Research Article

State-Space Geometry, Statistical Fluctuations, and Black Holes in String Theory

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We study the state-space geometry of various extremal and nonextremal black holes in string theory. From the notion of the intrinsic geometry, we offer a state-space perspective to the black hole vacuum fluctuations. For a given black hole entropy, we explicate the intrinsic geometric meaning of the statistical fluctuations, local and global stability conditions, and long range statistical correlations. We provide a set of physical motivations pertaining to the extremal and nonextremal black holes, namely, the meaning of the chemical geometry and physics of correlation. We illustrate the state-space configurations for general charge extremal black holes. In sequel, we extend our analysis for various possible charge and anticharge nonextremal black holes. From the perspective of statistical fluctuation theory, we offer general remarks, future directions, and open issues towards the intrinsic geometric understanding of the vacuum fluctuations and black holes in string theory.

1. Introduction

In this paper, we study statistical properties of the charged and anticharged black hole configurations in string theory. Specifically, we illustrate that the components of the vacuum fluctuations define a set of local pair correlations against the parameters, for example, charges, anticharges, mass, and angular momenta. Our consideration follows from the notion of the thermodynamic geometry, mainly introduced by Weinhold [1, 2] and Ruppeiner [3–9]. Importantly, this framework provides a simple platform to geometrically understand the statistical nature of local pair correlations and underlying structures pertaining to the vacuum phase transitions. In diverse contexts, the state-space geometric perspective offers an understanding of the phase structures of mixtures of gases, black hole configurations [10–26], generalized uncertainty principle [27], strong interactions, for example, hot QCD [28], quarkonium configurations [29], and some other systems, as well.

The main purpose of the present paper is to consider the state-space properties of various possible extremal and nonextremal black holes in string theory, in general. String theory [30], as the most promising framework to understand all possible fundamental interactions, celebrates the physics of black holes, in both the zero and the nonzero temperature domains. Our consideration hereby plays a crucial role in understanding the possible phases and stability of the string theory vacua. A further motivation follows from the consideration of the string theory black holes; namely, $\mathcal{N} = 2$ supergravity arises as a low energy limit of the Type II string theory solution, admitting extremal black holes with the zero Hawking temperature and a nonzero macroscopic attractor entropy.

A priori, the entropy depends on a large number of scalar moduli arising from the compactification of the 10-dimensional theory down to the 4-dimensional physical spacetime. This involves a 6-dimensional compactifying manifold. Interesting string theory compactifications involve T^6, $K_3 \times T^2$, and Calabi-Yau manifolds. The macroscopic entropy exhibits a fixed point behavior under the radial flow of the scalar fields. In such cases, the near horizon geometry of an extremal black hole turns out to be an

$AdS_2 \times S^2$ manifold which describes the Bertotti-Robinson vacuum associated with the black hole. The area of the black hole horizon is A and thus the macroscopic entropy [31–42] is given as $S_{\text{macro}} = \pi |Z_\infty|^2$. This is known as the Ferrara-Kallosh-Strominger attractor mechanism, which, as the macroscopic consideration, requires a validity from the microscopic or statistical basis of the entropy. In this concern, there have been various investigations on the physics of black holes, for example, horizon properties [43, 44], counting of black hole microstates [45–47], spectrum of half-BPS states in $\mathcal{N} = 4$ supersymmetric string theory [48], and fractionation of branes [49]. From the perspective of the fluctuation theory, our analysis is intended to provide the nature of the statistical structures of the extremal and nonextremal black hole configurations. The attractor configurations exist for the extremal black holes, in general. However, the corresponding nonextremal configurations exist in the throat approximation. In this direction, it is worth mentioning that there exists an extension of Sen entropy function formalism for $D_1 D_5$ and $D_2 D_6 NS_5$ nonextremal configurations [50–52]. In the throat approximation, these solutions, respectively, correspond to Schwarzschild black holes in $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^2 \times S^1 \times T^4$. In relation with the intrinsic state-space geometry, we will explore the statistical understanding of the attractor mechanism and the moduli space geometry and explain the vacuum fluctuations of the black brane configurations.

In this paper, we consider the state-space geometry of the spherical horizon topology black holes in four spacetime dimensions. These configurations carry a set of electric magnetic charges (q_i, p_i). Due to the consideration of Strominger and Vafa [53], these charges are associated with an ensemble of weakly interacting D-branes. Following [53–59], it turns out that the charges (q_i, p_i) are proportional to the number of electric and magnetic branes, which constitute the underlying ensemble of the chosen black hole. In the large charge limit, namely, when the number of such branes becomes large, we have treated the logarithm of the degeneracy of states of the statistical configuration as the Bekenstein-Hawking entropy of the associated string theory black holes. For the extremal black holes, the entropy is described in terms of the number of the constituent D-branes. For example, the two charge extremal configurations can be examined in terms of the winding modes and the momentum modes of an excited string carrying n_1 winding modes and n_p momentum modes. Correspondingly, the state-space geometry of the nonextremal black holes is described by adding energy to the extremal D-branes configurations. This renders as the contribution of the clockwise and anticlockwise momenta in the Kaluza-Klein scenarios and that of the antibrane charges in general to the black hole entropy.

From the perspective of black hole thermodynamics, we describe the structure of the state-space geometry of four-dimensional extremal and nonextremal black holes in a given duality frame. Thus, when we take arbitrary variations over the charges (q_i, p_i) on the electric and magnetic branes, the underlying statistical fluctuations are described by only the numbers of the constituent electric and magnetic branes.

From the perspective of the intrinsic state-space geometry, if one pretends that the notion of statistical fluctuations applies to intermediate regimes of the moduli space, then the attractor horizon configurations require an embedding to the higher dimensional intrinsic Riemanian manifold. Physically, such a higher dimensional manifold can be viewed as a possible blow-up of the attractor fixed point phase-space to a nontrivial moduli space. From the perspective of thermodynamic Ruppenier geometry, we have offered future directions and open issues in the conclusion. We leave the explicit consideration of these matters open for further research.

In Section 2, we define the general notion of vacuum fluctuations. This offers the physical meaning of the state-space geometry. In Section 3, we provide a brief review of statistical fluctuations. In particular, for a given black hole entropy, we firstly explicate the statistical meaning of state-space surface and then offer the general meaning of the local and global stability conditions and long range statistical correlations. In Section 4, we provide a set of physical motivations pertaining to the extremal and nonextremal black holes, the meaning of Wienhold chemical geometry, and the physics of correlation. In Section 5, we consider state-space configurations pertaining to the extremal black holes and explicate our analysis for the two and three charge configurations. In Section 6, we extend the above analysis for the four, six, and eight charge-anticharge nonextremal black holes. Finally, Section 7 provides general remarks, conclusion and outlook, and future directions and open issues towards the application of string theory.

2. Definition of State-Space Geometry

Considering the fact that the black hole configurations in string theory introduce the notion of vacuum, it turns out for any thermodynamic system, that there exist equilibrium thermodynamic states given by the maxima of the entropy. These states may be represented by points on the state-space. Along with the laws of the equilibrium thermodynamics, the theory of fluctuations leads to the intrinsic Riemannian geometric structure on the space of equilibrium states [8, 9]. The invariant distance between two arbitrary equilibrium states is inversely proportional to the fluctuations connecting the two states. In particular, a less probable fluctuation means that the states are far apart. For a given set of states $\{X_i\}$, the state-space metric tensor is defined by

$$g_{ij}(X) = -\partial_i \partial_j S(X_1, X_2, \ldots, X_n). \tag{1}$$

A physical motivation of (1) can be given as follows. Up to the second order approximation, the Taylor expansion of the entropy $S(X_1, X_2, \ldots, X_n)$ yields

$$S - S_0 = -\frac{1}{2} \sum_{i=1}^{n} g_{ij} \Delta X^i \Delta X^j, \tag{2}$$

where

$$g_{ij} := -\frac{\partial^2 S(X_1, X_2, \ldots, X_n)}{\partial X^i \partial X^j} = g_{ji} \tag{3}$$

is called the state-space metric tensor. In the present investigation, we consider the state-space variables $\{X_1, X_2, \ldots, X_n\}$ as the parameters of the ensemble of the microstates of the underlying microscopic configuration (e.g., conformal field theory [60], black hole conformal field theory [61], and hidden conformal field theory [62, 63]), which defines the corresponding macroscopic thermodynamic configuration. Physically, the state-space geometry can be understood as the intrinsic Riemannian geometry involving the parameters of the underlying microscopic statistical theory. In practice, we will consider the variables $\{X_1, X_2, \ldots, X_n\}$ as the parameters, namely, charges, anticharges, and others if any, of the corresponding low energy limit of the string theory, for example, $\mathcal{N} = 2$ supergravity. In the limit, when all the variables, namely, $\{X_1, X_2, \ldots, X_n\}$, are thermodynamic, the state-space metric tensor equation (1) reduces to the corresponding Ruppenier metric tensor. In the discrete limit, the relative coordinates ΔX^i are defined as $\Delta X^i := X^i - X_0^i$, for given $\{X_0^i\} \in M_n$. In the Gaussian approximation, the probability distribution has the following form:

$$P(X_1, X_2, \ldots, X_n) = A \exp\left(-\frac{1}{2} g_{ij} \Delta X^i \Delta X^j\right). \quad (4)$$

With the normalization

$$\int \prod_i dX_i P(X_1, X_2, \ldots, X_n) = 1, \quad (5)$$

we have the following probability distribution:

$$P(X_1, X_2, \ldots, X_n) = \frac{\sqrt{g(X)}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} g_{ij} dX^i \otimes dX^j\right), \quad (6)$$

where g_{ij} now, in a strict mathematical sense, is properly defined as the inner product $g(\partial/\partial X^i, \partial/\partial X^j)$ on the corresponding tangent space $T(M_n) \times T(M_n)$. In this connotation, the determinant of the state-space metric tensor,

$$g(X) := \|g_{ij}\|, \quad (7)$$

can be understood as the determinant of the corresponding matrix $[g_{ij}]_{n \times n}$. For a given state-space manifold (M_n, g), we will think of $\{dX^i\}$ as the basis of the cotangent space $T^*(M_n)$. In the subsequent analysis, by taking an account of the fact that the physical vacuum is neutral, we will choose $X_0^i = 0$.

3. Statistical Fluctuations

3.1. Black Hole Entropy. As a first exercise, we have illustrated thermodynamic state-space geometry for the two charge extremal black holes with electric charge q and magnetic charge p. The next step has thence been to examine the thermodynamic geometry at an attractor fixed point(s) for the extremal black holes as the maxima of their macroscopic entropy $S(q, p)$. Later on, the state-space geometry of nonextremal counterparts has as well been analyzed. In this investigation, we demonstrate that the state-space correlations of nonextremal black holes modulate relatively more swiftly to an equilibrium statistical basis than those of the corresponding extremal solutions.

3.2. State-Space Surface. The Ruppenier metric on the state-space (M_2, g) of two charge black holes is defined by

$$g_{qq} = -\frac{\partial^2 S(q, p)}{\partial q^2}, \qquad g_{qp} = -\frac{\partial^2 S(q, p)}{\partial q \partial p},$$

$$g_{pp} = -\frac{\partial^2 S(q, p)}{\partial p^2}. \quad (8)$$

Subsequently, the components of the state-space metric tensor are associated with the respective statistical pair correlation functions. It is worth mentioning that the coordinates on the state-space manifold are the parameters of the microscopic boundary conformal field theory which is dual the black hole space-time solution. This is because the underlying state-space metric tensor comprises of the Gaussian fluctuations of the entropy which is the function of the number of the branes and antibranes. For the chosen black hole configuration, the local stability of the underlying statistical system requires both principle minors to be positive. In this setup, the diagonal components of the state-space metric tensor, namely, $\{g_{x_i x_i} \mid x_i = (n, m)\}$, signify the heat capacities of the system. This requires that the diagonal components of the state-space metric tensor

$$g_{x_i x_i} > 0, \quad i = n, m, \quad (9)$$

be positive definite. In this investigation, we discuss the significance of the above observation for the eight parameter nonextremal black brane configurations in string theory. From the notion of the relative scaling property, we will demonstrate the nature of the brane-brane pair correlations; namely, from the perspective of the intrinsic Riemannian geometry, the stability properties of the eight parameter black branes are examined from the positivity of the principle minors of the space-state metric tensor. For the Gaussian fluctuations of the two charge equilibrium statistical configurations, the existence of a positive definite volume form on the state-space manifold $(M_2(R), g)$ imposes such a global stability condition. In particular, the above configuration leads to a stable statistical basis if the determinant of the state-space metric tensor,

$$\|g\| = S_{nn} S_{mm} - S_{nm}^2, \quad (10)$$

remains positive. Indeed, for the two charge black brane configurations, the geometric quantities corresponding to the underlying state-space manifold elucidate typical features of the Gaussian fluctuations about an ensemble of equilibrium brane microstates. In this case, we see that the Christoffel connections on the (M_2, g) are defined by

$$\Gamma_{ijk} = g_{ij,k} + g_{ik,j} - g_{jk,i}. \quad (11)$$

The only nonzero Riemann curvature tensor is

$$R_{qpqp} = \frac{N}{D}, \quad (12)$$

where

$$N := S_{pp}S_{qqq}S_{qpp} + S_{qp}S_{qqp}S_{qpp}$$
$$+ S_{qq}S_{qqp}S_{ppp} - S_{qp}S_{qqq}S_{ppp} \qquad (13)$$
$$- S_{qq}S_{qpp}^2 - S_{pp}S_{qqp}^2,$$

$$D := \left(S_{qq}S_{pp} - S_{qp}^2 \right)^2. \qquad (14)$$

The scalar curvature and the corresponding R_{ijkl} of an arbitrary two-dimensional intrinsic state-space manifold $(M_2(R), g)$ may be given as

$$R(q, p) = \frac{2}{\|g\|} R_{qpqp}(q, p). \qquad (15)$$

3.3. Stability Conditions. For a given set of variables $\{X^1, X^2, \ldots, X^n\}$, the local stability of the underlying state-space configuration demands the positivity of the heat capacities:

$$\left\{ g_{ii}\left(X^i\right) > 0; \forall i = 1, 2, \ldots, n \right\}. \qquad (16)$$

Physically, the principle components of the state-space metric tensor $\{g_{ii}(X^i) \mid i = 1, 2, \ldots, n\}$ signify a set of definite heat capacities (or the related compressibilities), whose positivity apprises that the black hole solution complies an underlying, locally in equilibrium, statistical configuration. Notice further that the positivity of principle components is not sufficient to insure the global stability of the chosen configuration and thus one may only achieve a locally stable equilibriumstatistical configuration. In fact, the global stability condition constraint over the allowed domain of the parameters of black hole configurations requires that all the principle components and all the principle minors of the metric tensor must be strictly positive definite [6]. The above stability conditions require that the following set of equations must be simultaneously satisfied:

$$p_0 := 1,$$

$$p_1 := g_{11} > 0,$$

$$p_2 := \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} > 0,$$

$$p_3 := \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} > 0, \qquad (17)$$

$$\vdots$$

$$p_n := \|g\| > 0.$$

3.4. Long Range Correlations. The thermodynamic scalar curvature of the state-space manifold is proportional to the correlation volume [6]. Physically, the scalar curvature signifies the interaction(s) of the underlying statistical system.

Ruppenier has in particular noticed for the black holes in general relativity that the scalar curvature

$$R(X) \sim \xi^d, \qquad (18)$$

where d is the spatial dimension of the statistical system and the ξ fixes the physical scale [6]. The limit $R(X) \rightarrow \infty$ indicates the existence of certain critical points or phase transitions in the underlying statistical system. The fact that "all the statistical degrees of freedom of a black hole live on the black hole event horizon" signifies that the state-space scalar curvature, as the intrinsic geometric invariant, indicates an average number of correlated Plank areas on the event horizon of the black hole [8]. In this concern, [9] offers interesting physical properties of the thermodynamic scalar curvature and phase transitions in Kerr-Newman black holes. Ruppeiner [6] has further conjectured that the global correlations can be expressed by the following arguments: (a) the zero state-space scalar curvature indicates certain bits of information on the event horizon, fluctuating independently of each other; (b) the diverging scalar curvature signals a phase transition indicating highly correlated pixels of the information.

4. Some Physical Motivations

4.1. Extremal Black Holes. The state-space of the extremal black hole configuration is a reduced space comprising of the states which respect the extremality (BPS) condition. The state-spaces of the extremal black holes show an intrinsic geometric description. Our intrinsic geometric analysis offers a possible zero temperature characterization of the limiting extremal black brane attractors. From the gauge/gravity correspondence, the existence of state-space geometry could be relevant to the boundary gauge theories, which have finitely many countable sets of conformal field theory states.

4.2. Nonextremal Black Holes. We will analyze the state-space geometry of nonextremal black holes by the addition of antibrane charge(s) to the entropy of the corresponding extremal black holes. To interrogate the stability of a chosen black hole system, we will investigate the question that the underlying metric $g_{ij}(X_i) = -\partial_i\partial_j S(X_1, X_2, \ldots, X_n)$ should provide a nondegenerate state-space manifold. The exact dependence varies case to case. In the next section, we will proceed in our analysis with an increasing number of the brane charges and antibrane charges.

4.3. Chemical Geometry. The thermodynamic configurations of nonextremal black holes in string theory with small statistical fluctuations in a "canonical" ensemble are stable if the following inequality holds:

$$\left\| \partial_i\partial_j S(X_1, X_2, \ldots, X_n) \right\| < 0. \qquad (19)$$

The thermal fluctuations of nonextremal black holes, when considered in the canonical ensemble, give a closer approximation to the microcanonical entropy:

$$S = S_0 - \frac{1}{2} \ln\left(CT^2\right) + \cdots. \qquad (20)$$

In (20), the S_0 is the entropy in the "canonical" ensemble and C is the specific heat of the black hole statistical configuration. At low temperature, the quantum effects dominate and the above expansion does not hold anymore. The stability condition of the canonical ensemble is just $C > 0$. In other words, the Hessian function of the internal energy with respect to the chemical variables, namely, $\{x_1, x_2, \ldots, x_n\}$, remains positive definite. Hence, the energy as the function of the $\{x_1, x_2, \ldots, x_n\}$ satisfies the following condition:

$$\left\| \partial_i \partial_j E(x_1, x_2, \ldots, x_n) \right\| > 0. \tag{21}$$

The state-space coordinates $\{X^i\}$ and intensive chemical variables $\{x_i\}$ are conjugate to each other. In particular, the $\{X^i\}$ are defined as the Legendre transform of $\{x_i\}$, and thus we have

$$X^i := \frac{\partial S(x)}{\partial x_i}. \tag{22}$$

4.4. Physics of Correlation. Geometrically, the positivity of the heat capacity $C > 0$ turns out to be the positivity condition of $g_{ij} > 0$, for a given i. In many cases, the state-space stability restriction on the parameters of the black hole corresponds to the situation away from the extremality condition; namely, $r_+ = r_-$. Far from the extremality condition, even at the zero antibrane charge or angular momentum, we find that there is a finite value of the thermodynamic scalar curvature, unlike the nonrotating or only brane-charged configurations. It turns out that the state-space geometry of the two charge extremal configurations is flat. Thus, the Einstein-Hilbert contributions lead to a noninteracting statistical system. At the tree level, some black hole configurations turn out to be ill-defined, as well. However, we anticipate that the corresponding state-space configuration would become well-defined when a sufficient number of higher derivative corrections [64–67] are taken into account with respect to the α'-corrections and the string loop l_s corrections. For the BTZ black holes [13], we notice that the large entropy limit turns out to be the stability bound, beyond which the underlying quantum effects dominate.

For the black hole in string theory, the Ricci scalar of the state-space geometry is anticipated to be positive definite with finitely many higher order corrections. For nonextremal black brane configurations, which are far from the extremality condition, such effects have been seen from the nature of the state-space scalar curvature $R(S(X_1, X_2, \ldots, X_n))$. Indeed, [12, 14] indicate that the limiting state-space scalar curvature $R(S(X_1, X_2, \ldots, X_n))|_{\text{no anticharge}} \neq 0$ gives a set of stability bounds on the statistical parameters. Thus, our consideration yields a classification of the domain of the parameters and global correlation of a nonextremal black hole.

4.5. String Theory Perspective. In this subsection, we recall a brief notion of entropy of a general string theory black brane configuration from the viewpoint of the counting of the black hole microstates [53, 53–59, 68]. Given a string

theory configuration, the choice of compactification [30] chosen is the factorization of the type $\mathcal{M}_{(3,1)} \times M_6$, where M_6 is a compact internal manifold. From the perspective of statistical ensemble theory, we will express the entropy of a nonextremal black hole as the function of the numbers of branes and antibranes. Namely, for the charged black holes, the electric and magnetic charges (q_i, p_i) form a coordinate chart on the state-space manifold. In this case, for a given ensemble of D-branes, the coordinate q_i is defined as the number of the electric branes and p_i as the number of the magnetic branes. Towards the end of this paper, we will offer further motivation for the consideration of the state-space geometry of large charged nonspherical horizon black holes in spacetime dimensions $D \geq 5$. In this concern, [68] plays a central role towards the formation of the lower dimensional black hole configuration. Namely, for the torus compatifications, the exotic branes play an important role concerning the physical properties of supertubes, the D_0-F_1 system and associated counting of the black hole microstates.

In what follows, we consider the four-dimensional string theory black holes in a given duality basis of the charges (q_i, p_i). From the perspective of string theory, the exotic branes and nongeometric configurations offer interesting fronts for the black holes in three spacetime dimensions. In general, such configurations could carry a dipole or a higher pole charge, and they leave the four-dimension black hole configuration asymptotically flat. In fact, for the spacetime dimensions $D \geq 4$, [68] shows that a charge particle corresponds to an underlying gauge field, modulo U-duality transformations. From the perspective of nonextremal black holes, by taking appropriate boundary condition, namely, the unit asymptotic limit of the harmonic function which defines the spacetime metric, one can choose the spacetime regions such that the supertube effects arising from nonexotic branes can effectively be put off in an asymptotically flat space [68]. This allows one to compute the Arnowitt-Deser-Misner (ADM) mass of the asymptotic black hole. From the viewpoint of the statistical investigation, the dependence of the mass to the entropy of a nonextremal black hole comes from the contribution of the antibranes to the counting degeneracy of the states.

5. Extremal Black Holes in String Theory

5.1. Two Charge Configurations. The state-space geometry of the two charge extremal configurations is analyzed in terms of the winding modes and the momentum modes of an excited string carrying n_1 winding modes and n_p momentum modes. In the large charge limit, the microscopic entropy obtained by the degeneracy of the underlying conformal field theory states reduces to the following expression:

$$S_{\text{micro}} = 2\sqrt{2n_1 n_p}. \tag{23}$$

The microscopic counting can be accomplished by considering an ensemble of weakly interacting D-branes [54]. The counting entropy and the macroscopic attractor entropy of the two charge black holes in string theory which have a n_4

number of D_4 branes and a n_0 number of D_0 branes match and thus we have

$$S_{\text{micro}} = 2\pi \sqrt{n_0 n_4} = S_{\text{macro}}. \quad (24)$$

In this case, the components of underlying state-space metric tensor are

$$g_{n_0 n_0} = \frac{\pi}{2n_0} \sqrt{\frac{n_4}{n_0}}, \qquad g_{n_0 n_4} = -\frac{\pi}{2} \frac{1}{\sqrt{n_0 n_4}},$$
$$g_{n_4 n_4} = \frac{\pi}{2n_4} \sqrt{\frac{n_0}{n_4}}. \quad (25)$$

The diagonal pair correlation functions remain positive definite:

$$g_{n_i n_i} > 0 \quad \forall i \in \{0, 4\} \mid n_i > 0, \quad g_{n_4 n_4} > 0, \quad \forall (n_0, n_4). \quad (26)$$

For distinct $i, j \in \{0, 4\}$, the state-space pair correlation functions admit

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{n_i}{n_j}. \quad (27)$$

The global properties of fluctuating two charge D_0-D_4 extremal configurations are determined by possible principle minors. The first minor constraint $p_1 > 0$ directly follows from the positivity of the first component of metric tensor:

$$p_1 = \frac{\pi}{2n_0} \sqrt{\frac{n_4}{n_0}}. \quad (28)$$

The determinant of the metric tensor $p_2 := g(n_0, n_4)$ vanishes identically for all allowed values of the parameters. Thus, the leading order large charge extremal black branes having (i) a n_0 number of D_0-branes and a n_4 number of D_4 or (ii) excited strings with a n_1 number of windings and a n_p number of momenta, where either set of charges forms local coordinates on the state-space manifold, find degenerate intrinsic state-space configurations. For a given configuration entropy $S_0 := 2\pi c$, the constant entropy curve can be depicted as the rectangular hyperbola

$$n_0 n_4 = c^2. \quad (29)$$

The intrinsic state-space configuration depends on the attractor values of the scalar fields which arise from the chosen string compactification. Thus, the possible state-space Ruppenier geometry may become well-defined against further higher derivative α'-corrections. In particular, the determinant of the state-space metric tensor may take positive/negative definite values over the domain of brane charges. We will illustrate this point in a bit more detail in the subsequent consideration with a higher number of charges and anticharges.

5.2. Three Charge Configurations. From the consideration of the two derivative Einstein-Hilbert action, [53] shows that the leading order entropy of the three charge D_1-D_5-P extremal black holes is

$$S_{\text{micro}} = 2\pi \sqrt{n_1 n_5 n_p} = S_{\text{macro}}. \quad (30)$$

The concerned components of state-space metric tensor are given in **Appendix A**. Hereby, it follows further that the local state-space metric constraints are satisfied as

$$g_{n_i n_i} > 0 \quad \forall i \in \{1, 5, p\} \mid n_i > 0. \quad (31)$$

For distinct $i, j \in \{1, 5\}$ and p, the list of relative correlation functions is depicted in **Appendix A**. Further, we see that the local stabilities pertaining to the lines and two-dimensional surfaces of the state-space manifold are measured as

$$p_1 = \frac{\pi}{2n_1} \sqrt{\frac{n_5 n_p}{n_1}}, \qquad p_2 = -\frac{\pi^2}{4n_1 n_5^2 n_p} \left(n_p^2 n_1 + n_5^3\right). \quad (32)$$

The stability of the entire equilibrium phase-space configurations of the D_1-D_5-P extremal black holes is determined by the $p_3 := g$ determinant of the state-space metric tensor:

$$\|g\| = -\frac{1}{2}\pi^3 \left(n_1 n_5 n_p\right)^{-1/2}. \quad (33)$$

The universal nature of statistical interactions and the other properties concerning Maldacena, Strominger, and Witten (MSW) rotating black branes [55] are elucidated by the state-space scalar curvature:

$$R\left(n_1, n_5, n_p\right) = \frac{3}{4\pi \sqrt{n_1 n_5 n_p}}. \quad (34)$$

The constant entropy (or scalar curvature) curve defining the state-space manifold is the higher dimensional hyperbola:

$$n_1 n_5 n_p = c^2, \quad (35)$$

where c takes respective values of $(c_S, c_R) = (S_0/2\pi, 3/4\pi R_0)$. In [12, 14, 17, 18], we have shown that similar results hold for the state-space configuration of the four charge extremal black holes.

6. Nonextremal Black Holes in String Theory

6.1. Four Charge Configurations. The state-space configuration of the nonextremal D_1–D_5 black holes is considered with nonzero momenta along the clockwise and anticlockwise directions of the Kaluza-Klein compactification circle S^1. Following [56], the microscopic entropy and the macroscopic entropy match for given total mass and brane charges :

$$S_{\text{micro}} = 2\pi \sqrt{n_1 n_5} \left(\sqrt{n_p} + \sqrt{n_{\bar{p}}}\right) = S_{\text{macro}}. \quad (36)$$

The state-space covariant metric tensor is defined as a negative Hessian matrix of the entropy with respect to the number of D_1, D_5 branes $\{n_i \mid i = 1, 5\}$ and clockwise-anticlockwise

Kaluza-Klein momentum charges $\{n_p, \overline{n_p}\}$. Herewith, we find that the components of the metric tensor take elegant forms. The corresponding expressions are given in Appendix B. As in the case of the extremal configurations, the state-space metric satisfies the following constraints:

$$g_{n_i n_i} > 0, \quad \forall i = 1, 5; \qquad g_{n_a n_a} > 0, \quad \forall a = p, \overline{p}. \quad (37)$$

Furthermore, the scaling relations for distinct $i, j \in \{1, 5\}$ and p, concerning the list of relative correlation functions, are offered in Appendix B. In this case, we find that the stability criteria of the possible surfaces and hypersurfaces of the underlying state-space configuration are determined by the positivity of the following principle minors:

$$p_0 = 1, \qquad p_1 = \frac{\pi}{2} \sqrt{\frac{n_5}{n_1^3}} \left(\sqrt{n_p} + \sqrt{\overline{n_p}} \right),$$

$$p_2 = 0, \qquad p_3 = -\frac{1}{2n_p} \frac{\pi^3}{\sqrt{n_1 n_5}} \left(\sqrt{n_p} + \sqrt{\overline{n_p}} \right). \quad (38)$$

The complete local stability of the full nonextremal D_1–D_5 black brane state-space configuration is ascertained by the positivity of the determinant of the metric tensor:

$$g\left(n_1, n_5, n_p, \overline{n_p}\right) = -\frac{1}{4} \frac{\pi^4}{\left(n_p \overline{n_p}\right)^{3/2}} \left(\sqrt{n_p} + \sqrt{\overline{n_p}} \right)^2. \quad (39)$$

The global state-space properties concerning the four charge nonextremal D_1–D_5 black holes are determined by the regularity of the invariant scalar curvature:

$$R\left(n_1, n_5, n_p, \overline{n_p}\right) = \frac{9}{4\pi\sqrt{n_1 n_5}} \left(\sqrt{n_p} + \sqrt{\overline{n_p}} \right)^{-6} f\left(n_p, \overline{n_p}\right), \quad (40)$$

where the function $f(n_p, \overline{n_p})$ of two momenta $(n_p, \overline{n_p})$ running in opposite directions of the Kaluza-Klein circle S^1 has been defined as

$$f\left(n_p, \overline{n_p}\right) := n_p^{5/2} + 10 n_p^{3/2} \overline{n_p} + 5 n_p^{1/2} \overline{n_p}^2 + 5 n_p^2 \overline{n_p}^{1/2} + 10 n_p \overline{n_p}^{3/2} + \overline{n_p}^{5/2}. \quad (41)$$

By noticing the Pascal coefficient structure in (41), we see that the above function $f(n_p, \overline{n_p})$ can be factorized as

$$f\left(n_p, \overline{n_p}\right) = \left(n_p + \overline{n_p}\right)^5. \quad (42)$$

Thus, (40) leads to the following state-space scalar curvature:

$$R\left(n_1, n_5, n_p, \overline{n_p}\right) = \frac{9}{4\pi\sqrt{n_1 n_5}} \times \left(\frac{1}{\sqrt{n_p} + \sqrt{\overline{n_p}}} \right). \quad (43)$$

In the large charge limit, the nonextremal D_1–D_5 black branes have a nonvanishing small scalar curvature function on the state-space manifold (M_4, g). This implies an almost everywhere weakly interacting statistical basis. In this case, the constant entropy hypersurface is defined by the curve

$$\frac{c^2}{n_1 n_5} = \left(\sqrt{n_p} + \sqrt{\overline{n_p}} \right)^2. \quad (44)$$

As in the case of two charge D_0-D_4 extremal black holes and D_1-D_5-P extremal black holes, the constant c takes the same value of $c := S_0^2/4\pi^2$. For a given state-space scalar curvature k, the constant state-space curvature curves take the following form:

$$f\left(n_p, \overline{n_p}\right) = k\sqrt{n_1 n_5} \left(\sqrt{n_p} + \sqrt{\overline{n_p}} \right)^6. \quad (45)$$

6.2. Six Charge Configurations. We now extrapolate the state-space geometry of four charge nonextremal D_1–D_5 solutions for nonlarge charges, where we are no longer close to an ensemble of supersymmetric states. In [57], the computation of the entropy of all such special extremal and near-extremal black hole configurations has been considered. The leading order entropy as a function of charges $\{n_i\}$ and anticharges $\{m_i\}$ is

$$S\left(n_1, m_1, n_2, m_2, n_3, m_3\right)$$
$$:= 2\pi \left(\sqrt{n_1} + \sqrt{m_1} \right) \left(\sqrt{n_2} + \sqrt{m_2} \right) \left(\sqrt{n_3} + \sqrt{m_3} \right). \quad (46)$$

For given charges $i, j \in A_1 := \{n_1, m_1\}$; $k, l \in A_2 := \{n_2, m_2\}$; and $m, n \in A_3 := \{n_3, m_3\}$, the intrinsic state-space pair correlations are in precise accordance with the underlying macroscopic attractor configurations which are being disclosed in the special leading order limit of the nonextremal D_1–D_5 solutions. The components of the covariant state-space metric tensor over generic nonlarge charge domains are not difficult to compute, and, indeed, we have offered their corresponding expressions in Appendix C.

For all finite (n_i, m_i), $i = 1, 2, 3$, the components involving brane-brane state-space correlations $g_{n_i n_i}$ and antibrane-antibrane state-space correlations $g_{m_i m_i}$ satisfy the following positivity conditions:

$$g_{n_i n_i} > 0, \qquad g_{m_i m_i} > 0. \quad (47)$$

The distinct $\{n_i, m_i \mid i \in \{1, 2, 3\}\}$ describing six charge string theory black holes have three types of relative pair correlation functions. The corresponding expressions of the relative statistical correlation functions are given in Appendix C.

Notice hereby that the scaling relations remain similar to those obtained in the previous case, except that (i) the number of relative correlation functions has been increased, and (ii) the set of cross ratios, namely, $\{g_{ij}/g_{kl}, g_{kl}/g_{mn}, g_{ij}/g_{mn}\}$ being zero in the previous case, becomes ill-defined for the six charge state-space configurations. Inspecting the specific pair of distinct charge sets A_i and A_j, there are now 24 types of nontrivial relative correlation functions. The set of principle

components denominator ratios computed from the above state-space metric tensor reduces to

$$\frac{g_{ij}}{g_{kk}} = 0, \quad \forall i, j, k \in \{n_1, m_1, n_2, m_2, n_3, m_3\}. \tag{48}$$

For given $i, j \in A_1 := \{n_1, m_1\}$; $k, l \in A_2 := \{n_2, m_2\}$; $m, n \in A_3 := \{n_3, m_3\}$, and $g_{n_i m_i} = 0$, there are the total 15 types of trivial relative correlation functions. There are five such trivial ratios in each family $\{A_i \mid i = 1, 2, 3\}$. The local stability of the higher charged string theory nonextremal black holes is given by

$$p_1 = \frac{\pi}{2n_1^{3/2}} \left(\sqrt{n_2} + \sqrt{m_2}\right) \left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$p_2 = \frac{1}{4} \frac{\pi^2}{(n_1 m_1)^{3/2}} \left(\sqrt{n_2} + \sqrt{m_2}\right)^2 \left(\sqrt{n_3} + \sqrt{m_3}\right)^2,$$

$$p_3 = \frac{1}{8} \frac{\pi^3}{(n_1 m_1 n_2)^{3/2}} \sqrt{m_2} \left(\sqrt{n_3} + \sqrt{m_3}\right)^3 \tag{49}$$

$$\times \left(\sqrt{n_2} + \sqrt{m_2}\right) \left(\sqrt{n_1} + \sqrt{m_1}\right),$$

$$p_4 = 0.$$

The principle minor p_5 remains nonvanishing for all values of charges on the constituent brane and antibranes. In general, by an explicit calculation, we find that the hyper-surface minor p_5 takes the following nontrivial value:

$$p_5 = -\frac{1}{8} \frac{\pi^5}{(n_1 m_1 n_2 m_2)^{3/2} n_3} \left(\sqrt{n_1} + \sqrt{m_1}\right)^3 \tag{50}$$

$$\times \left(\sqrt{n_2} + \sqrt{m_2}\right)^3 \left(\sqrt{n_3} + \sqrt{m_3}\right)^3.$$

Specifically, for an identical value of the brane and antibrane charges, the minor p_5 reduces to

$$p_5(k) = -64 \frac{\pi^5}{k^{5/2}}. \tag{51}$$

The global stability on the full state-space configuration is carried forward by computing the determinant of the metric tensor:

$$\|g\| = -\frac{1}{16} \frac{\pi^6}{(n_1 m_1 n_2 m_2 n_3 m_3)^{3/2}} \left(\sqrt{n_1} + \sqrt{m_1}\right)^4 \tag{52}$$

$$\times \left(\sqrt{n_2} + \sqrt{m_2}\right)^4 \left(\sqrt{n_3} + \sqrt{m_3}\right)^4.$$

The underlying state-space configuration remains nondegenerate for the domain of given nonzero brane antibrane charges, except for extreme values of the brane and antibrane charges $\{n_i, m_i\}$, when they belong to the set

$$B := \{(n_1, n_2, n_3, m_1, m_2, m_3) \mid \tag{53}$$

$$(n_i, m_i) = (0, 0), (\infty, \infty), \text{ some } i\},$$

among the given brane-antibrane pairs $\{(n_1, m_1), (n_2, m_2), (n_3, m_3)\}$. The component $R_{n_1 n_2 m_3 m_4}$ diverges at the roots of the two variables polynomials defined as the functions of brane and antibrane charges:

$$f_1(n_2, m_2) = n_2^4 m_2^3 + 2(n_2 m_2)^{7/2} + n_2^3 m_2^4,$$

$$f_2(n_3, m_3) = m_3^{9/2} n_3^4 + n_3^4 m_3^{9/2}. \tag{54}$$

However, the component R_{n_3, m_3, n_3, m_3} with an equal number of brane and antibrane charges diverges at a root of a single higher degree polynomial:

$$f(n_1, m_1, n_2, m_2, n_3, m_3)$$

$$:= n_2^4 m_2^3 n_3^{9/2} m_3^4 + n_2^4 m_2^3 n_3^4 m_3^{9/2}$$

$$+ 2n_2^{7/2} m_2^{7/2} n_3^{9/2} m_3^4 + 2n_2^{7/2} m_2^{7/2} n_3^4 m_3^{9/2} \tag{55}$$

$$+ n_2^3 m_2^4 n_3^{9/2} m_3^4 + n_2^3 m_2^4 n_3^4 m_3^{9/2}.$$

Herewith, from the perspective of state-space global invariants, we focus on the limiting nature of the underlying ensemble. Thus, we may choose the equal charge and anticharge limit by defining $m_i := k$ and $n_i := k$ for the calculation of the Ricci scalar. In this case, we find the following small negative curvature scalar:

$$R(k) = -\frac{15}{16} \frac{1}{\pi k^{3/2}}. \tag{56}$$

Further, the physical meaning of taking an equal value of the charges and anticharges lies in the ensemble theory, namely, in the thermodynamic limit, all the statistical fluctuations of the charges and anticharges approach to a limiting Gaussian fluctuations. In this sense, we can take the average over the concerned individual Gaussian fluctuations. This shows that the limiting statistical ensemble of nonextremal nonlarge charge D_1–D_5 solutions yields an attractive state-space configuration. Finally, such a limiting procedure is indeed defined by considering the standard deviations of the equal integer charges and anticharges, and thus our interest in calculating the limiting Ricci scalar in order to know the nature of the long range interactions underlying in the system.

For a given entropy S_0, the constant entropy hypersurface is again some nonstandard curve:

$$\left(\sqrt{n_1} + \sqrt{m_1}\right) \left(\sqrt{n_2} + \sqrt{m_2}\right) \left(\sqrt{n_3} + \sqrt{m_3}\right) = c, \tag{57}$$

where the real constant c takes the precise value of $S_0/2\pi$.

6.3. Eight Charge Configurations. From the perspective of the higher charged and anticharged black hole configurations in string theory, let us systematically analyze the underlying statistical structures. In this case, the state-space configuration of the nonextremal black hole involves finitely many non-trivially circularly fibered Kaluza-Klein monopoles. In this process, we enlist the complete set of nontrivial relative state-space correlation functions of the eight charged anticharged

configurations, with respect to the lower parameter configurations, as considered in [12, 14]. There have been calculations of the entropy of the extremal, near-extremal, and general nonextremal solutions in string theory; see, for instance, [58, 59]. Inductively, the most general charge anticharge nonextremal black hole has the following entropy:

$$S\left(n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4\right) = 2\pi \prod_{i=1}^{4} \left(\sqrt{n_i} + \sqrt{m_i}\right). \tag{58}$$

For the distinct $i, j, k \in \{1, 2, 3, 4\}$, we find that the components of the metric tensor are

$$g_{n_i n_i} = \frac{\pi}{2 n_i^{3/2}} \prod_{j \neq i} \left(\sqrt{n_j} + \sqrt{m_j}\right),$$

$$g_{n_i n_j} = -\frac{\pi}{2 \left(n_i n_j\right)^{1/2}} \prod_{i \neq k \neq j} \left(\sqrt{n_k} + \sqrt{m_k}\right),$$

$$g_{n_i m_i} = 0,$$

$$g_{n_i m_j} = -\frac{\pi}{2 \left(n_i m_j\right)^{1/2}} \prod_{i \neq k \neq j} \left(\sqrt{n_k} + \sqrt{m_k}\right), \tag{59}$$

$$g_{m_i m_i} = \frac{\pi}{2 m_i^{3/2}} \prod_{j \neq i} \left(\sqrt{n_j} + \sqrt{m_j}\right),$$

$$g_{m_i m_j} = -\frac{\pi}{2 \left(m_i m_j\right)^{1/2}} \prod_{i \neq k \neq j} \left(\sqrt{n_k} + \sqrt{m_k}\right).$$

From the above depiction, it is evident that the principle components of the state-space metric tensor $\{g_{n_i n_i}, g_{m_i m_i} \mid i = 1, 2, 3, 4\}$ essentially signify a set of definite heat capacities (or the related compressibilities) whose positivity in turn apprises that the black brane solutions comply with an underlying equilibrium statistical configuration. For an arbitrary number of the branes $\{n_i\}$ and antibranes $\{m_i\}$, we find that the associated state-space metric constraints as the diagonal pair correlation functions remain positive definite. In particular, $\forall i \in \{1, 2, 3, 4\}$; it is clear that we have the following positivity conditions:

$$g_{n_i n_i} > 0 \mid n_i, m_i > 0, \qquad g_{m_i m_i} > 0 \mid n_i, m_i > 0. \tag{60}$$

As observed in [12, 14], we find that the ratios of diagonal components vary inversely with a multiple of a well-defined factor in the underlying parameters, namely, the charges and anticharges, which changes under the Gaussian fluctuations, whereas the ratios involving off diagonal components in effect uniquely inversely vary in the parameters of the chosen set A_i of equilibrium black brane configurations. This suggests that the diagonal components weaken in a relatively controlled fashion into an equilibrium, in contrast with the off diagonal components, which vary over the domain of associated parameters defining the D_1-D_5-P-KK nonextremal nonlarge charge configurations. In short, we can easily substantiate, for the distinct $x_i := (n_i, m_i) \mid i \in \{1, 2, 3, 4\}$ describing eight (anti)charge string theory black holes, that the relative

pair correlation functions have distinct types of relative correlation functions. Apart from the zeros, infinities, and similar factorizations, we see that the nontrivial relative correlation functions satisfy the following scaling relations:

$$\frac{g_{x_i x_i}}{g_{x_j x_j}} = \left(\frac{x_j}{x_i}\right)^{3/2} \frac{\sqrt{n_j} + \sqrt{m_j}}{\sqrt{n_i} + \sqrt{m_i}},$$

$$\frac{g_{x_i x_j}}{g_{x_k x_l}} = \left(\frac{x_i x_j}{x_k x_l}\right)^{-1/2} \frac{\prod_{i \neq p \neq j} \left(\sqrt{n_p} + \sqrt{m_p}\right)}{\prod_{k \neq q \neq l} \left(\sqrt{n_q} + \sqrt{m_q}\right)}, \tag{61}$$

$$\frac{g_{x_i x_i}}{g_{x_i x_k}} = -\sqrt{\left(\frac{x_k}{x_i^2}\right)} \frac{\prod_{p \neq i} \left(\sqrt{n_p} + \sqrt{m_p}\right)}{\prod_{i \neq q \neq k} \left(\sqrt{n_q} + \sqrt{m_q}\right)}.$$

As noticed in [12, 14], it is not difficult to analyze the statistical stability properties of the eight charged anticharged nonextremal black holes; namely, we can compute the principle minors associated with the state-space metric tensor and thereby argue that all the principle minors must be positive definite, in order to have a globally stable configuration. In the present case, it turns out that the above black hole is stable only when some of the charges and/or anticharges are held fixed or take specific values such that $p_i > 0$ for all the dimensions of the state-space manifold. From the definition of the Hessian matrix of the associated entropy concerning the most general nonextremal nonlarge charged black holes, we observe that some of the principle minors p_i are indeed nonpositive. In fact, we discover a uniform local stability criteria on the three-dimensional hypersurfaces, two-dimensional surface, and the one-dimensional line of the underlying state-space manifold. In order to simplify the factors of the higher principle, we may hereby collect the powers of each factor $(\sqrt{n_i} + \sqrt{m_i})$ appearing in the expression of the entropy. With this notation, Appendix D provides the corresponding principle minors for the most general nonextremal nonlarge charged anticharged black hole in string theory involving finitely many nontrivially circularly fibered Kaluza-Klein monopoles.

Notice that the heat capacities, as the diagonal components g_{ii}, surface minor p_2, hypersurface minors p_3, p_5, p_6, and p_7, and the determinant of the state-space metric tensor, as the highest principle minor p_8 are examined as the functions of the number of branes n and antibranes m. Thus, they describe the nature of the statistical fluctuations in the vacuum configuration. The corresponding scalar curvature is offered for an equal number of branes and antibranes ($n = m$), which describes the nature of the long range statistical fluctuations. As per the above evaluation, we have obtained the exact expressions for the components of the metric tensor, principle minors, determinant of the metric tensor, and the underlying scalar curvature of the fluctuating statistical configuration of the eight parameter black holes in string theory. Qualitatively, the local and the global correlation properties of the limiting vacuum configuration can be realized under the statistical fluctuations. The first seven principle minors describe the local stability properties, and the last minor describes the global ensemble stability.

The scalar curvature describes the corresponding phase space stability of the eight parameter black hole configuration. In general, there exists an akin higher degree polynomial equation on which the Ricci scalar curvature becomes null, and exactly on these points the state-space configuration of the underlying nonlarge charge nonextremal eight charge black hole system corresponds to a noninteracting statistical system. In this case, the corresponding state-space manifold (M_8, g) becomes free from the statistical interaction with a vanishing state-space scalar curvature. As in case of the six charge configuration, we find interestingly that there exists an attractive configuration for the equal number of branes $n := k$ and antibranes $m := k$. In the limit of a large k, the corresponding system possesses a small negative value of the state-space scalar curvature:

$$R(k) = -\frac{21}{32}\frac{1}{\pi k^2}. \tag{62}$$

Interestingly, it turns out that the system becomes noninteracting in the limit of $k \to \infty$. For the case of the $n = k = m$, we observe that the corresponding principle minors reduce to the following constant values:

$$\{p_i\}_{i=1}^8 = \{4\pi, 16\pi^2, 32\pi^3, 0, -2048\pi^5, -16384\pi^6, \\ -163840\pi^7, -1048576\pi^8\}. \tag{63}$$

In this case, we find that the limiting underlying statistical system remains stable when at most three of the parameters, namely, $\{n_i = k = m_i\}$, are allowed to fluctuate. Herewith, we find for the case of $n := k$ and $m := k$ that the state-space manifold of the eight parameter brane and antibrane configuration is free from critical phenomena, except for the roots of the determinant. Thus, the regular state-space scalar curvature is comprehensively universal for the nonlarge charge nonextremal black brane configurations in string theory. In fact, the above perception turns out to be justified from the typical state-space geometry, namely, the definition of the metric tensor as the negative Hessian matrix of the duality invariant expression of the black brane entropy. In this case, we may nevertheless easily observe, for a given entropy S_0, that the constant entropy hypersurface is given by the following curve:

$$\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_2} + \sqrt{m_2}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right)\left(\sqrt{n_4} + \sqrt{m_4}\right) = c, \tag{64}$$

where c is a real constant taking the precise value of $S_0/2\pi$. Under the vacuum fluctuations, the present analysis indicates that the entropy of the eight parameter black brane solution defines a nondegenerate embedding in the viewpoints of intrinsic state-space geometry. The above state-space computations determine an intricate set of statistical properties, namely, pair correlation functions and correlation volume, which reveal the possible nature of the associated parameters prescribing an ensemble of microstates of the dual conformal field theory living on the boundary of the black brane solution. For any black brane configuration, the above computation hereby shows that we can exhibit

the state-space geometric acquisitions with an appropriate comprehension of the required parameters, for example, the charges and anticharges $\{n_i, m_i\}$, which define the coordinate charts. From the consideration of the state-space geometry, we have analyzed state-space pair correlation functions and the notion of stability of the most general nonextremal black hole in string theory. From the perspective of the intrinsic Riemannian geometry, we find that the stability of these black branes has been divulged from the positivity of principle minors of the space-state metric tensor.

Herewith, we have explicitly extended the state-space analysis for the four charge and four anticharge nonextremal black branes in string theory. The present consideration of the eight parameter black brane configurations, where the underlying leading order statistical entropy is written as a function of the charges $\{n_i\}$ and anticharges $\{m_i\}$, describes the stability properties under the Gaussian fluctuations. The present consideration includes all the special cases of the extremal and near-extremal configurations with a fewer number of charges and anticharges. In this case, we obtain the standard pattern of the underlying state-space geometry and constant entropy curve as that of the lower parameter nonextremal black holes. The local coordinate of the state-space manifold involves four charges and four anticharges of the underlying nonextremal black holes. In fact, the conclusion to be drawn remains the same, as the underlying state-space geometry remains well-defined as an intrinsic Riemannian manifold $N := M_8 \setminus \widetilde{B}$, where $\widetilde{B}$ is the set of roots of the determinant of the metric tensor. In particular, the state-space configuration of eight parameter black brane solutions remains nondegenerate for various domains of nonzero brane antibrane charges, except for the values, when the brane charges $\{n_i\}$ and antibrane charges $\{m_i\}$ belong to the set

$$\widetilde{B} := \{(n_1, n_2, n_3, n_4, m_1, m_2, m_3, m_4) \mid \\ (n_i, m_i) = (0, 0), (\infty, \infty)\}, \tag{65}$$

for a given brane-antibrane pair, $i \in \{1, 2, 3, 4\}$. Our analysis indicates that the leading order statistical behavior of the black brane configurations in string theory remains intact under the inclusion of the Kaluza-Klein monopoles. In short, we have considered the eight charged anticharged string theory black brane configuration and analyzed the state-space pair correlation functions, relative scaling relations, stability conditions, and the corresponding global properties. Given a general nonextremal black brane configuration, we have exposed (i) for what conditions the considered black hole configuration is stable, (ii) how its state-space correlations scale in terms of the numbers of branes and antibranes.

7. Conclusion and Outlook

The Ruppenier geometry of two charge leading order extremal black holes remains flat or ill-defined. Thus, the statistical systems are, respectively, noninteracting or require higher derivative corrections. However, an addition of the third brane charge and other brane and antibrane charges

indicates an interacting statistical system. The statistical fluctuations in the canonical ensemble lead to an interacting statistical system, as the scalar curvature of the state-space takes a nonzero value. We have explored the state-space geometric description of the charged extremal and associated charged anticharged nonextremal black holes in string theory.

Our analysis illustrates that the stability properties of the specific state-space hypersurface may exactly be exploited in general. The definite behavior of the state-space properties, as accounted in the specific cases, suggests that the underlying hypersurfaces of the state-space configuration include the intriguing mathematical feature. Namely, we find well-defined stability properties for the generic extremal and nonextremal black brane configurations, except for some specific values of the charges and anticharges. With and without the large charge limit, we have provided explicit forms of the higher principle minors of the state-space metric tensor for various charged, anticharged, extremal and nonextremal black holes in string theory. In this concern, the state-space configurations of the string theory black holes are generically well-defined and indicate an interacting statistical basis. Interestingly, we discover the state-space geometric nature of all possible general black brane configurations. From the very definition of the intrinsic metric tensor, the present analysis offers a definite stability character of string theory vacua.

Significantly, we notice that the related principle minors and the invariant state-space scalar curvature classify the underlying statistical fluctuations. The scalar curvature of a class of extremal black holes and the corresponding nonextremal black branes is everywhere regular with and without the stringy α'-corrections. A nonzero value of the state-space scalar curvature indicates an interacting underlying statistical system. We find that the antibrane corrections modify the state-space curvature, but do not induce phase transitions. In the limit of an extremal black hole, we construct the intrinsic geometric realization of a possible thermodynamic description at the zero temperature.

Importantly, the notion of the state-space of the considered black hole follows from the corresponding Wald and Cardy entropies. The microscopic and macroscopic entropies match in the large charge limit. From the perspective of statistical fluctuations, we anticipate the intrinsic geometric realization of two point local correlation functions and the corresponding global correlation length of the underlying conformal field theory configurations. In relation to the gauge-gravity correspondence and extremal black holes, our analysis describes state-space geometric properties of the corresponding boundary gauge theory.

General Remarks. For distinct $\{i, j\}$, the state-space pair correlations of an extremal configurations scale as

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{X_j}{X_i}\right)^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{X_i}{X_j}. \qquad (66)$$

In general, the black brane configurations in string theory can be categorized as per their state-space invariants. The underlying subconfigurations turn out to be well-defined over possible domains, whenever there exists a respective set of nonzero state-space principle minors. The underlying full configuration turns out to be everywhere well-defined, whenever there exists a nonzero state-space determinant. The underlying configuration corresponds to an interacting statistical system, whenever there exists a nonzero state-space scalar curvature. The intrinsic state-space manifold of extremal/nonextremal and supersymmetric/nonsupersymmetric string theory black holes may intrinsically be described by an embedding:

$$\left(M_{(n)}, g\right) \hookrightarrow \left(M_{(n+1)}, \tilde{g}\right). \qquad (67)$$

The extremal state-space configuration may be examined as a restriction to the full counting entropy with an intrinsic state-space metric tensor $g \mapsto \tilde{g}|_{r_+=r_-}$. Furthermore, the state-space configurations of the supersymmetric black holes may be examined as the BPS restriction of the full space of the counting entropy with an understanding that the intrinsic state-space metric tensor is defined as $g := \tilde{g}|_{M=M_0}$. From the perspective of string theory, the restrictions $r_+ = r_-$ and $M = M_0(P_i, Q_i)$ should be understood as the fact that it has been applied to an assigned entropy of the nonextremal/nonsupersymmetric (or nearly extremal/nearly supersymmetric) black brane configuration. This allows one to compute the fluctuations in ADM mass of the black hole. In the viewpoint of the present research on the state-space geometry, it is worth mentioning that the dependence of the mass to the entropy of a nonextremal black hole comes from the contribution of the antibranes, see, for instance, Section 4.5, and so we may examine the corresponding Weinhold chemical geometry, as mentioned in Section 4.3.

Future Directions and Open Issues. The state-space instabilities and their relation to the dual microscopic conformal field theories could open up a number of new realizations. The state-space perspective includes the following issues.

(i) Multicenter Gibbons-Hawking solutions [69, 70] with generalized base space manifolds having a mixing of positive and negative residues, see [71, 72] for a perspective development of state-space geometry by invoking the role of foaming of black holes and plumbing the Abyss for the microstates counting of black rings.

(ii) Dual conformal field theories and string duality symmetries, see [61] for a quantum mechanical perspective of superconformal black holes and [73, 74] for the origin of gravitational thermodynamics and the role of giant gravitons in conformal field theory.

(iii) Stabilization against local and/or global perturbations, see [75–80] for black brane dynamics, stability, and critical phenomena. Thus, the consideration of state-space geometry is well-suited for examining the domain of instability. This includes Gregory-Laflamme (GL) modes, chemical potential

fluctuations, electric-magnetic charges and dipole charges, rotational fluctuations, and the thermodynamic temperature fluctuations for the near-extremal and nonextremal black brane solutions. We leave this perspective of the state-space geometry open for a future research.

In general, various D dimensional black brane configurations, see, for instance, [75–80] for black rings in $D > 5$ spacetime dimensions with $S^1 \times S^{D-3}$ horizon topology, and the higher horizon topologies, for example, $S^1 \times S^1 \times S^2$, $S^3 \times S^3$, and so forth offer a platform to extend the consideration of the state-space geometry.

On the other hand, the bubbling black brane solutions, namely, Lin, Lunin, and Maldacena (LLM) geometries [81], are interesting from the perspective of Mathur's Fuzzball conjecture(s). From the perspective of the generalized hyper-Kähler manifolds, Mathur's conjecture [82–85] reduces to classifying and counting asymptotically flat four-dimensional hyper Kähler manifolds [71] which have moduli regions of uniform signature $(+, +, +, +)$ and $(-, -, -, -)$.

Finally, the new physics at the length of the Planck scale anticipates an analysis of the state-space configurations. In particular, it materializes that the state-space geometry may be explored with the parameters of the foam geometries [71], and the corresponding empty space virtual black holes, see [81] for the notion of bubbling AdS space and 1/2 BPS geometries. In such cases, the local and global statistical correlations, among the parameters of the microstates of the black hole conformal field theory [60, 61], would involve the foams of two spheres. From the perspective of the string theory, the present exploration thus opens up an avenue for learning new insights into the promising structures of the black brane space-time configurations at very small scales.

Appendices

In these appendices, we provide explicit forms of the state-space correlation arising from the metric tensor of the charged (non)extremal (non)large black holes in string theory. In fact, our analysis illustrates that the stability properties of the specific state-space hypersurface may exactly be exploited in general. The definite behavior of state-space properties, as accounted in the concerned main sections, suggests that the various intriguing hypersurfaces of the state-space configuration include the nice feature that they do have definite stability properties, except for some specific values of the charges and anticharges.

As mentioned in the main sections, these configurations are generically well-defined and indicate an interacting statistical basis. Herewith, we discover that the state-space geometry of the general black brane configurations in string theory indicates the possible nature of the underlying statistical fluctuations. Significantly, we notice from the very definition of the intrinsic metric tensor that the related statistical pair correlation functions and relative statistical correlation functions take the following exact expressions.

A. Correlations for Three Charge Configurations

Following the notion of the fluctuations, we see from the Hessian of the entropy equation (30) that the components of state-space metric tensor are

$$
\begin{aligned}
&g_{n_1 n_1} = \frac{\pi}{2n_1}\sqrt{\frac{n_5 n_p}{n_1}}, \qquad g_{n_1 n_5} = -\frac{\pi}{2}\sqrt{\frac{n_p}{n_1 n_5}}, \\
&g_{n_1 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 n_p}}, \qquad g_{n_5 n_5} = \frac{\pi}{2n_5}\sqrt{\frac{n_1 n_p}{n_5}}, \qquad \text{(A.1)} \\
&g_{n_5 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 n_p}}, \qquad g_{n_p n_p} = \frac{\pi}{2n_p}\sqrt{\frac{n_1 n_5}{n_p}}.
\end{aligned}
$$

For distinct $i, j \in \{1, 5\}$ and p, the list of relative correlation functions follows the scaling relations:

$$
\begin{aligned}
&\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{pp}} = \left(\frac{n_p}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{ij}} = -\left(\frac{n_j}{n_i}\right), \\
&\frac{g_{ii}}{g_{ip}} = -\left(\frac{n_p}{n_i}\right), \qquad \frac{g_{ip}}{g_{jp}} = \left(\frac{n_j}{n_i}\right), \qquad \frac{g_{ii}}{g_{jp}} = -\left(\frac{n_j n_p}{n_i^2}\right), \\
&\frac{g_{ip}}{g_{pp}} = -\left(\frac{n_p}{n_i}\right), \qquad \frac{g_{ij}}{g_{ip}} = \left(\frac{n_p}{n_j}\right), \qquad \frac{g_{ij}}{g_{pp}} = -\left(\frac{n_p^2}{n_i n_j}\right).
\end{aligned}
$$

(A.2)

B. Correlations for Four Charge Configurations

For the given entropy as in (36), we find that the components of the metric tensor are

$$
\begin{aligned}
&g_{n_1 n_1} = \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right), \\
&g_{n_1 n_5} = -\frac{\pi}{2\sqrt{n_1 n_5}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right), \\
&g_{n_1 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 n_p}}, \qquad g_{n_1 \overline{n_p}} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 \overline{n_p}}}, \\
&g_{n_5 n_5} = \frac{\pi}{2}\sqrt{\frac{n_1}{n_5^3}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right), \qquad g_{n_5 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 n_p}}, \\
&g_{n_5 \overline{n_p}} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 \overline{n_p}}}, \qquad g_{n_p n_p} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{n_p^3}}, \\
&g_{n_p \overline{n_p}} = 0, \qquad g_{\overline{n_p} \overline{n_p}} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{\overline{n_p}^3}}.
\end{aligned}
$$

(B.1)

For distinct $i, j \in \{1, 5\}$, and $k, l \in \{p, \overline{p}\}$ describing four charge nonextremal D_1–D_5-P-$\overline{P}$ black holes, the statistical pair correlations consist of the following scaling relations:

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{kk}} = \frac{n_k}{n_i^2}\sqrt{n_k}\left(\sqrt{n_p} + \sqrt{n_{\overline{p}}}\right),$$

$$\frac{g_{ii}}{g_{ij}} = -\frac{n_j}{n_i},$$

$$\frac{g_{ii}}{g_{ik}} = -\frac{\sqrt{n_k}}{n_i}\left(\sqrt{n_p} + \sqrt{n_{\overline{p}}}\right),$$

$$\frac{g_{ik}}{g_{jk}} = \frac{n_j}{n_i}, \qquad \frac{g_{ii}}{g_{jk}} = -\frac{n_j}{n_i^2}\sqrt{n_k}\left(\sqrt{n_p} + \sqrt{n_{\overline{p}}}\right), \tag{B.2}$$

$$\frac{g_{ik}}{g_{kk}} = -\frac{n_k}{n_i}, \qquad \frac{g_{ij}}{g_{ik}} = \frac{\sqrt{n_k}}{n_j}\left(\sqrt{n_p} + \sqrt{n_{\overline{p}}}\right),$$

$$\frac{g_{ij}}{g_{kk}} = -\frac{n_k}{n_i n_j}\sqrt{n_k}\left(\sqrt{n_p} + \sqrt{n_{\overline{p}}}\right).$$

Notice that the list of other mixed relative correlation functions concerning the nonextremal D_1–D_5-P-$\overline{P}$ black holes read as

$$\frac{g_{ik}}{g_{il}} = \sqrt{\frac{n_l}{n_k}}, \qquad \frac{g_{ik}}{g_{jl}} = \frac{n_j}{n_i}\sqrt{\frac{n_l}{n_k}}, \qquad \frac{g_{kl}}{g_{ij}} = 0,$$

$$\frac{g_{kl}}{g_{ii}} = 0, \qquad \frac{g_{kk}}{g_{ll}} = \left(\frac{n_l}{n_k}\right)^{3/2}, \qquad \frac{g_{kl}}{g_{kk}} = 0. \tag{B.3}$$

C. Correlations for Six Charge Configurations

Over generic nonlarge charge domains, we find from the entropy equation (46) that the components of the covariant state-space metric tensor are given by the following expressions:

$$g_{n_1 n_1} = \frac{\pi}{2n_1^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right), \qquad g_{n_1 m_1} = 0,$$

$$g_{n_1 n_2} = -\frac{\pi}{2\sqrt{n_1 n_2}}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{n_1 m_2} = -\frac{\pi}{2\sqrt{n_1 m_2}}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{n_1 n_3} = -\frac{\pi}{2\sqrt{n_1 n_3}}\left(\sqrt{n_2} + \sqrt{m_2}\right),$$

$$g_{n_1 m_3} = -\frac{\pi}{2\sqrt{n_1 m_3}}\left(\sqrt{n_2} + \sqrt{m_2}\right),$$

$$g_{m_1 m_1} = \frac{\pi}{2m_1^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{m_1 n_2} = -\frac{\pi}{2\sqrt{m_1 n_2}}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{m_1 m_2} = -\frac{\pi}{2\sqrt{m_1 m_2}}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{m_1 n_3} = -\frac{\pi}{2\sqrt{m_1 n_3}}\left(\sqrt{n_2} + \sqrt{m_2}\right),$$

$$g_{m_1 m_3} = -\frac{\pi}{2\sqrt{m_1 m_3}}\left(\sqrt{n_2} + \sqrt{m_2}\right),$$

$$g_{n_2 n_2} = \frac{\pi}{2n_2^{3/2}}\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{n_2 m_2} = 0, \qquad g_{n_2 n_3} = -\frac{\pi}{2\sqrt{n_2 n_3}}\left(\sqrt{n_1} + \sqrt{m_1}\right)$$

$$g_{n_2 m_3} = -\frac{\pi}{2\sqrt{n_2 m_3}}\left(\sqrt{n_1} + \sqrt{m_1}\right),$$

$$g_{m_2 m_2} = \frac{\pi}{2m_2^{3/2}}\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$g_{m_2 n_3} = -\frac{\pi}{2\sqrt{m_2 n_3}}\left(\sqrt{n_1} + \sqrt{m_1}\right),$$

$$g_{m_2 m_3} = -\frac{\pi}{2\sqrt{m_2 m_3}}\left(\sqrt{n_1} + \sqrt{m_1}\right),$$

$$g_{n_3 n_3} = \frac{\pi}{2n_3^{3/2}}\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_2} + \sqrt{m_2}\right), \qquad g_{n_3 m_3} = 0,$$

$$g_{m_3 m_3} = \frac{\pi}{2m_3^{3/2}}\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_2} + \sqrt{m_2}\right). \tag{C.1}$$

In this case, from the definition of the relative statistical correlation functions, for $i, j \in \{n_1, m_1\}$, and $k, l \in \{n_2, m_2\}$, the relative correlation functions satisfy the following scaling relations:

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{j}{i}\right)^{3/2}, \qquad \frac{g_{ii}}{g_{kk}} = \left(\frac{k}{i}\right)^{3/2}\left(\frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}}\right),$$

$$\frac{g_{ij}}{g_{ii}} = 0,$$

$$\frac{g_{ii}}{g_{ik}} = -\frac{\sqrt{k}}{i}\left(\sqrt{n_2} + \sqrt{m_2}\right), \qquad \frac{g_{ik}}{g_{jk}} = \sqrt{\frac{j}{i}},$$

$$\frac{g_{ii}}{g_{jk}} = -\frac{\sqrt{jk}}{i^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right),$$

$$\frac{g_{kk}}{g_{ik}} = -\frac{\sqrt{i}}{k}\left(\sqrt{n_2} + \sqrt{m_2}\right), \qquad \frac{g_{ij}}{g_{ik}} = 0, \qquad \frac{g_{ij}}{g_{kk}} = 0. \tag{C.2}$$

The other concerned relative correlation functions are

$$\frac{g_{ik}}{g_{il}} = \sqrt{\frac{l}{k}}, \qquad \frac{g_{ik}}{g_{jl}} = \sqrt{\frac{jl}{ik}}, \qquad \frac{g_{ij}}{g_{kl}} = \text{n.d.},$$

$$\frac{g_{kl}}{g_{ii}} = 0, \qquad \frac{g_{kk}}{g_{ll}} = \left(\frac{l}{k}\right)^{3/2}, \qquad \frac{g_{kl}}{g_{kk}} = 0. \tag{C.3}$$

For $k, l \in \{n_2, m_2\}$, and $m, n \in \{n_3, m_3\}$, we have

$$\frac{g_{kk}}{g_{mm}} = \left(\frac{m}{k}\right)^{3/2}\left(\frac{\sqrt{n_3} + \sqrt{m_3}}{\sqrt{n_2} + \sqrt{m_2}}\right),$$

$$\frac{g_{kl}}{g_{kk}} = 0, \qquad \frac{g_{kk}}{g_{km}} = -\frac{\sqrt{m}}{k}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$\frac{g_{km}}{g_{lm}} = \sqrt{\frac{l}{k}}, \qquad \frac{g_{kk}}{g_{lm}} = -\frac{\sqrt{lm}}{k^{3/2}}\left(\sqrt{n_3} + \sqrt{m_3}\right), \tag{C.4}$$

$$\frac{g_{mm}}{g_{km}} = -\frac{\sqrt{k}}{m}\left(\sqrt{n_2} + \sqrt{m_2}\right),$$

$$\frac{g_{kl}}{g_{km}} = 0, \qquad \frac{g_{kl}}{g_{mm}} = 0.$$

The other concerned relative correlation functions are

$$\frac{g_{km}}{g_{kn}} = \sqrt{\frac{n}{m}}, \qquad \frac{g_{km}}{g_{ln}} = \sqrt{\frac{ln}{km}}, \qquad \frac{g_{kl}}{g_{mn}} = \text{n.d.},$$

$$\frac{g_{mn}}{g_{kk}} = 0, \qquad \frac{g_{mm}}{g_{nn}} = \left(\frac{n}{m}\right)^{3/2}, \qquad \frac{g_{mn}}{g_{mm}} = 0. \tag{C.5}$$

However, for $i, j \in \{n_1, m_1\}$, and $m, n \in \{n_3, m_3\}$, we have

$$\frac{g_{ii}}{g_{mm}} = \left(\frac{m}{i}\right)^{3/2}\left(\frac{\sqrt{n_3} + \sqrt{m_3}}{\sqrt{n_1} + \sqrt{m_1}}\right), \qquad \frac{g_{ij}}{g_{ii}} = 0,$$

$$\frac{g_{ii}}{g_{im}} = -\frac{\sqrt{m}}{i}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$\frac{g_{im}}{g_{jm}} = \sqrt{\frac{j}{i}}, \qquad \frac{g_{ii}}{g_{jm}} = -\frac{\sqrt{jm}}{i^{3/2}}\left(\sqrt{n_3} + \sqrt{m_3}\right),$$

$$\frac{g_{mm}}{g_{im}} = -\frac{\sqrt{i}}{m}\left(\sqrt{n_1} + \sqrt{m_1}\right), \tag{C.6}$$

$$\frac{g_{ij}}{g_{im}} = 0, \qquad \frac{g_{ij}}{g_{mm}} = 0, \qquad \frac{g_{im}}{g_{in}} = \sqrt{\frac{n}{m}},$$

$$\frac{g_{im}}{g_{jn}} = \sqrt{\frac{jn}{im}}, \qquad \frac{g_{ij}}{g_{mn}} = \text{n.d.},$$

$$\frac{g_{mn}}{g_{ii}} = 0, \qquad \frac{g_{mn}}{g_{mm}} = 0.$$

D. Principle Minors for Eight Charge Configurations

For the entropy equation (58) of the most general nonextremal nonlarge charged anticharged black hole in string involving finitely many nontrivially circularly fibered Kaluza-Klein monopoles, the principle minors take the following expressions:

$$p_1 = \frac{\pi}{2n_1^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right)\left(\sqrt{n_4} + \sqrt{m_4}\right),$$

$$p_2 = \frac{\pi^2}{4(n_1 m_1)^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)^2\left(\sqrt{n_3} + \sqrt{m_3}\right)^2$$
$$\times \left(\sqrt{n_4} + \sqrt{m_4}\right)^2,$$

$$p_3 = \frac{\pi^3}{8(n_1 m_1 n_2)^{3/2}}\left(\sqrt{n_3} + \sqrt{m_3}\right)^3\left(\sqrt{n_4} + \sqrt{m_4}\right)^3$$
$$\times \left(\sqrt{n_2} + \sqrt{m_2}\right)\sqrt{m_2}\left(\sqrt{n_1} + \sqrt{m_1}\right),$$

$$p_4 = 0,$$

$$p_5 = -\frac{\pi^5}{8(n_1 n_2 m_2 m_1)^{3/2}n_3}\left(\sqrt{n_2} + \sqrt{m_2}\right)^3$$
$$\times \left(\sqrt{n_3} + \sqrt{m_3}\right)^3\left(\sqrt{n_4} + \sqrt{m_4}\right)^5\left(\sqrt{n_1} + \sqrt{m_1}\right)^3,$$

$$p_6 = -\frac{\pi^6}{16(n_1 n_2 m_1 m_2 n_3 m_3)^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)^4$$
$$\times \left(\sqrt{n_3} + \sqrt{m_3}\right)^4\left(\sqrt{n_4} + \sqrt{m_4}\right)^6\left(\sqrt{n_1} + \sqrt{m_1}\right)^4,$$

$$p_7 = -\frac{\pi^7}{32(n_1 m_1 n_2 m_2 n_3 m_3 n_4)^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)^5$$
$$\times \left(\sqrt{n_3} + \sqrt{m_3}\right)^5\left(\sqrt{n_4} + \sqrt{m_4}\right)^5\left(4\sqrt{n_4} + \sqrt{m_4}\right)$$
$$\times \left(\sqrt{n_1} + \sqrt{m_1}\right)^5,$$

$$p_8 = -\frac{\pi^8}{16\left(\prod_{i=1}^{4} n_i m_i\right)^{3/2}}\left(\sqrt{n_2} + \sqrt{m_2}\right)^6$$
$$\times \left(\sqrt{n_3} + \sqrt{m_3}\right)^6\left(\sqrt{n_4} + \sqrt{m_4}\right)^6\left(\sqrt{n_1} + \sqrt{m_1}\right)^6. \tag{D.1}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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