\left[P\left(K_a\right)\right], \tag{43}$$

where $\gamma < 1$, rather than (41).

In view of the foregoing arguments (2.1), the difference between (43) and (41) is insignificant and it is perfectly correct to use (41) instead of (43).

(2.2) Now we consider the opposite case or the transition to the ultraviolet limit

$$a \longrightarrow l_{\min} = \kappa l_{\min}. \tag{44}$$

That is,

$$a = \kappa l_{\min}. \tag{45}$$

Here κ is on the order of 1.

Taking into consideration point (1.1) stating that in this case models for different values of the parameter γ are coincident, by formula (40) for any γ, we have

$$\left|\left(\widetilde{\delta l}\right)_{\gamma}\right|_{\min} = \left|\left(\widetilde{\delta l}\right)_{1}\right|_{\min} = l_{\min}. \tag{46}$$

But in this case the Gaussian curvature K_a is not a "small value" continuously dependent on a, taking, according to (39), a discrete series of values $K_a, K_{a_{\pm1}}, K_{a_{\pm2}}, \ldots$.

Yet (25), similar to (32), (33) is valid in the semiclassical approximation only, that is, at low energies.

Then, in accordance with the above arguments, the limiting transition to high energies (44) gives a discrete chain of equations or a single equation with a discrete set of solutions as follows:

$$\begin{gathered} -f'\left(K_a\right)K_a^2 - \frac{1}{2}K_a = \Theta\left(K_a\right); \\ -f'\left(K_{a\pm1}\right)K_{a\pm1}^2 - \frac{1}{2}K_{a\pm1} = \Theta\left(K_{a\pm1}\right); \end{gathered} \tag{47}$$

and so on. Here $\Theta(K_a)$ is some function that in the limiting transition to low energies must reproduce the low-energy result to a high degree of accuracy; that is, $P(K_a)$ appears for $a \gg l_{\min}$ from formula (33):

$$\lim_{K_a \to 0} \Theta\left(K_a\right) = P\left(K_a\right). \tag{48}$$

In general, $\Theta(K_a)$ may lack coincidence with the high-energy limit of the momentum-energy tensor trace (if any):

$$\lim_{a \to l_{\min}} P\left(K_a\right). \tag{49}$$

At the same time, when we naturally assume that the static spherically-symmetric horizon space-time with the radius of several Planck's units (45) is nothing else but a micro black hole, then the high-energy limit (49) is existing and the replacement of $\Theta(K_a)$ by $P(K_a)$ in r.h.s. of the foregoing equations is possible to give a hypothetical gravitational equation for the event horizon micro black hole. But a question arises, for which values of the parameter a (45) (or K_a) is this valid and what is a minimal value of the parameter $\gamma = \gamma(a)$ in this case.

In all the cases under study, (2.1) and (2.2), the deformation parameter α_a (17) (λ_a (18)) is, within the constant factor, coincident with the Gaussian curvature K_a (resp., $\sqrt{K_a}$) that is in essence continuous in the low-energy case and discrete in the high-energy case.

In this way the above-mentioned example shows that, despite the absence of infinitesimal spatial-temporal increments owing to the existence of $l_{\min}$ and the essential "discreteness" of a theory, this discreteness at low energies is not felt, the theory being actually continuous. The indicated discreteness is significant only in the case of high (Planck's) energies.

In [78] it is shown that the Einstein equation for horizon spaces in the differential form may be written as a thermodynamic identity (the first principle of thermodynamics) ([78], formula (119)):

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T} \underbrace{\frac{c^3}{G\hbar} d\left(\frac{1}{4}4\pi a^2\right)}_{dS} \underbrace{-\frac{1}{2}\frac{c^4 da}{G}}_{-dE} = \underbrace{Pd\left(\frac{4\pi}{3}a^3\right)}_{PdV}, \tag{50}$$

where, as noted above, T is temperature of the horizon surface, S is corresponding entropy, E is internal energy, and V is space volume.

Note that, because of the existing $l_{\min}$, practically all quantities in (50) (except of T) represent GEQ. Apparently, the radius of a sphere $r = a$, its volume V, and entropy represent such quantities

$$S = \frac{4\pi a^2}{4 l_P^2} = \frac{\pi a^2}{l_P^2}, \tag{51}$$

within the constant factor $1/4l_P^2$ equal to a sphere with the radius a.

Because of this, there are no infinitesimal increments of these quantities, that is, da, dV, dS. And, provided $l_{\min}$ is involved, the Einstein equation for the above-mentioned case in the differential form (50) makes no sense and is useless. If da may be, purely formerly, replaced by $l_{\min}$, then, as the quantity $l_{\min}$ is fixed, it is obvious that "dS" and "dV" in (50) will be growing as a and a^2, respectively. And at low energies, that is, for large values of $a \gg l_{\min}$, this naturally leads to infinitely large rather than infinitesimal values.

In a similar way it is easily seen that the "Entropic Approach to Gravity" [82] in the present formalism is invalid within the scope of the minimal length theory. In fact, the "main instrument" in [82] is a formula for the infinitesimal variation dN in the bit numbers N on the holographic screen $\mathcal{S}$ with the radius R and with the surface area A ([82], formula (4.18)):

$$dN = \frac{c^3}{G\hbar} dA = \frac{dA}{l_P^2}. \tag{52}$$

As $N = A/l_P^2$ and A represents GEQ, it is clear that N is also GEQ and hence neither dA nor dN makes sense.

It is obvious that infinitesimal variations of the screen surface area dA are possible only in a continuous theory involving no $l_{\min}$.

When $l_{\min} \propto l_P$ is involved, the minimal variation ΔA is evidently associated with a minimal variation in the radius R

$$R \longrightarrow R \pm l_{\min} \tag{53}$$

is dependent on R and growing as R for $R \gg l_{\min}$:

$$\begin{aligned}\Delta_{\pm} A(R) &= \left(A\left(R \pm l_{\min}\right) - A(R)\right) \propto \left(\frac{\pm 2R}{l_{\min}} + 1\right) \\ &= \pm 2N_R + 1,\end{aligned} \tag{54}$$

where $N_R = R/l_{\min}$, as indicated above.

But, as noted above, a minimal increment of the radius R equal to $|\Delta_{\min} R| = l_{\min} \propto l_P$ corresponds only to the case of maximal (Planck's) energies or, similarly, to the parameter $\gamma = 1$ in formula (21). However, in [82], the considered low energies are far from the Planck energies and hence in this case in (21) $\gamma < 1$, (53), and (54) are, respectively, replaced by

$$R \longrightarrow R \pm N_R^{1-\gamma} l_{\min}, \tag{55}$$

$$\begin{aligned}\Delta_{\pm} A(R) &= \left(A\left(R \pm N_R^{1-\gamma} l_{\min}\right) - A(R)\right) \propto \pm N_R^{2-\gamma} + N_R^{2-2\gamma} \\ &= N_R^{2-2\gamma}\left(\pm N_R^{\gamma} + 1\right).\end{aligned} \tag{56}$$

An increase of r.h.s in (56) with the growth of R (or identically of N_R) for $R \gg l_{\min}$ is obvious.

So, if $l_{\min}$ is involved, formula (4.18) from [82] makes no sense similar to other formulae derived on its basis (4.19), (4.20), (4.22), (5.32)–(5.34),... in [82] and similar to the derivation method for Einstein's equations proposed in this work.

Proceeding from the principal parameters of this work α_l (or λ_l), the fact is obvious and is supported by formula (19) given in this paper, meaning that

$$\alpha_R^{-1} \sim A. \tag{57}$$

That is, small variations of α_R (low energies) result in large variations of α_R^{-1}, as indicated by formula (54).

In fact, we have *no-go theorems*.

The last statements concerning dS, dN may be explicitly interpreted using the language of a quantum information theory as follows: due to the existence of the minimal length $l_{\min}$, the minimal area $l_{\min}^2$ and volume $l_{\min}^3$ are also involved, and that means "quantization" of the areas and volumes. As, up to the known constants, the "bit number" N from (52) and the entropy S from (51) are nothing else but

$$S = \frac{A}{4 l_{\min}^2}, \qquad N = \frac{A}{l_{\min}^2}, \tag{58}$$

it is obvious that there is a "minimal measure" for the "amount of data" that may be referred to as "one bit" (or "one qubit").

The statement that there is no such quantity as dN (and, resp., dS) is equivalent to claiming the absence of 0.25 bit, 0.001 bit, and so on.

This inference completely conforms to the Hooft-Susskind Holographic Principle (HP) [60–63] that includes two main statements as follows.

(a) All information contained in a particular spatial domain is concentrated at the boundary of this domain.

(b) A theory for the boundary of the spatial domain under study should contain maximally one degree of freedom per Planck's area l_P^2.

In fact (but not explicitly) HP implicates the existence of $l_{\min} = l_P$. The existence of $l_{\min} \propto l_P$ totally conforms to HP, providing its generalization. Specifically, without the loss of generality, l_P^2 in point (b) may be replaced by $l_{\min}^2$.

So, the principal inference of this work is as follows: provided the minimal length $l_{\min}$ is involved, its existence must be taken into consideration not only at high but also at low energies, both in a quantum theory and in gravity. This becomes apparent by rejection of the infinitesimal quantities associated with the spatial-temporal variations $dx_\mu, \ldots$. In other words, with the involvement of $l_{\min}$, the general relativity (GR) must be replaced by a (still unframed) minimal length gravitation theory that may be denoted as $\mathrm{Grav}^{l_{\min}}$. In their results GR and $\mathrm{Grav}^{l_{\min}}$ should be very close but, as regards their mathematical apparatus (instruments), these theories are absolutely different.

Besides, $\mathrm{Grav}^{l_{\min}}$ should offer a rather natural transition from high to low energies

$$\left[N_L \approx 1\right] \longrightarrow \left[N_L \gg 1\right], \tag{59}$$

and vice versa

$$\left[N_L \gg 1\right] \longrightarrow \left[N_L \approx 1\right], \tag{60}$$

where N_L is integer from formula (7) determining the characteristics scale of the lengths L (energies $E \sim 1/L \propto 1/N_L$).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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