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Research Article

An Analytical Study of the Nonsinglet Spin Structure Function $g_1^{\mathrm{NS}}(x,t)$ Up to NLO in the DGLAP Approach at Small x

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A next-to-leading order QCD calculation of nonsinglet spin structure function $g_1^{\mathrm{NS}}(x,t)$ at small x is presented using the analytical methods: Lagrange's method and method of characteristics. The compatibility of these analytical approaches is tested by comparing the analytical solutions with the available polarized global fits.

1. Introduction

Study of flavour nonsinglet and singlet evolution equations with next-to-leading order corrections in α_s helps us to understand accurately the spin content of the nucleon. The spin-dependent DGLAP [1–4] evolution equations provide us the basic framework to study the polarized quark and gluon structure functions which finally give us polarized proton and neutron structure functions. Apart from the discussions about the numerical solutions [5–17] of DGLAP equations, analytical approaches towards these evolution equations at small x are also available in literature [18–23] with reasonable phenomenological success.

There are many QCD working groups continuously upgrading the QCD parameterization for the polarized global fits [24–32]. NNPDF [24, 29] is a new approach to PDF fitting based on Monte Carlo sampling and Neural Networks. HOPPET is an x-space evolution code which provides polarized PDFs for longitudinally polarised evolution up to NLO [30]. These Parton Distribution Function (PDF) evolution programs are used by the QCD working group to set some benchmark results in recent QCD NLO analysis.

In this work we extend our present analytical analysis up to NLO, obtain analytical solutions of spin-dependent DGLAP evolution equations at small x, and calculate Q^2 evolution for nonsinglet structure function, as well as making a comparative study of the two analytical methods.

The structure of the paper is as follows: Section 1 is the Introduction, Section 2 describes the formalism part, and in Section 3, we discuss our findings and show the results, and Section 4 contains the conclusion.

2. Formalism

2.1. Approximation of DGLAP Equation at Small x. The polarized nonsinglet structure function $g_1^{\mathrm{NS}}(x,t)$ evolves independently of the polarized singlet and gluon distribution in DGLAP framework. The evolution equation for $g_1^{\mathrm{NS}}(x,t)$ is [33]

$$\frac{\partial \Delta q^{\mathrm{NS}}}{\partial t} = \frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{dz}{z}\Delta P_{\mathrm{qq}}(z)\,\Delta q^{\mathrm{NS}}\left(\frac{x}{z},t\right). \qquad (1)$$

Here $\Delta P_{\mathrm{qq}}(z)$ is the polarized splitting kernel [34–36], α_s is the NLO running coupling constant, and $t=\ln(Q^2/\Lambda^2)$. The quark-quark splitting function P_{qq} can be expressed as

$$\Delta P_{\mathrm{qq}}(x) = \frac{\alpha_s(Q^2)}{2\pi}\Delta P_{\mathrm{qq}}^0(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2\Delta P_{\mathrm{qq}}^1(x) + \mathcal{O}\left(\Delta P_{\mathrm{qq}}^2(x)\right), \qquad (2)$$

where $\Delta P^{0}_{\mathrm{qq}}(x)$, $\Delta P^{1}_{\mathrm{qq}}(x)$ are LO and NLO quark-quark splitting functions, respectively.

Introducing a variable $u = 1 - z$ and expanding the argument $\Delta q^{\mathrm{NS}}(x/z, t)$ on the r.h.s. of (1) in a Taylor series as well as neglecting the higher order terms we get two approximate relations, respectively,

$$\Delta q^{\mathrm{NS}}\left(\frac{x}{z}, t\right) = \Delta q^{\mathrm{NS}}(x,t) + xu\frac{\partial}{\partial x}\Delta q^{\mathrm{NS}}(x,t), \tag{3}$$

$$\Delta q^{\mathrm{NS}}\left(\frac{x}{z}, t\right) = \Delta q^{\mathrm{NS}}(x,t) + \left(\frac{x}{z} - x\right)\frac{\partial}{\partial x}\Delta q^{\mathrm{NS}}(x,t). \tag{4}$$

The levels of approximation of (3) and (4) are as discussed in [37]. Using both (3) and (4) in (1) separately and putting the expressions for NLO polarized splitting function at small x [34–36, 38], we get two reduced forms of (1) as a function of x and t. We can express them as

$$\begin{aligned}\frac{\partial \Delta q^{\mathrm{NS}}(x,t)}{\partial t} &= \left[A_i(x)T(t) + B_i(x)T^2(t)\right]\frac{\partial \Delta q^{\mathrm{NS}}(x,t)}{\partial x} \\ &\quad + \left[C_i(x)T(t) + D_i(x)T^2(t)\right]\Delta q^{\mathrm{NS}}(x,t)\end{aligned} \tag{5}$$

with $i = 1, 2$. Here we introduce the running coupling constant up to NLO as

$$T(t) = \frac{2}{\beta_0 t}\left[1 - \frac{\beta_1}{\beta_0^2 t}\log t\right] \tag{6}$$

with

$$\begin{aligned}\beta_0 &= \frac{33 - 2n_f}{3}, \\ \beta_1 &= 102 - \frac{38n_f}{3}.\end{aligned} \tag{7}$$

The functions $A_i(x)$, $B_i(x)$, $C_i(x)$, and $D_i(x)$ are, respectively, $i = 1, 2$

Case 1. For the PDE derived using (3) ($i = 1$),

$$\begin{aligned}A_1(x) &= \frac{4}{3}\left(x\log\frac{1}{x} - x + x^2\right), \\ B_1(x) &= x(x-1)\left(-\frac{20\pi^2}{27} + \frac{256}{9}\right) \\ &\quad + x\log\frac{1}{x}\left(-\frac{20\pi^2}{27} + \frac{260}{9}\right) + \frac{10}{9}x^2\log\frac{1}{x}, \\ C_1(x) &= \frac{4}{3}\left(\frac{1}{2} + \log\frac{1}{x}\right), \\ D_1(x) &= \left(-\frac{20\pi^2}{27} + \frac{260}{9}\right)\log\frac{1}{x}.\end{aligned} \tag{8}$$

Case 2. For the PDE derived using (4) ($i = 2$),

$$\begin{aligned}A_2(x) &= \frac{4}{3}\left(1 - x - x\log\frac{1}{x}\right), \\ B_2(x) &= \left(-\frac{20\pi^2}{27} + \frac{236}{9}\right)(1-x) \\ &\quad + \left(\frac{20\pi^2}{27} - \frac{260}{9}\right)x\log\frac{1}{x} + \frac{106}{9}\log\frac{1}{x}, \\ C_2(x) &= \frac{4}{3}\left(\frac{1}{2} + \log\frac{1}{x}\right), \\ D_2(x) &= \left(-\frac{20\pi^2}{27} + \frac{260}{9}\right)\log\frac{1}{x}.\end{aligned} \tag{9}$$

Unlike in LO, (5) cannot be solved analytically. Hence, as in [39, 40], we introduce an assumption which linearizes $T^2(t)$ as

$$T^2(t) = T^0 T(t), \tag{10}$$

where T^0 is a numerical parameter to be obtained from Q^2 range under consideration as has been done in [39, 40]. We will make a detailed study of this parameter later in this present work.

We now solve (5) analytically by Lagrange's method [41] and then by method of characteristics [42, 43].

2.2. Solution by Lagrange's Method. To obtain solutions of the PDE (5), we recast it in the form

$$\frac{dt}{Q_i(t)} = \frac{dx}{P_i(x)} = \frac{d\Delta q^{\mathrm{NS}}(x,t)}{R_i(x,t)\Delta q^{\mathrm{NS}}(x,t)} \tag{11}$$

with the forms of $P_i(x)$, $Q_i(t)$, and $R_i(x,t)$ ($i = 1, 2$) given as

$$\begin{aligned}P_i(x) &= -\left[A_i(x)T(t) + B_i(x)T^2(t)\right], \\ Q_i(t) &= 1, \\ R_i(x,t) &= \left[C_i(x)T(t) + D_i(x)T^2(t)\right].\end{aligned} \tag{12}$$

Two independent solutions u_i and v_i ($i = 1, 2$) (say) are to be obtained for this auxiliary system, so that the genenral solution of (11) can be written as

$$f(u_i, v_i) = 0, \tag{13}$$

f being an arbitrary function of u_i and v_i ($i = 1, 2$). It yields for both Cases 1 and 2 ($i = 1, 2$) that

$$\begin{aligned}u_i &= t^{(1+(b/t))}\chi_i^{\mathrm{NS}}(x), \\ v_i &= \Delta q^{\mathrm{NS}}(x,t)\,\xi_i^{\mathrm{NS}}(x).\end{aligned} \tag{14}$$

We define $\chi_i^{\mathrm{NS}}(x)$ and $\xi_i^{\mathrm{NS}}(x)$ as, with $i = 1, 2$,

$$\begin{aligned}\chi_i^{\mathrm{NS}}(x) &= \exp\left[\frac{b}{t} - \frac{1}{a}N_i^{\mathrm{NS}}(x)\right], \\ \xi_i^{\mathrm{NS}}(x) &= \exp\left[-M_i^{\mathrm{NS}}(x)\right],\end{aligned} \tag{15}$$

where $a = 2/\beta_0, b = \beta_1/\beta_0$. In (15),

$$N_i^{\mathrm{NS}}(x) = -\int \frac{dx}{A_i(x) + B_i(x) T^0}, \tag{16}$$

$$M_i^{\mathrm{NS}}(x) = -\int \frac{dx\left[C_i(x) + D_i(x) T^0\right]}{\left[A_i(x) + B_i(x) T^0\right]} \tag{17}$$

with ($i = 1, 2$). Demanding the linearity of the solution for $\Delta q^{\mathrm{NS}}(x,t)$ we get the possible form of $f(u_i, v_i)$ as

$$u_i + \alpha v_i = \beta, \tag{18}$$

where α and β are constants to be determined using the appropriate boundary conditions. Using relations equation (18) and physically plausible boundary conditions we get the solution for $\Delta q^{\mathrm{NS}}(x,t)$ at NLO as follows.

Case 1. Using relation equation (18),

$$\Delta q_1^{\mathrm{NS}}(x,t) = \Delta q^{\mathrm{NS}}(x,t_0)\Big|_{\mathrm{NLO}} \left(\frac{t}{t_0}\right) \Gamma_1(x,t), \tag{19}$$

$$\begin{aligned}\Gamma_1(x,t) = {} & \left(t^{b/t} \exp\left[\frac{b}{t} - \frac{N_1^{\mathrm{NS}}(1)}{a}\right]\right. \\ & \left. - t^{b/t} \exp\left[\frac{b}{t} - \frac{N_1^{\mathrm{NS}}(x)}{a}\right]\right) \\ & \times \left(\left(\frac{t}{t_0}\right) t^{b/t} \exp\left[\frac{b}{t} - \frac{N_1^{\mathrm{NS}}(1)}{a}\right]\right. \\ & \left. - t_0^{b/t_0} \exp\left[\frac{b}{t_0} - \frac{N_1^{\mathrm{NS}}(x)}{a}\right]\right)^{-1}.\end{aligned} \tag{20}$$

We can put (19) in the form as

$$\Delta q_1^{\mathrm{NS}}(x,t) = \Delta q^{\mathrm{NS}}(x,t_0)\Big|_{\mathrm{NLO}} \left(\frac{t}{t_0}\right)^{1+\epsilon_1(x,t)}, \tag{21}$$

where

$$\epsilon_1(x,t) = \frac{\log \Gamma_1(x,t)}{\log(t/t_0)}. \tag{22}$$

The term $(t/t_0)^{\epsilon_1(x,t)}$ gives us the measure of NLO effect on polarized structure function $\Delta q^{\mathrm{NS}}(x,t)$ at small x for the solution equation (21).

Case 2. Here we use (4). In this case also we get the following.

Using (18),

$$\Delta q_2^{\mathrm{NS}}(x,t) = \Delta q^{\mathrm{NS}}(x,t_0)\Big|_{\mathrm{NLO}} \left(\frac{t}{t_0}\right) \Gamma_2(x,t) \tag{23}$$

with

$$\begin{aligned}\Gamma_2(x,t) = {} & \left(t^{b/t} \exp\left[\frac{b}{t} - \frac{N_2^{\mathrm{NS}}(1)}{a}\right]\right. \\ & \left. - t^{b/t} \exp\left[\frac{b}{t} - \frac{N_2^{\mathrm{NS}}(x)}{a}\right]\right) \\ & \times \left(\left(\frac{t}{t_0}\right) t^{b/t} \exp\left[\frac{b}{t} - \frac{N_2^{\mathrm{NS}}(1)}{a}\right]\right. \\ & \left. - t_0^{b/t_0} \exp\left[\frac{b}{t_0} - \frac{N_2^{\mathrm{NS}}(x)}{a}\right]\right)^{-1}.\end{aligned} \tag{24}$$

As in Case 1, we express (23) in a form as

$$\Delta q_2^{\mathrm{NS}}(x,t) = \Delta q^{\mathrm{NS}}(x,t_0)\Big|_{\mathrm{NLO}} \left(\frac{t}{t_0}\right)^{1+\epsilon_2(x,t)} \tag{25}$$

with

$$\epsilon_2(x,t) = \frac{\log \Gamma_2(x,t)}{\log(t/t_0)} \tag{26}$$

giving us the measure of NLO effect at small x for the solution equation (25).

Thus we get two analytical solutions of (5) by Lagrange's method for $\Delta q^{\mathrm{NS}}(x,t)$ at NLO at small x, given by (21) and (25).

2.3. Approximate Analytical Forms of $\Gamma_1(x,t)$ and $\Gamma_2(x,t)$ at Small x. To obtain the analytical forms of (22) and (26) we need the analytical forms of $\Gamma_1(x,t)$ and $\Gamma_2(x,t)$. To that end we need explicit corresponding forms of $\chi_1^{\mathrm{NS}}(x)$, $\chi_2^{\mathrm{NS}}(x)$, $N_1^{\mathrm{NS}}(x)$, and $N_2^{\mathrm{NS}}(x)$. It is possible only in the very small x limit. In the very small x region ($\log(1/x) \gg x\log(1/x) \gg x$) the analytical form of $N_1^{\mathrm{NS}}(x)$, as defined in (16), can be obtained as

$$N_1^{\mathrm{NS}}(x) \approx -\int \frac{dx}{x z_1 \log(1/x) - z_2 x} \tag{27}$$

which after integration yields

$$N_1^{\mathrm{NS}}(x) \approx \frac{\log(z_2 - z_1 \log(1/x))}{z_1}. \tag{28}$$

Here

$$\begin{aligned} z_1 &= \frac{4}{3} + T^0 \lambda_1, \\ z_2 &= \frac{4}{3} + T^0 \lambda_2 \end{aligned} \tag{29}$$

with

$$\begin{aligned} \lambda_1 &= -\frac{20\pi^2}{27} + \frac{260}{9}, \\ \lambda_2 &= -\frac{20\pi^2}{27} + \frac{256}{9}. \end{aligned} \tag{30}$$

Under a similar small x approximation as for $N_1^{NS}(x)$, the $N_2^{NS}(x)$ takes the form

$$N_2^{NS}(x) \approx -\int \frac{dx}{((4/3)+T^0 c)+(106/9)\log(1/x)\,T^0}, \quad (31)$$

where

$$c = \frac{236}{9} - \frac{20\pi^2}{27}. \quad (32)$$

To obtain an analytical form of $N_2^{NS}(x)$ we need an additional approximation as used in deriving equation (27), which yields

$$\log\frac{1}{x} \gg \frac{(4/3)+T^0 c}{(106/9)\,T^0}. \quad (33)$$

This yields, $N_2^{NS}(x)$,

$$N_2^{NS}(x) = \frac{9}{106T^0} Ei\left(-\log\frac{1}{x}\right), \quad (34)$$

where

$$Ei(-z) = C + \log z + \int_0^z \frac{e^{-t}-1}{t}dt \quad (z>0). \quad (35)$$

Here C is Euler's constant [44, 45] and has the value $C = 0.577215$. Taking first three terms in the series expansion of the integral $\int_0^z (e^{-t}-1/t)dt$ [44] and regularizing the value of x at $x = x_0$, we get

$$\begin{aligned} Ei\left(-\log\frac{1}{x}\right) &= C + \log\log\frac{1}{x} + \log\left(\frac{\log(1/x)}{\log(1/x_0)}\right) \\ &\quad + \frac{\log(1/x_0)}{\log(1/x)} + \frac{1}{2.2!}\left(\frac{(\log(1/x))^2}{(\log(1/x_0))^2}\right) \\ &\quad + \log\frac{\log(1/x_0)}{\log(1/x)}. \end{aligned} \quad (36)$$

Using above equation $N_2^{NS}(x)$ takes the form

$$\begin{aligned} N_2^{NS}(x) &= \frac{9}{106T^0}\left[C + \log\log\frac{1}{x} + \log\left(\frac{\log(1/x)}{\log(1/x_0)}\right)\right. \\ &\quad + \frac{\log(1/x_0)}{\log(1/x)} + \frac{1}{2.2!}\left(\frac{(\log(1/x))^2}{(\log(1/x_0))^2}\right) \\ &\quad \left. + \log\frac{\log(1/x_0)}{\log(1/x)}\right]. \end{aligned} \quad (37)$$

2.4. Solution by Method of Characteristics. To use this method, it is convenient if (5) can be rewritten as

$$\begin{aligned} t\frac{\partial\Delta q^{NS}(x,t)}{\partial t} &= \frac{2}{\beta_0}u(t)\left[A_i(x)+B_i(x)T^0\right]\frac{\partial\Delta q^{NS}(x,t)}{\partial x} \\ &\quad + \frac{2}{\beta_0}u(t)\left[C_i(x)+D_i(x)T^0(t)\right]\Delta q^{NS}(x,t). \end{aligned} \quad (38)$$

Further the running coupling constant $T(t)$ as defined by (6) is reexpressed as

$$T(t) = \frac{2}{\beta_0 t}u(t) \quad (39)$$

with

$$u(t) = \left(1 - \frac{\beta_1}{\beta_0^2 t}\log t\right). \quad (40)$$

In the method of characteristics, the original set of variables (x,t) are changed to a new set of variables (s,τ) and the PDE becomes an ordinary differential equation with respect to either s or τ. Now along the characteristic curve the PDE (38) becomes an ODE and takes the form

$$\frac{d\Delta q_i^{NS}}{ds} + \widehat{C}_i(s,\tau)\,\Delta q_i^{NS} = 0 \quad (41)$$

with

$$\widehat{C}_i(s,\tau) = \frac{2}{\beta_0}u(t)\left[C_i(x)+D_i(x)T^0\right], \quad (42)$$

where C_i and D_i $(i = 1,2)$ are as defined in our earlier section.

Case 1. Using (3) $(i = 1)$,

$$s_1 = \log\left(\frac{t}{t_0}\right), \quad (43)$$

$$\tau_1 = \exp\left[x\left(\frac{t}{t_0}\right)^{\alpha_1}\right], \quad (44)$$

where

$$\alpha_1 = \frac{4u(t)\left(-18+T^0\left(-933+10\pi^2\right)\right)}{27\beta_0}. \quad (45)$$

In obtaining (44), we used "ultra small x" limit. Integrating (41) along the characteristic curve and then going back to (x,t) using (44), the solution for $\Delta q^{NS}(x,t)$ at NLO comes out as

$$\Delta q_1^{NS}(x,t) = \Delta q^{NS}(x,t_0)\left(\frac{t}{t_0}\right)^{\Lambda_1(x,t)}, \quad (46)$$

$$\Lambda_1(x,t) = \frac{1}{\log(t/t_0)}\log\left(\frac{\Delta q^{NS}(\tau_1)}{\Delta q^{NS}(x,t_0)}\right) + \frac{\gamma_1}{\log(t/t_0)}, \quad (47)$$

$$\begin{aligned} \gamma_1(x,t) &= -\frac{2}{\beta_0}u(t)\left\{\frac{4}{3}\left(\frac{1}{2}+\log\frac{1}{x}\right)\right. \\ &\quad \left. - T^0\left(\frac{20\pi^2}{27}\log x - \frac{308}{9}\log x\right)\right\}. \end{aligned} \quad (48)$$

Case 2. Using (4) ($i = 2$), solutions of the characteristic equations for the PDE (38) are

$$s_2 = \log\left(\frac{t}{t_0}\right), \tag{49}$$

$$\tau_2 = \exp\left[\left(\omega + \log x\right)\left(\frac{t}{t_0}\right)^{-\alpha_2} - \omega\right], \tag{50}$$

where

$$\omega = \frac{2\left(-9 + 192T^0 + 5\pi^2 T^0\right)}{129T^0}, \qquad \alpha_2 = \frac{2u(t)}{\beta_0}\frac{86T^0}{9\exp[-\omega]}. \tag{51}$$

Thus the solution of the equation for charateristic curve leads us to the solution for $\Delta q^{\mathrm{NS}}(x,t)$ at NLO (using (4)) as

$$\Delta q_2^{\mathrm{NS}}(x,t) = \Delta q^{\mathrm{NS}}(x,t_0)\left(\frac{t}{t_0}\right)^{\Lambda_2(x,t)}, \tag{52}$$

$$\Lambda_2(x,t) = \frac{1}{\log(t/t_0)}\log\left(\frac{\Delta q^{\mathrm{NS}}(\tau_2)}{\Delta q^{\mathrm{NS}}(x,t_0)}\right) + \frac{\gamma_2}{\log(t/t_0)}, \tag{53}$$

$$\gamma_2(x,t) = -\frac{2}{\beta_0}u(t)\left\{\frac{4}{3}\left(\frac{1}{2} + \log\frac{1}{x}\right) + T^0\left(\frac{280}{9}\log x - \frac{20\pi^2}{27}\log x\right)\right\}. \tag{54}$$

Since the expressions of τ (τ_1 and τ_2) are different (44) and (50), in this case too we have two solutions (46) and (52), corresponding to the level of approximations equations (3) and (4).

In the next section we will discuss the relative merits of the four solutions.

3. Results and Discussion

3.1. The Parameter T^0. We have defined the running coupling constant as given in (10). Here T^0 is a parameter to be determined numerically for the particular Q^2 range under study. There are many illustrated values for this numerical parameter T^0 available in literature, some of which are phenomenologically justified [39, 40, 46]. It is reasonable to identify $2\pi T^0$ as the average value of the coupling constant for the particular Q^2 range under study [39]. Taking $Q^2 = 2\,\mathrm{GeV}^2$, for all the ranges within the perturbative region, CCFR range ($1.3\,\mathrm{GeV}^2 \leq Q^2 \leq 12\,\mathrm{GeV}^2$) yields $T^0 = 0.027$ [39].

Again within the range x and Q^2, ($0.01 \leq x \leq 0.0489$) and ($1.496\,\mathrm{GeV}^2 \leq Q^2 \leq 13.39\,\mathrm{GeV}^2$) for E665 as well as ($0.0045 \leq x \leq 0.14$) and ($0.75\,\mathrm{GeV}^2 \leq Q^2 \leq 20\,\mathrm{GeV}^2$) for NMC, the valid range of T^0 is found to be ($0.08 \leq T^0 \leq 0.25$) [40]. It is also observed that, in the range ($0\,\mathrm{GeV}^2 \leq Q^2 \leq 50\,\mathrm{GeV}^2$), as per requirement of the range of data compared, choice of a suitable value of T^0 ($T^0 = 0.108$), can minimize the error [46].

However such approach does not yield any definite NLO effect to be compared on LO. In stead it leads to an NLO analysis with an additional parameter fitted from data. In this work we will rather find a theoretical limit on T^0 in the relative range of Q^2 compatible with the perturbative expectation.

It is to be noted that the approximation for linearising $T(t)$ as per equation (10) is exactly true only for very small variation of $T(t)$ with T^0. That is, this approach is applicable only in a very limited range of Q^2. If a comparison of the prediction of the model is done in a large Q^2 range the assumption is expected to break down. So the best way to fit the T^0 is to consider experimentally accessible Q^2 range and find the upper and lower limits of $T(t)$ and to consider its average value.

In the experimentally available Q^2 range for g_1^{NS} for HERMES [47] ($2\,\mathrm{GeV}^2 \leq Q^2 \leq 14\,\mathrm{GeV}^2$)$T^0$ is found to be in the range ($0.035 \leq T^0 \leq 0.055$). So taking the average value of both the upper and lower limit of T^0 we derive the value of T^0 as $T^0 = 0.045$.

3.2. Constraints on the Analytical Solutions. The analytical solutions at small x using Lagrange's method have two more restrictions.

(i) ($z_2 - z_1\log(1/x)$) should be positive in (28).

(ii) Allowed x region should satisfy (33) for any given T^0.

For $T^0 = 0.045$, since $z_1 = 2.30$ and $z_2 = 2.28$, (28) yields $x > 0.367$, which is outside the the expected small x region. In fact for any positive value of T^0 there is no small x solution for ($z_2 - z_1\log(1/x) > 1$). This implies that a physically plausible analytical solution by Lagrange's method at NLO equation (5) with ($i = 1$) at small x does not exist. Equation (21) is therfore not pursued further.

In case of $N_2^{\mathrm{NS}}(x)$, on the other hand we use (33) with the value of $T^0 = 0.045$ and obtain the limiting value of x to be $x \leq 0.016$, up to which the analytical solution (25) is valid. We also take it to be the regularized value of x_0 in (37).

(iii) Due to nonintegral exponent of $(1-x)^{\alpha_i}$ factor in the standard PDF forms [25–28, 48], $\Delta q_2^{\mathrm{NS}}(\tau_1)$ in (46) becomes in general complex. Hence these cannot be used in the solution given by (46) which was obtained by method of characteristics. Hence (46) is no more considered for comparative study.

We are left with (25) and (52) to pursue the comparative analysis of our analytical methods.

3.3. Comparison with Exact Results. The formalism developed above is valid at small x, $x\log(1/x) \gg x$, which yields that x should be $x \ll \exp^{-1} \simeq 0.367$. On the other hand (25) has specific small x range of validity, $x \leq 0.016$, while (52) does not have such defining limits. We therefore study

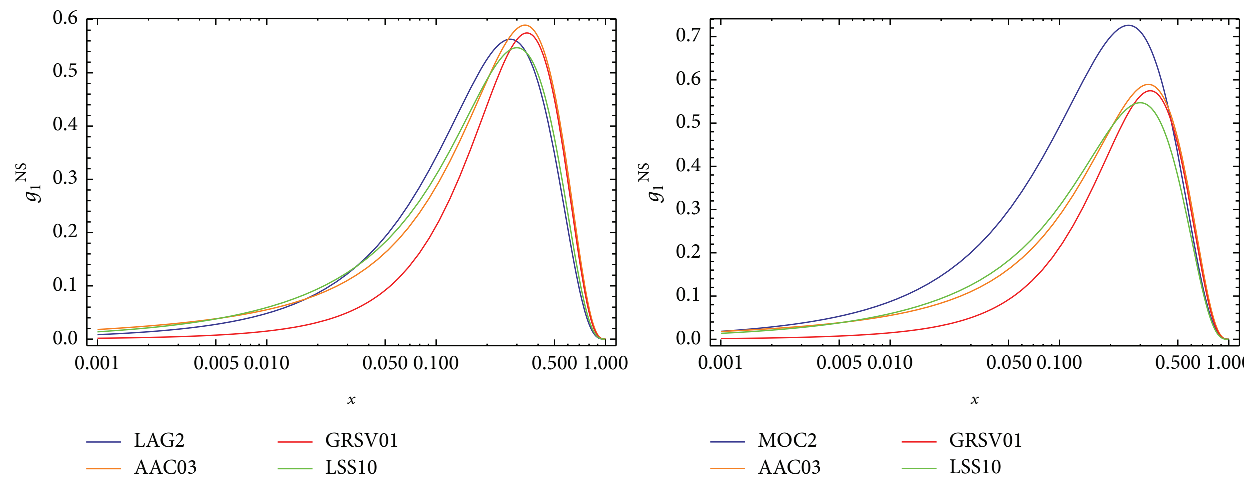

Figure 1: Polarized nonsinglet structure function $g_1^{NS}(x,t)$ as function of x at $Q^2 = 2\,\mathrm{GeV}^2$ at NLO, resulting, respectively, from our analytical calulations and their comparison with global fits to test the accuracy of our calculation.

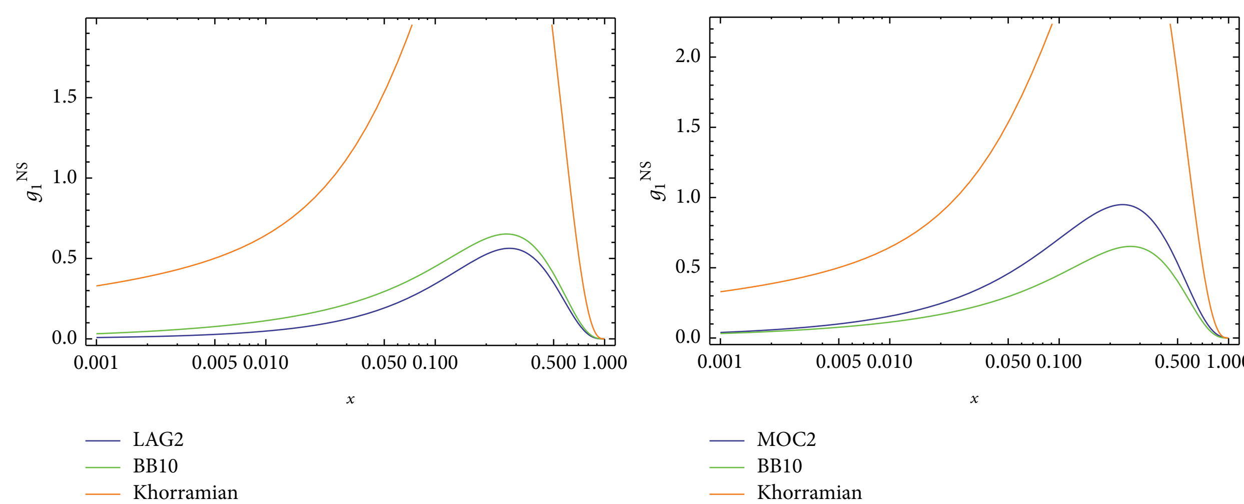

Figure 2: Polarized nonsinglet structure function $g_1^{NS}(x,t)$ as function of x at $Q^2 = 4\,\mathrm{GeV}^2$ at NLO, resulting, respectively, from our analytical calulations and their comparison with global fits to test the accuracy of our calculation.

the two solutions within ($x \leq 10^{-2}$), while comparing with exact results.

We compare our results for g_1^{NS} ((25) and (52)) with AAC03, GRSV01, LSS10, BB10, and recent Khorramian polarized NLO global fits [25–28, 48] at two different Q^2 values. In this calculation, we choose $\Lambda_{\mathrm{QCD}}^{\mathrm{NLO}} = 0.366$ GeV for $n_f = 3$ and $Q^2 \leq m_c^2$ and $\Lambda_{\mathrm{QCD}}^{\mathrm{NLO}} = 0.311$ GeV for $n_f = 4$ and $Q^2 \geq m_c^2$. The number of active flavors n_f is fixed by the number of quarks with $m_q^2 \leq Q^2$ taking $m_c = 1.43$ GeV as in [27].

In Figure 1, (25) and (52) are shown separately with the AAC03, GRSV01, and LSS10 global fits at $Q^2 = 2\,\mathrm{GeV}^2$. We observe that, within our valid small x range, (25) compares better as (52) evolves rapidly as we approach $x = 10^{-2}$.

Figure 2 shows a similar comparison of our solutions at $Q^2 = 4\,\mathrm{GeV}^2$ with the available NLO exact solutions by BB10 and Khorramian group [28, 48]. It is observed that in the small x region both the analytical solutions evolve with a good agreement with the theoretical prediction by Blumlein and Bottcher. It is to be noted that a preference of one solution over the other is not possible phenomenologically from Figure 2. Again our analytical models are not consistent with the global fit developed by Khorramian group [48].

In Figure 3, we therefore plot both solutions together and address if any of these two solutions fares better in the range $x \leq 10^{-2}$, which is the common range of validity for both. From it, we observe that at both Q^2, within the x range $x \leq 10^{-2}$, the solution given by (25) is more consistent with the exact theoretical predictions, than the solution by (52), thus giving Lagrange's method an edge over method of characteristics as a more appropriate method for obtaining small x analytical solutions in polarized NLO case also.

In Figures 1–3, we have taken the minimum cut-off of x range $x = 10^{-3}$, since resummation effects might be

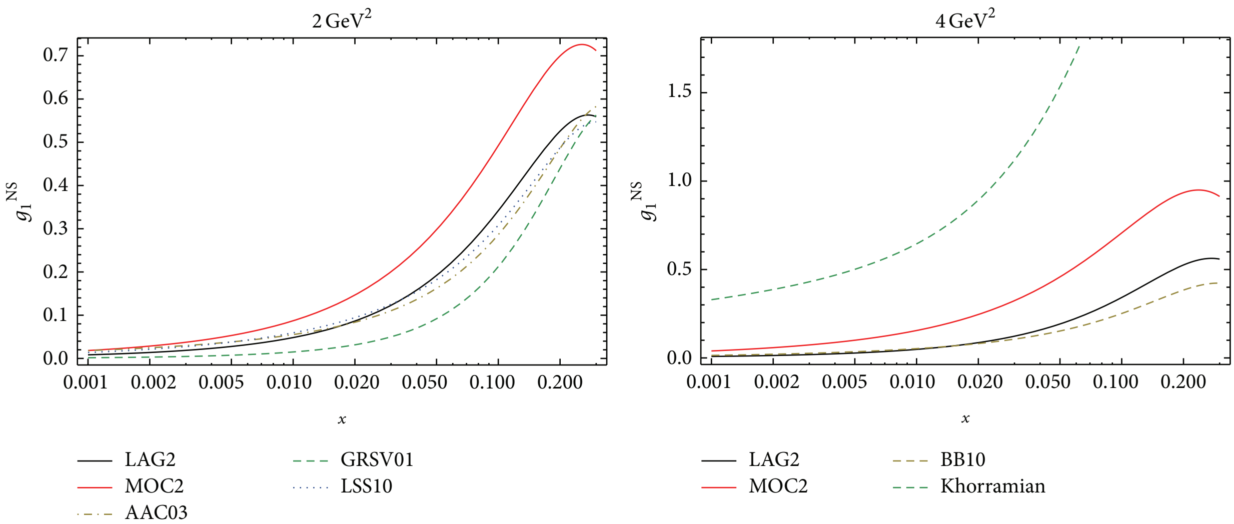

FIGURE 3: $g_1^{NS}(x,t)$ from (25) and (52) as function of x at $Q^2 = 2, 4\,\mathrm{GeV}^2$, respectively, at NLO.

important in such very small x region [49] below $x < 10^{-3}$ and DGLAP equations fail to describe these resummation effects.

4. Conclusion

In this comparative study, we have obtained analytical solutions for the polarized nonsinglet structure function $g_1^{NS}(x,t)$ at NLO, using the two analytical methods: Lagrange's method and method of characteristics. Due to physical constraints these two methods lead us to only two suitable solutions for $g_1^{NS}(x,t)$ at NLO, valid for small x. In this particular work, we have compared our analytical solutions only with the polarized global fits to test the consistency and plotted our solutions against x for two different Q^2 values, approximately in the range $(10^{-3} \le x \le 10^{-2})$. Within this range, it is observed that Lagrange's method is more consistent over the method of characteristics. Though we considered two levels of approximations as given in (3) and (4), our analysis indicates that only one approximation equation (4) leads us to physically plausible analytical solutions at small x, although theoritically former one (3) is preferred. Instead of various numerical methods, our method too proves out to be workable alternative to study these polarized evolution equations at small x.

A new insight in the structure functions drawn from this analytical approach is this: the (x,Q^2) dependence of the exponents of (t/t_0) in (25) and (52) plays a decisive role in selecting the kinematic range of phenomenological validity of the solutions in certain (x,Q^2) range. Again as both solutions (25) and (52) show identical behaviour numerically at (x,Q^2) range, we can infer that their exponents, $\Lambda_2(x,t)$ and $(1+\epsilon_2(x,t))$, are almost equal in that considered kinematic range. Moreover using recent works available [50, 51] the comaparative study of these analytical methods can be extended up to NNLO.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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