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Research Article

Dirac Equation in the Presence of Hartmann and Ring-Shaped Oscillator Potentials

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The importance of the energy spectrum of bound states and their restrictions in quantum mechanics due to the different methods have been used for calculating and determining the limit of them. Comparison of Schrödinger-like equation obtained by Dirac equation with the nonrelativistic solvable models is the most efficient method. By this technique, the exact relativistic solutions of Dirac equation for Hartmann and Ring-Shaped Oscillator Potentials are accessible, when the scalar potential is equal to the vector potential. Using solvable nonrelativistic quantum mechanics systems as a basic model and considering the physical conditions provide the changes in the restrictions of relativistic parameters based on the nonrelativistic definitions of parameters.

1. Introduction

Since the advent of quantum mechanics, several methods have been developed in order to find the exact energy spectrum of bound states in stationary quantum systems. The knowledge of these spectrums is necessary for several applications in many fields of physics and theoretical chemistry [1–4]. Such encouraging results have arisen some studies on the potential within the frame work of common wave equations of both nonrelativistic and relativistic wave equations, that is, including Schrödinger, Duffin-Kemmer-Petiau (DKP), Klein-Gordon, or Dirac equations [5–10]. There are some noncentral separable potentials in spherical coordinates which are of considerable interest and are practical in the branches of science such as chemistry and nuclear physics. Hartmann potential introduced by Hartmann is one of the noncentral potentials, which can be realized by adding a potential proportional to Coulomb potential [11–16]. This potential was suggested to describe the energy spectrum of Ring-Shaped Potential obtained by replacing the Coulomb part of Hartmann potential with a Harmonic Oscillator term and that is called a Ring-Shaped Oscillator Potential, which is investigated to find discrete spectrum and integrals of motions [17–22]. The relativistic linear interaction, which is called the relativistic oscillator due to the similarity with the nonrelativistic harmonic oscillator, has been subject of many successful theoretical studies. Such a space has interesting property and algebra; for example, there are some articles in which a free particle has been studied in different situations; Dirac oscillator system that is initiated by a relativistic fermion is subjected to linear vector potential [23–25]. In this article, for solving Dirac equation with Hartmann and Ring-Shaped Oscillator Potentials in three dimensions, equality of scalar and vector potentials can constitute a couple of differential equations for the spinor components [26, 27]. One of them is the second-order differential equation for the upper spinor and the lower spinor can be gotten from the first-order differential equation based on the upper spinor. Since Hartmann and Ring-Shaped Oscillator Potentials contained two radial and angular parts, the second-order differential equation is considered in the spherical polar coordinates. With separation of the second-order differential equation, there are two Schrödinger-like equations in r and θ coordinates. Moreover, the normalized solution of the polar angular part is considered as an exponential function based on φ coordinate and separating constant, because there is not any part of φ coordinate in the potential function. There exists one-dimensional solvable Schrödinger equation in the nonrelativistic quantum mechanics for the determined potential which can be expanded to the Schrödinger-like equation and is derived from Dirac equation [28, 29]. In the radial part of differential equation, the relativistic energy spectrum can be gotten by comparison with the nonrelativistic solvable Schrödinger equation. In this comparison, the relativistic energy spectrum is obtained based on the nonrelativistic energy spectrum and the wave function of the nonrelativistic space will be considered for calculation of the relations between nonrelativistic and relativistic parameters. The mentioned method can be used on the angular part of differential equation. The relations of parameters between the two models are confirmed to the changes in the restriction of parameters. The new restriction of parameters and separating constants ensure the physical conditions. The paper is organized as follows: assuming $V(\vec{r}) = S(\vec{r})$, the couple of differential equations can be obtained for the spinor components and the second-order differential equation can be separated to the three coordinates in Section 2. The radial part of Dirac equation and the relativistic energy spectrum that is associated with the radial part have been investigated in Section 3. The angular part of Dirac equation for the potential that is related to θ coordinate according to different function of θ and the relativistic parameters have been paid attention in Section 4. Finally, in Section 5, the brief of method has been presented.

2. The General Form of Hartmann and Ring-Shaped Oscillator Potentials in Dirac Equation

The generalized Hartmann and Ring-Shaped Oscillator Potentials are defined as follows [27]:

$$V(r,\theta) = V(r) + \frac{f(\theta)}{2r^2}. \tag{1}$$

The radial part of potential can be considered as Coulomb and harmonic oscillator potentials [17, 18]:

$$\begin{aligned} V(r) &= -\frac{1}{2}\left(\frac{V_0\lambda}{r}\right), \\ V(r) &= Kr^2, \end{aligned} \tag{2}$$

where V_0, λ, and K are free parameters with respect to the relevant potentials. Different types of functions are assumed for the angular part of potential so that the exact solvable models can be provided from Dirac equation. In (1), θ and r are polar angular and radial in spherical coordinates of hydrogen atom.

Time-independent Dirac equation for arbitrary scalar and vector potentials is given by differential equation:

$$\begin{aligned} &\left[c\vec{\alpha}.\vec{P} + \beta\left(Mc^2 + \vec{S}\left(\vec{r}\right)\right)\right]\psi\left(\vec{r}\right) \\ &\quad = \left[\varepsilon - V\left(\vec{r}\right)\right]\psi\left(\vec{r}\right). \end{aligned} \tag{3}$$

The following parameters are defined in (3):

$$\begin{aligned} \vec{P} &= -i\hbar\vec{\nabla}, \\ \alpha &\equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \\ \beta &\equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \end{aligned} \tag{4}$$

where $\vec{\sigma}$ and I are vector Pauli spin matrix and the identity matrix, respectively. Use the Pauli-Dirac representation as

$$\psi\left(\vec{r}\right) = \begin{pmatrix} \varphi\left(\vec{r}\right) \\ \chi\left(\vec{r}\right) \end{pmatrix}, \tag{5}$$

where $\varphi(r)$ and $\psi(r)$ are spinor components. The following set of coupled equations for the spinor components can be gotten:

$$c\vec{\sigma}.\vec{P}\chi\left(\vec{r}\right) = \left[\varepsilon - V\left(\vec{r}\right) - Mc^2 - S\left(\vec{r}\right)\right]\varphi\left(\vec{r}\right), \tag{6}$$

$$c\vec{\sigma}.\vec{P}\varphi\left(\vec{r}\right) = \left[\varepsilon - V\left(\vec{r}\right) + Mc^2 + S\left(\vec{r}\right)\right]\chi\left(\vec{r}\right). \tag{7}$$

Assume that $S(\vec{r}) = V(\vec{r})$ and $S(\vec{r}) = -V(\vec{r})$ due to combining (6) and (7) and provide the situations for obtaining the second-order differential equations according to one of the components so that another component can be gotten by using the first differential equation based on the determined component. Since in the case where $S(\vec{r}) = -V(\vec{r})$ the treatment of (6) and (7) is quite equivalent to the case where $S(\vec{r}) = V(\vec{r})$, the case where $S(\vec{r}) = V(\vec{r})$ is considered and then the results of that case will be expanded to the second case [26, 27].

The state $S(\vec{r}) = V(\vec{r})$ allows making two differential equations for each component:

$$\chi\left(\vec{r}\right) = \left[\frac{c\vec{\sigma}.\vec{P}}{\varepsilon + Mc^2}\right]\varphi\left(\vec{r}\right), \tag{8}$$

$$\begin{aligned} &\left[c^2\vec{P}^2 + 2\left(\varepsilon + Mc^2\right)V\left(\vec{r}\right)\right]\varphi\left(\vec{r}\right) \\ &\quad = \left[\varepsilon^2 - M^2c^4\right]\varphi\left(\vec{r}\right). \end{aligned} \tag{9}$$

Schrödinger-like equation is obtained for the component $\varphi(\vec{r})$ by considering the definitions of $\vec{P}$ and $V(\vec{r})$ in (9):

$$\begin{aligned} &\left[-\hbar^2c^2\vec{\nabla}^2 - \left(\varepsilon + Mc^2\right)\left(\frac{V_0\lambda}{r} - \frac{f(\theta)}{r^2}\right)\right]\varphi\left(\vec{r}\right) \\ &\quad = \left[\varepsilon^2 - M^2c^4\right]\varphi\left(\vec{r}\right). \end{aligned} \tag{10}$$

Assuming a solution as

$$\varphi\left(\vec{r}\right) = \frac{1}{r}u(r)\Theta(\theta)\Phi(\varphi), \tag{11}$$

(10) can be separated to three differential equations in the three dimensions φ, r, and θ:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2, \quad (12)$$

$$-\frac{d^2u(r)}{dr^2} + \left[\frac{\rho}{r^2} + 2\frac{\left(\varepsilon + Mc^2\right)}{\hbar^2c^2}V(r)\right]u(r) = \frac{\left(\varepsilon^2 - M^2c^4\right)}{\hbar^2c^2}u(r), \quad (13)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) - \left[\frac{m^2}{\sin^2\theta} + \left(\varepsilon + M^2c^4\right)f(\theta) - s\right]\Theta(\theta) = 0, \quad (14)$$

where m and ρ/r^2 are separation factors.

The normalized solution of (12) which satisfies the boundary conditions becomes

$$\Phi(\varphi) = -\frac{1}{\sqrt{2\pi}}e^{im\varphi}, \quad m = 0, \pm1, \pm2, \ldots. \quad (15)$$

3. The Radial Part Solutions of Dirac Equation

In this section, radial part of wave function (13) will be analyzed by corresponding to Generalized Laguerre differential equation. Coulomb and harmonic oscillator potentials are two potentials that are considered in (13), respectively. For these potentials, (13) can be converted to Generalized Laguerre differential equation with exact solution of Generalized Laguerre polynomials. In the first case, substituting the radial part of potential as Coulomb potential in (13) [26, 27],

$$\frac{d^2u(r)}{dr^2} + \left[\frac{\left(\varepsilon^2 - M^2c^4\right)}{\hbar^2c^2} - \frac{\rho}{r^2} + \left(\frac{\varepsilon + Mc^2}{\hbar^2c^2}\right)\frac{V_0\lambda}{r}\right]u(r) = 0, \quad (16)$$

and considering units system ($\hbar = 2m = 1$), (16) can be compared to the following nonrelativistic solvable model [28, 29]:

$$\frac{d^2u_{n,l}(r)}{dr^2} + (E - V(r))\,u_{n,l}(r) = 0. \quad (17)$$

Indeed, comparing radial Schrödinger-like equation to nonrelativistic Schrödinger equation according to Coulomb potential with exact solution based on Generalized Laguerre polynomials, the results of nonrelativistic equation can be expanded to relativistic models. Nonrelativistic model for Coulomb potential has the following form:

$$\frac{d^2u(r)}{dr^2} + \left[-\frac{e^4}{4(n+l+1)} - \frac{l(l+1)}{r^2} + \frac{e^2}{r}\right]u(r) = 0. \quad (18)$$

Therefore, relativistic parameters can be connected to nonrelativistic parameters as follows:

$$\rho = l(l+1), \quad (19)$$

$$\left(\frac{\varepsilon + Mc^2}{c^2}\right)V_0\lambda = e^2, \quad (20)$$

$$\frac{\varepsilon^2 - M^2c^4}{c^2} = -\frac{e^4}{4(n+l+1)^2}. \quad (21)$$

Since $e^2c^2/4(n+l+1)^2 > 0$, relation of parameters (20) causes the condition $|\varepsilon| < M^2c^2$. Relativistic energy can be calculated based on defined parameters in nonrelativistic solvable model by Combining above relations of parameters. Assuming that $\tau = V_0\lambda/2c(n+l+1)$, relativistic energy can be obtained as follows [26, 27]:

$$\varepsilon = Mc^2\frac{1-\tau^2}{1+\tau^2}. \quad (22)$$

In nonrelativistic model, the exact solution is considered for (17) as [28, 29]

$$u_{n,l}(r) = f(r)\,F(g(r)), \quad (23)$$

where F(g(r)) is a special function based on the internal function $g(r)$. Generalized Laguerre polynomials are orthogonal polynomials that are satisfied in (18). Therefore, that function can be expanded to Schrödinger-like equation (16) of radial part. Since $\alpha > -1$ in Generalized Laguerre polynomials $L_n^{\alpha}(g(r))$ and $\alpha = 2l+1$ in nonrelativistic model, condition of $l > -1$ is satisfied in Generalized Laguerre polynomials. Therefore, according to relation of $\alpha = 2l+1$ in relativistic model, $\rho < 0$ and $\rho > 0$ are considered for $-1 < l < 0$ and $l > 0$, respectively. In the last angular part section, it will be shown that relativistic energy is calculated based on nonrelativistic energy and term of $\rho + 1/4$. Since the sign of nonrelativistic energy term is cleared, determining term of $\rho + 1/4$ is very important because of the condition $|\varepsilon| < M^2c^2$. The term of $\rho + 1/4$ should be signed for defined different l parameter. According to parametric relation of $\rho = l(l+1)$, $\rho + 1/4$ will be positive for each l that is defined in the problem. It means that the sign of term $\rho = l(l+1)$ separated the limit of l parameter. The ρ relativistic parameter will be restricted by $\rho \geq 0$ for $l > 0$ and $-1/4 \leq \rho \leq 0$ for $-1 < l < 0$. Since there is term of $n+l+1$ in energy spectrum and for $n = l+1$ singularity happens in the wave function, in Generalized Laguerre polynomials related to differential equation (16), parameter n is transformed to $n-l-1$. Thus energy spectrum will be restricted and the problem of singularity will disappear. If the following nonnormalized wave function is associated with differential equation (18) for internal function $g(r) = ((e^2/(n+l+1))r)$ [28, 29]:

$$u_{n,l}(r) \propto g^{(l+1)}\exp\left(-\frac{g}{2}\right)L_n^{2l+1}(g(r)), \quad (24)$$

the radial wave function is considered to differential equation (16) as follows:

$$u_{n,l}^{(1)}(r) \propto (2kr)^{l+1}\exp(-kr)\,L_{n-l-1}^{2l+1}(2kr), \quad (25)$$

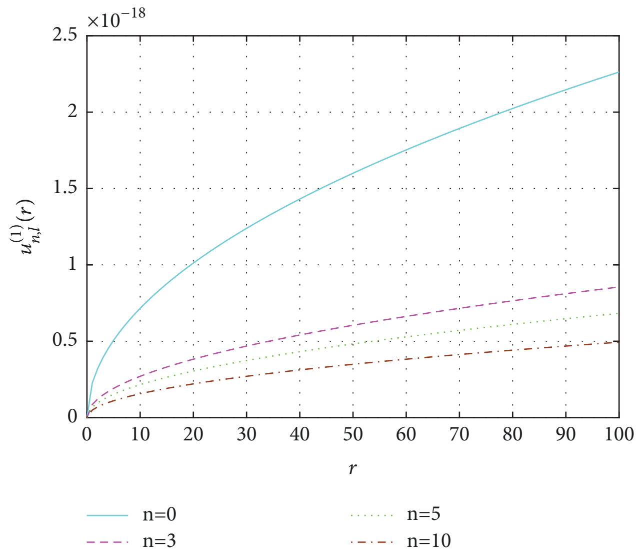

FIGURE 1: $u_{n,l}^{(1)}(r)$ versus r with $l = -0.5$.

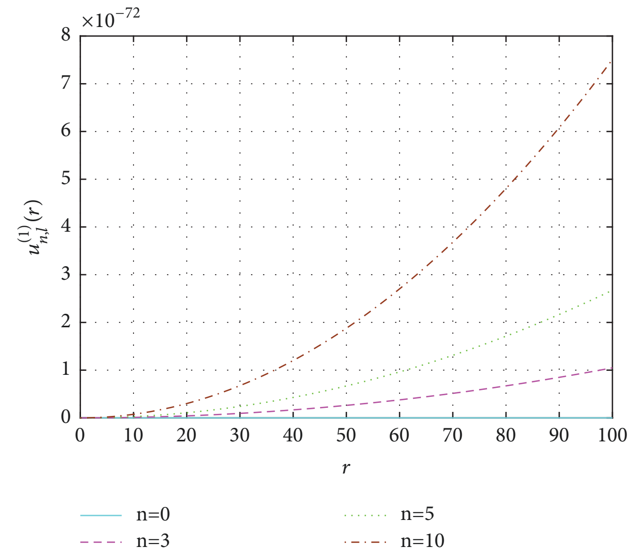

FIGURE 2: $u_{n,l}^{(1)}(r)$ versus r with $l = 1$.

where $k = e^2/2(n+l+1)$ and $0 < r < +\infty$. The wave function that is satisfied in Schrödinger-like equation must be physically acceptable. Physical wave functions are satisfied in the usual square-integrability condition as $\int_{x_1}^{x_2} |\Psi_n(x)|^2 dx < \infty$ for energy bound state to ensure Hermiticity of Hamiltonian in Hilbert space spanned by its eigenfunctions. Since this integral must be finite, the wave functions have a constant value or zero at the endpoints of definition internal of $\overrightarrow{V}$ potential. Therefore, solutions of Schrödinger-like equation should be checked at the endpoints of $[x_1, x_2]$ interval, therefore providing square-integrability condition and investigating physical situations in the wave functions. It is seen that wave function (25) is a square-integrability function at the endpoints of $[0, +\infty]$ interval, so that $u_{n,l}(r) \longrightarrow 0$ when $r \longrightarrow 0$ and $r \longrightarrow +\infty$ for the range $l > 0$ and $-1 < l < 0$. Thus it will be physically acceptable wave function in restriction of l parameter. $u_{n,l}^{(1)}(r)$ wave function in restriction of l parameter for $l = -0.5$ and $l = 1$ based on r is displaced in Figures 1 and 2.

In the second case, the following differential equation is obtained from (13) for harmonic oscillator as radial part of potential:

$$\frac{d^2u(r)}{dr^2} + \left[\frac{(\varepsilon^2 - M^2c^4)}{\hbar^2c^2} - \frac{\rho}{r^2} - \left(\frac{\varepsilon + Mc^2}{\hbar^2c^2}\right)(2kr^2)\right]u(r) = 0, \tag{26}$$

where $k > 0$. Nonrelativistic solvable model based on (17) which can be compared to (26) has the following form ($\hbar = 2m = 1$) [28, 29]:

$$\frac{d^2u(r)}{dr^2} + \left[2n\omega + \left(l + \frac{3}{2}\right)\omega - \frac{l(l+1)}{r^2} - \frac{1}{4}\omega^2r^2\right]u(r) = 0, \tag{27}$$

where $\omega > 0$. The relations of parameters between (26) and (27) are as follows:

$$\rho = l(l+1), \tag{28}$$

$$\frac{2k(\varepsilon + Mc^2)}{c^2} = \frac{1}{4}\omega^2, \tag{29}$$

$$\frac{\varepsilon^2 - M^2c^4}{c^2} = 2n\omega + \left(l + \frac{3}{2}\right)\omega. \tag{30}$$

Since $L_n^\alpha(g(r))$ Generalized Laguerre polynomials for $\alpha > -1$ are related to (27) as an exact solution and α parameter is defined as $\alpha = l + 1/2$, l parameter will be restricted by $l > -3/2$. The relations of parameters (28) and (30) emphasize the conditions of $\rho + 1/4 \geq 0$ and $|\varepsilon| < Mc^2$. Relativistic energy that is related to nonrelativistic energy for harmonic oscillator can be gotten by combining relations (29) and (30) as follows:

$$(\varepsilon - Mc^2)^2(\varepsilon + Mc^2) = 8kc^2\left(2n + l + \frac{3}{2}\right), \tag{31}$$

where (31) is a third-order equation of ε. In nonrelativistic solvable model, the wave function that is associated with (27) is [28, 29]

$$u_{n,l}(r) \propto g^{(l+1)/2} \exp\left(-\frac{g}{2}\right) L_n^{(l+1/2)}(g(r)), \tag{32}$$

where $g(r) = (1/2)\omega r^2$. By comparing two nonrelativistic and relativistic models, the wave function (32) can be expanded

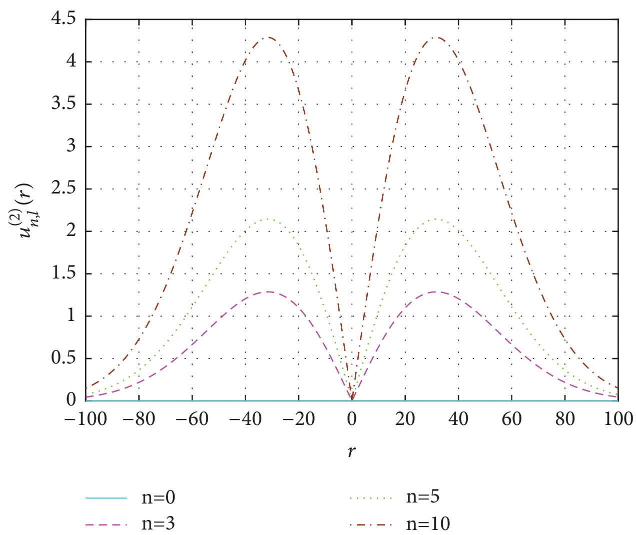

FIGURE 3: $u^{(2)}_{n,l}(r)$ versus r with $l = 0$ and $\omega = 10^{-3}$.

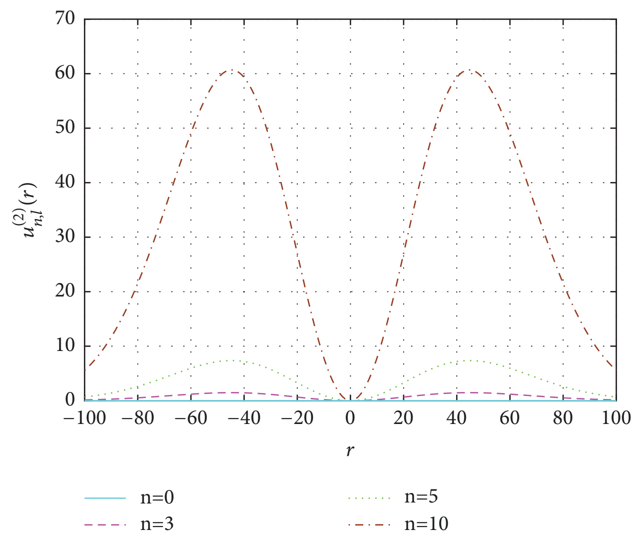

FIGURE 4: $u^{(2)}_{n,l}(r)$ versus r with $l = 1$ and $\omega = 10^{-3}$.

to (26). Therefore, radial part of spinor wave function can be corresponded to (26) as follows:

$$u^{(2)}_{n,l}(r) \propto \left(\frac{1}{2}\omega r^2\right)^{(l+1)/2} \exp\left(-\frac{1}{4}\omega r^2\right) L^{(l+1/2)}_{n}\left(\frac{1}{2}\omega r^2\right), \quad (33)$$

where $-\infty < r < +\infty$. In the investigation of square-integrability condition, it is obvious that wave function (33) is limited as $u_{n,l}(r) \longrightarrow 0$ when $r \longrightarrow -\infty$ and $r \longrightarrow +\infty$ in restriction of l parameter that has been introduced as $l > -3/2$. $u^{(2)}_{n,l}(r)$ wave function for $\omega = 10^{-3}$ and in restriction of l parameter for $l = 0$ and $l = 1$ is indicated in Figures 3 and 4.

4. The Angular Part Solutions of Dirac Equation

As mentioned before, for Hartmann and Ring-Shaped Oscillator Potentials, angular part of Dirac equation is [26, 27]

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) - \left[\frac{m^2}{\sin^2\theta} + \left(\varepsilon + M^2c^4\right) f(\theta) - s\right]\Theta(\theta) = 0. \quad (34)$$

Assuming that $\Theta(\theta) = H(\theta)/\sin^{1/2}\theta$, first-order differential term can be vanished from differential equation (34). This transformation can provide the condition that Schrödinger-like equation is accessible from (34). Considering the mentioned transformation, (34) can be converted to

$$\frac{d^2H(\theta)}{d\theta^2} + \left[-\frac{\left(m^2 - 1/4\right)}{\sin^2\theta} - \left(\varepsilon + Mc^2\right) f(\theta) + \rho + \frac{1}{4}\right] H(\theta) = 0. \quad (35)$$

In comparing (35) with the following Schrödinger solvable equation ($\hbar = 2m = 1$) [28, 29],

$$\frac{d^2H(x)}{dx^2} + \left[E - V(x)\right] H(x) = 0, \quad (36)$$

(35) will be solvable according to different types of $f(\theta)$. It means that the solution of (36) for nonrelativistic energy spectrum and different potentials will be expanded to Schrödinger-like equation obtained from Dirac equation. In this comparison, relativistic parameters can be connected to nonrelativistic parameters. Furthermore, it should be mentioned that this method is useable for special functions of $f(\theta)$. Therefore, $f(\theta)$ functions that can be solved in these techniques are as follows [26, 27]:

$$f_1(\theta) = \frac{\gamma + \beta\cos\theta + \alpha\cos^2\theta}{\sin^2\theta}, \quad (37)$$

$$f_2(\theta) = \frac{\gamma + \beta\cos^2\theta + \alpha\cos^4\theta}{\sin^2\theta\cos^2\theta}, \quad (38)$$

$$f_3(\theta) = \gamma + \beta\cot\theta + \alpha\cot^2\theta, \quad (39)$$

where α, β, and γ are arbitrary constant values. In other words, above functions are solvable functions that are considered as Hartmann and Ring-Shaped Oscillator Potentials. If $f_1(\theta)$ is substituted in (35) as

$$\frac{d^2H(\theta)}{d\theta^2}+\left\{\left[\left(-m^2+\frac{1}{4}\right)-\eta(\gamma+\alpha)\right]\csc^2\theta\right.$$
$$\left.-\eta\beta\csc\theta\cot\theta+\eta\alpha+\rho+\frac{1}{4}\right\}H(\theta)=0, \tag{40}$$

where $\eta = \varepsilon + Mc^2$, (40) can be compared to the following Schrödinger equation ($\hbar = 2m = 1$) [28, 29]:

$$\frac{d^2H(x)}{dx^2}+\left[-\left(\lambda^2+s^2-s\right)\csc^2 x\right.$$
$$\left.+\lambda(2s-1)\csc x\cot x+(s+n)^2\right]H(x)=0. \tag{41}$$

The relations of parameters between nonrelativistic solvable model and relativistic model will be obtained by comparing between (40) and (41) as

$$\eta\alpha+\rho+\frac{1}{4}=(s+n)^2, \tag{42}$$

$$\eta(\gamma+\alpha)+m^2-\frac{1}{4}=\lambda^2+s^2-s, \tag{43}$$

$$\eta\beta=-\lambda(2s-1). \tag{44}$$

The wave function that is related to (40) is written based on Jacobi polynomials $P_n^{(\mu,\nu)}(g(x))$, where $\mu > -1$, $\nu > -1$, and $n = 0, 1, 2, \ldots$. According to parameter definitions of $\mu = -\lambda + s - 1/2$ and $\nu = \lambda + s - 1/2$ in Jacobi polynomials, restrictions of s and λ parameters will be as $s > -1/2$ and $-(s + 1/2) < \lambda < (s + 1/2)$.

The relation between nonrelativistic energy and relativistic energy according to (42) causes ρ separation constant to be calculated as $\rho = (s+n)^2 - \alpha(\varepsilon + Mc^2) - 1/4$, so that condition of $\rho + 1/4 \geq 0$ causes

$$\varepsilon \leq \frac{1}{\alpha}(s+n)^2 - Mc^2. \tag{45}$$

Positive values may be provided for relativistic energy, if $\alpha > 0$. Nonnormalized wave function that is associated with the solvable differential equation (41) is [28, 29]

$$H(x)=(1-g)^{(s-\lambda)/2}(1+g)^{(s+\lambda)/2}$$
$$\cdot P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(g(x)), \tag{46}$$

where $g(x) = \cos x$. Considering function (46), the wave function is obtained for differential equation (40) as follows:

$$H(\theta)=(1-\cos\theta)^{(s-\lambda)/2}(1+\cos\theta)^{(s+\lambda)/2}$$
$$\cdot P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(\cos\theta). \tag{47}$$

According to $\Theta(\theta) = H(\theta)/\sin^{1/2}\theta$, angular part of Dirac equation is gotten as

$$\Theta^{(1)}(\theta)=2^{s-1}(\sin\theta)^{s-\lambda-1/2}(\cos\theta)^{s+\lambda-1/4}$$
$$\cdot P_n^{(-\lambda+s-1/2,\lambda+s-1/2)}(\cos\theta), \tag{48}$$

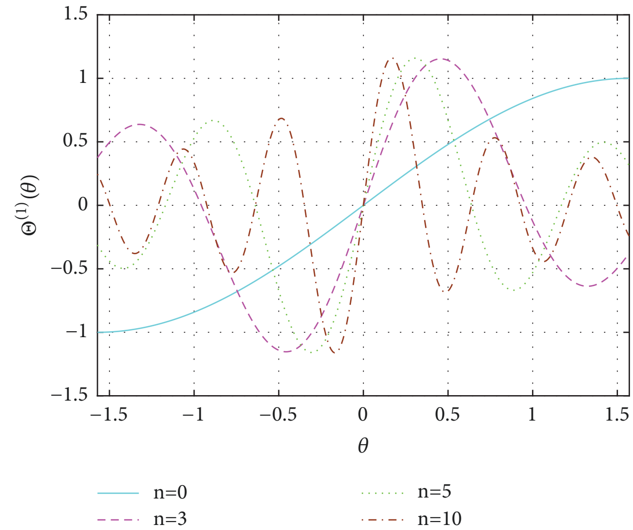

Figure 5: $\Theta^{(2)}(\theta)$ versus θ with $s = 1$ and $\lambda = -0.5$.

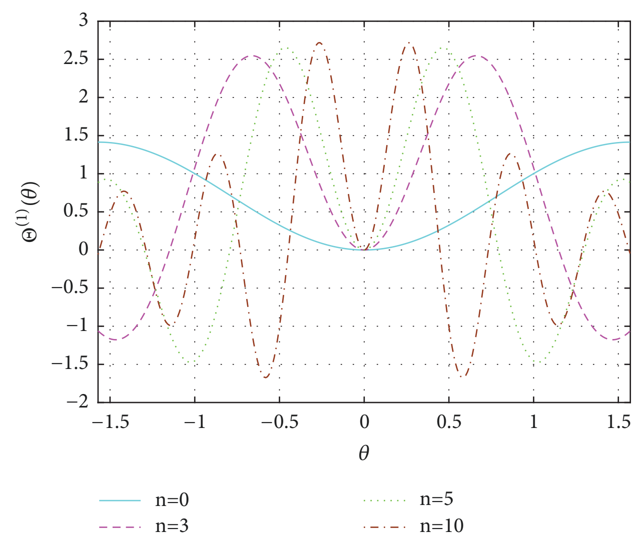

Figure 6: $\Theta^{(1)}(\theta)$ versus θ with $s = 1.5$ and $\lambda = -1$.

where $-\pi/2 \leq \theta \leq +\pi/2$. Wave function (48) will be zero, if θ variable is limited to the endpoints of interval; it means that when $\theta \longrightarrow \mp\pi/2$, the wave function is restricted as $\Theta(\theta) \longrightarrow 0$, although establishing of mentioned physical situations and also avoiding divergence of wave function (48) at $\theta = 0$ will cause restriction of λ and s parameters changing to $-(s+1/4) < \lambda < (s-1/2)$ for $s > 3/8$. $\Theta^{(1)}(\theta)$ wave function based on θ is depicted in restriction of λ and s parameters for $\lambda = -0.5$ and $s = 1$ in Figure 5 and $\lambda = -1$ and $s = 1.5$ in Figure 6.

The illustrated technique can be expanded to other functions of θ. For $f_2(\theta)$ function, angular part of Dirac equation has the following form:

$$\frac{d^2H(\theta)}{d\theta^2}+\left\{\left[\left(-m^2+\frac{1}{4}\right)-\eta(\gamma+\beta+\alpha)\right]\csc^2\theta\right.$$
$$\left.-\eta\gamma\sec^2\theta+\eta\alpha+\rho+\frac{1}{4}\right\}H(\theta)=0. \tag{49}$$

Perfect differential solvable equation ($\hbar = 2m = 1$) that can be used for this method is [28, 29]

$$\frac{d^2 H(x)}{dx^2} + \left[-\lambda(\lambda-1)\csc^2 x - s(s-1)\sec^2 x + (\lambda+s+2n)^2\right] H(x) = 0. \tag{50}$$

The following parameter relations are made by comparison between (49) and (50):

$$\eta\alpha + \rho + \frac{1}{4} = (\lambda+s+2n)^2, \tag{51}$$

$$\eta(\gamma+\beta+\alpha) + m^2 - \frac{1}{4} = \lambda(\lambda-1), \tag{52}$$

$$\eta\gamma = s(s-1). \tag{53}$$

Since Jacobi polynomials $P_n^{(\mu,\nu)}$ are associated with (50) for $\mu > -1$ and $\nu > -1$, λ and s will be restricted by $\lambda > -1/2$ and $s > -1/2$. Relation (51) confirms ρ parameter as $\rho = (\lambda + s + 2n)^2 - \eta\alpha - 1/4$ which can connect relativistic energy to perfect nonrelativistic parameters. The condition of $\rho + 1/4 \geq 0$ creates the following range of relativistic energy spectrum:

$$\varepsilon \leq \frac{1}{\alpha}(\lambda+s+2n)^2 - Mc^2. \tag{54}$$

If α parameter is considered as $\alpha > 0$, positive values may be gotten for relativistic energy spectrum.

Nonnormalized wave function that is satisfied in differential equation (50) for $g(x) = \cos(2x)$ is [28, 29]

$$H(x) = (1-g)^{\lambda/2}(1+g)^{s/2} P_n^{(\lambda-1/2, s-1/2)}(g(x)). \tag{55}$$

Function (55) can be expanded to differential equation (49) and considered as the exact solution of differential equation. The mentioned solution based on Jacobi polynomials is considered as

$$H(\theta) = (1-\cos 2\theta)^{\lambda/2}(1+\cos 2\theta)^{s/2} \cdot P_n^{(\lambda-1/2, s-1/2)}(\cos 2\theta), \tag{56}$$

so that angular part solution of Dirac equation can be constituted as

$$\Theta^{(2)}(\theta) = 2^{(\lambda+s)/2}(\sin\theta)^{\lambda-1/2}(\cos\theta)^s P_n^{(\lambda-1/2, s-1/2)}(\cos 2\theta), \tag{57}$$

where $-\pi/4 \leq \theta \leq +\pi/4$. Wave function (57) will be always constant value at the endpoints of defined interval for θ variable, but divergence of the wave function at $\theta = 0$ converts the restriction of λ parameter to $\lambda > 1/2$. Therefore, wave function (57) will be physical solution, if s and λ parameters are considered as $s > -1/2$ and $\lambda > 1/2$. In restriction $\lambda = 0.6$ and $s = -0.2$ and also $\lambda = 0.55$ and $s = 1$; $\Theta^{(2)}(\theta)$ wave function is performed in Figures 7 and 8, respectively.

Another function of $f_3(\theta)$ can be also analyzed by this method because there is a nonrelativistic solvable model that

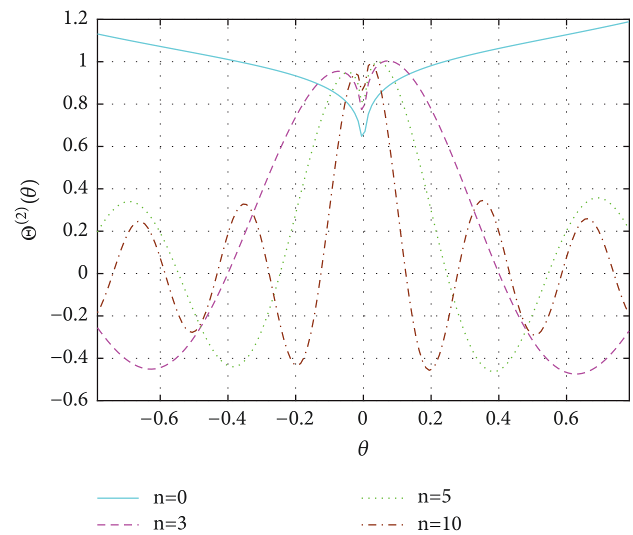

FIGURE 7: $\Theta^{(2)}(\theta)$ versus θ with $s = -0.2$ and $\lambda = 0.6$.

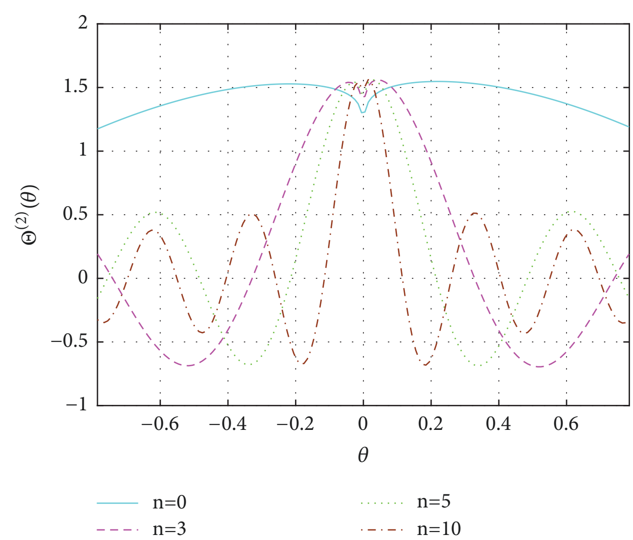

FIGURE 8: $\Theta^{(2)}(\theta)$ versus θ with $s = 1$ and $\lambda = 0.55$.

can correspond to this function as angular part solution of Dirac equation. Angular part of Dirac equation with $f_3(\theta)$ according to (35) is

$$\frac{d^2 H(\theta)}{d\theta^2} + \left\{\left[\left(-m^2 + \frac{1}{4}\right) - \eta\alpha\right]\csc^2\theta - \eta\beta\cot\theta - \eta(\gamma-\alpha) + \rho + \frac{1}{4}\right\} H(\theta) = 0. \tag{58}$$

For corresponding to nonrelativistic solvable model, the following Schrödinger equation ($\hbar = 2m = 1$) is considered [28, 29]:

$$\frac{d^2 H(x)}{dx^2} + \left[-s(s+1)\csc^2 x + 2\lambda\cot\theta + (s-n)^2 - \frac{\lambda^2}{(s-n)^2}\right] H(x) = 0. \tag{59}$$

The relativistic parameters in (58) connected to the nonrelativistic parameters in (59) are as follows:

$$\eta(\gamma-\alpha)-\left(\rho+\frac{1}{4}\right)=\frac{\lambda^2}{(s-n)^2}-(s-n)^2, \quad (60)$$

$$\eta\alpha+m^2-\frac{1}{4}=s(s+1), \quad (61)$$

$$\eta\beta=-2\lambda. \quad (62)$$

In the assumed solvable model, the limits of s and λ parameters are considered as $s>n-1$ and $-i(s-n)(s-n+1)<\lambda<i(s-n)(s-n+1)$. By using relation (60) ρ separation constant can be obtained as $\rho=\eta(\gamma-\alpha)+(s-n)^2-\lambda^2/(s-n)^2-1/4$. The range of relativistic energy will be of the following form, if the condition $\rho+1/4\geq 0$ is considered:

$$\varepsilon\leq\left(\frac{1}{\alpha-\gamma}\right)\left[(s-n)^2-\frac{\lambda^2}{(s-n)^2}\right]-Mc^2. \quad (63)$$

If $\alpha>\gamma$ is considered, it will be possible to calculate positive value for relativistic energy spectrum. The following nonnormalized wave function that is associated with Jacobi polynomials in the solvable model (59), for $g(x)=-i\cot x$, is [28, 29]

$$H(x)=\left(g^2-1\right)^{(s-n)/2}\cdot\exp\left(\frac{\lambda}{s-n}x\right)P_n^{(s-n+i(\lambda/(s-n)),s-n-i(\lambda/(s-n)))}(g(x)). \quad (64)$$

The above nonnormalized function based on Jacobi polynomials can be applied for differential equation (58) as follows:

$$H(\theta)=(-1)^{(s-n)/2}(\csc\theta)^{s-n}\exp\left(\frac{\lambda}{s-n}\theta\right)\cdot P_n^{(s-n+i(\lambda/(s-n)),s-n-i(\lambda/(s-n)))}(-i\cot\theta). \quad (65)$$

Finally, angular part solution of Dirac equation which was called $\Theta(\theta)$ is gotten as

$$\Theta^{(3)}(\theta)=(-1)^{(s-n)/2}(\csc\theta)^{s-n+1/2}\exp\left(\frac{\lambda}{s-n}\theta\right)\cdot P_n^{(s-n+i(\lambda/(s-n)),s-n-i(\lambda/(s-n)))}(-i\cot\theta), \quad (66)$$

where $0\leq\theta\leq\pi$. Restriction of s parameter will be $s<n-1/2$, if the boundary situations are considered for wave function (66) in the endpoints of interval as $\theta\longrightarrow 0$ and $\theta\longrightarrow\pi$ with no divergence at $\theta=\pi/2$. Therefore, wave function (66) is physically acceptable by providing the range of s and λ parameters as $n-1<s<n-1/2$ and $-i(s-n)(s-n+1)<\lambda<i(s-n)(s-n+1)$. $\Theta^{(3)}(\theta)$ wave function in restriction $\lambda=0.1i$ and $s=9.5$ and also $\lambda=1i$ and $s=9.5$ is displayed in Figures 9 and 10.

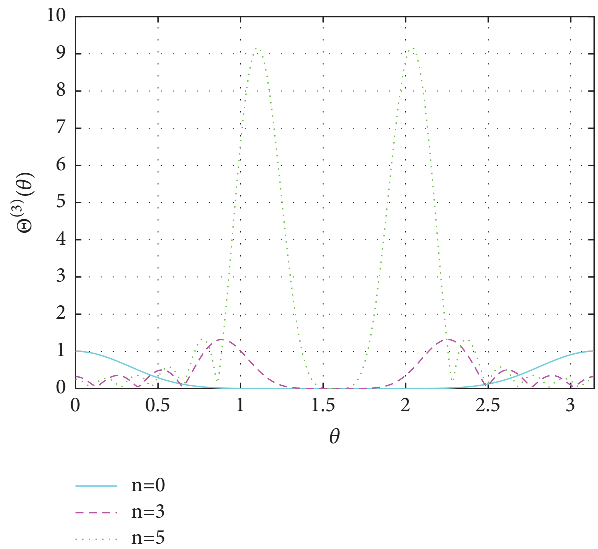

Figure 9: $\Theta^{(3)}(\theta)$ versus θ with $s=9.5$ and $\lambda=0.1i$.

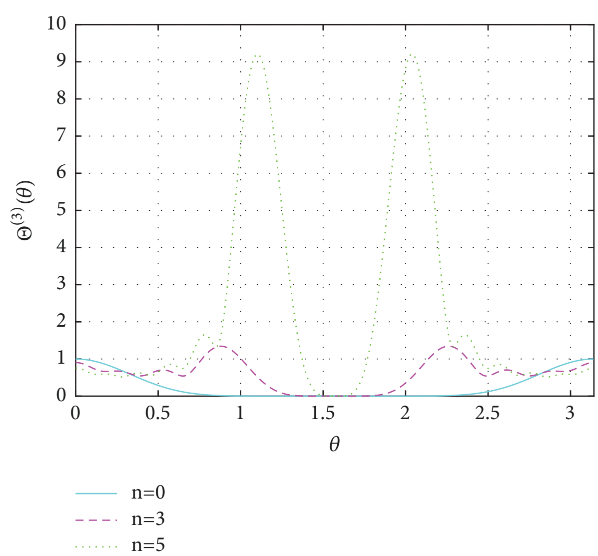

Figure 10: $\Theta^{(3)}(\theta)$ versus θ with $s=9.5$ and $\lambda=1i$.

5. Conclusion

The energy spectrum of bound states and spinor wave function of Dirac equation for Hartmann and Ring-Shaped Oscillator Potentials have been calculated by comparing the mentioned relativistic models with nonrelativistic systems. In radial and angular parts of Dirac equation, relativistic parameters and their restrictions have been investigated by considering the solutions of nonrelativistic models related to the problem and restrictions of nonrelativistic parameters. By this method, spinor wave functions are associated with orthogonal polynomials such as Generalized Laguerre polynomials and Jacobi polynomials in radial and angular parts of Dirac equation, respectively.

Conflicts of Interest

The author declares that she has no conflicts of interest.

References

[1] B. Bentag and L. Chetouani, “Path integral treatment of a noncentral potential,” *Czechoslovak Journal of Physics*, vol. 50, no. 5, pp. 593–606, 2000.

[2] Z. Q. Ma and B. W. Xu, “Quantum correction in exact quantization rules,” *EPL (Europhysics Letters)*, vol. 69, p. 685, 2005.

[3] S.-A. Yahiaoui and M. Bentaiba, “A group-theoretical method for Natanzon potentials in position-dependent mass background by means of conformal mappings,” *International Journal of Theoretical Physics*, vol. 48, no. 2, pp. 315–322, 2009.

[4] A. Gharbi and A. Bouda, “Energy spectra of Hartmann and ring-shaped oscillator potentials using the quantum Hamilton–Jacobi formalism,” *Physica Scripta*, vol. 88, Article ID 045007, 2013.

[5] H. Hassanabadi, B. H. Yazarloo, S. Zarrinkamar, and A. A. Rajabi, “Duffin-Kemmer-Petiau equation under a scalar Coulomb interaction,” *Physical Review C*, vol. 84, Article ID 064003, 2011.

[6] B. H. Yazarloo, H. Hassanabadi, and S. Zarrinkamar, “Oscillator strengths based on the Möbius square potential under Schrödinger equation,” *The European Physical Journal Plus*, vol. 127, p. 51, 2012.

[7] G. F. Wei and S. H. Dong, “Algebraic approach to pseudospin symmetry for the Dirac equation with scalar and vector modified Pöschl-Teller potentials,” *EPL (Europhysics Letters)*, vol. 87, Article ID 40004, 2009.

[8] S. Zarrinkamar, A. A. Rajabi, B. H. Yazarloo, and H. Hassanabadi, “Quasi-Analytical Solutions of DKP Equation under the Deng-Fan Interaction,” *Advances in High Energy Physics*, vol. 2012, 13 pages, 2012.

[9] S.-H. Dong, “Exact solutions of the two-dimensional Schrödinger equation with certain central potentials,” *International Journal of Theoretical Physics*, vol. 39, no. 4, pp. 1119–1128, 2000.

[10] H. Hassanabadi, A. N. Ikot, and S. Zarrinkamar, “Exact Solution of Klein-Gordon with the Pöschl-Teller Double-Ring-Shaped Coulomb Potential,” *Acta Physica Polonica A*, vol. 126, p. 647, 2014.

[11] H. Hartmann, “Die Bewegung eines Körpers in einem ringförmigen Potentialfeld,” *Theoretical Chemistry Accounts*, vol. 24, no. 2-3, pp. 201–206, 1972.

[12] A. Hautot, “Exact motion in noncentral electric fields,” *Journal of Mathematical Physics*, vol. 14, 1973.

[13] M. Kibler and P. Winternitz, “Dynamical invariance algebra of the Hartmann potential,” *Journal of Physics A: Mathematical and General*, vol. 20, no. 13, pp. 4097–4108, 1987.

[14] M. Kibler and P. Winternitz, “Periodicity and quasi-periodicity for super-integrable Hamiltonian systems,” *Physics Letters A*, vol. 147, no. 7, pp. 338–342, 1990.

[15] M. Kibler, G. H. Lamot, and P. Winternitz, “Classical trajectories for two ring-shaped potentials,” *International Journal of Quantum Chemistry*, vol. 43, p. 625, 1992.

[16] M. Hamzavi, H. Hassanabadi, and A. A. Rajabi, “Exact solutions of dirac equation with hartmann potential by nikiforov–uvarov method,” *International Journal of Modern Physics E*, vol. 19, p. 2189, 2010.

[17] S. W. Qian, B. W. Huang, D. Y. Wang, and Z. Y. Gu, “Supersymmetry and shape invariance of Hartmann potential and ring-shaped oscillator potential in the r and theta dimensions of spherical polar coordinates,” *Communications in Theoretical Physics*, vol. 38, p. 139, 2002.

[18] C. Berkdemir, “A novel angle-dependent potential and its exact solution,” *Journal of Mathematical Chemistry*, vol. 46, no. 1, pp. 139–154, 2009.

[19] C. Quesne, “A new ring-shaped potential and its dynamical invariance algebra,” *Journal of Physics A.: Mathematical and General*, vol. 21, no. 14, pp. 3093–3101, 1988.

[20] S. H. Dong, C. Y. Chen, and M. Lozada-Cassou, “Quantum properties of complete solutions for a new noncentral ring-shaped potential,” *International Journal of Quantum Chemistry*, vol. 105, p. 453, 2005.

[21] C.-Y. Chen and S.-H. Dong, “Exactly complete solutions of the Coulomb potential plus a new ring-shaped potential,” *Physics Letters A*, vol. 335, no. 5-6, pp. 374–382, 2005.

[22] A. D. Alhaidari, “Scattering and bound states for a class of non-central potentials,” *Journal of Physics A: Mathematical and General*, vol. 38, no. 15, pp. 3409–3429, 2005.

[23] Q. H. Liu, L. H. Tang, and D. M. Xun, “Geometric momentum: The proper momentum for a free particle on a two-dimensional sphere,” *Physical Review A: Atomic, Molecular and Optical Physics*, vol. 84, no. 4, 2011.

[24] N. Candemir and O. Bayrak, “Massive Dirac equation in asymmetric Hulthén potential,” *Journal of Mathematical Physics*, vol. 54, no. 4, Article ID 042104, 2013.

[25] Z. Sh. Zhang, Sh. F. Xiao, and D. M. Xun, “An enlarged canonical quantization scheme and quantization of a free particle on two-dimensional sphere,” *Communications in Theoretical Physics*, vol. 63, p. 19, 2015.

[26] G. Chen, “Solution of the Klein-Gordon for exponential scalar and vector potentials,” *Physics Letters A*, vol. 339, no. 3-5, pp. 300–303, 2005.

[27] A. de Souza Dutra and M. Hott, “Dirac equation exact solutions for generalized asymmetrical Hartmann potentials,” *Physics Letters A*, vol. 356, no. 3, pp. 215–219, 2006.

[28] G. Levái, “A search for the shape-invariant solvable potentials,” *Journal of Physics A: Mathematical and General*, vol. 22, no. 6, pp. 689–702, 1989.

[29] H. Panahi and Z. Bakhshi, “Solvable potentials with position-dependent effective mass and constant mass schrödinger equation,” *Acta Physica Polonica B*, vol. 41, p. 11, 2010.