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Research Article

Heun Functions Describing Bosons and Fermions on Melvin's Spacetime

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Employing a pseudo-orthonormal coordinate-free approach, the solutions to the Klein–Gordon and Dirac equations for particles in Melvin spacetime are derived in terms of Heun's biconfluent functions.

1. Introduction

The study of relativistic particles in static magnetic fields has a long history and is still attracting considerable attention, especially for cases where someone deals with curved manifolds.

Even though on Minkowski spacetime the relativistic behavior of an electron in various magnetostatic configurations is well understood (see, for example, Johnson and Lippmann's paper [1]), a weakness on curved spacetime regards the explicit gauge covariant formulation.

Recently, when dealing with slowly rotating neutron stars which have been termed as magnetars [2], it has been assumed that their huge magnetic induction in the core and crust, $B \sim 10^{14}$–10^{15} (G), is affecting the spacetime geometry. A way out could be the search for general relativistic solutions with the magnetic field considered as a perturbation of the spherically symmetric background [3]. Another way is to assume that magnetized metrics, as the one belonging to the Melvin class [4, 5], may be reliable candidates for describing these highly compact astrophysical objects with a dominant axial magnetic field [6].

Within a coordinate-dependent formulation, switching between canonical and pseudo-orthonormal basis, the above-mentioned authors are integrating the system of four coupled first-order differential equations, in the first approximation, neglecting the terms in higher orders of the polar radial coordinate ρ. Their solutions are expressed in terms of generalized Laguerre polynomials, similarly to the case of the Dirac equation in cylindrical coordinates on a flat manifold [7].

In the present work, we are applying a coordinates-free method to analyze the Klein–Gordon and Dirac equations describing particles evolving in Melvin's spacetime. Employing Cartan's formalism, we are computing all the essential geometrical objects for writing down the corresponding matter fields and Einstein's equations.

It turns out that the $SO(3,1) \times U(1)$-gauge covariant Klein–Gordon equation can be exactly solved, its solutions being given by the Heun biconfluent functions [8–10]. The same happens with the approximate expression of the second-order differential system derived from the Dirac equation.

The Heun functions, either general or confluent, are main targets of recent investigations and have been obtained for massless particles evolving in a Universe described by the metric function written as a nonlinear mixture of Schwarzschild, Melvine, and Bertotti-Robinson solutions [11].

2. The Geometry

Recently, in [12], the procedure of transforming a known static symmetric solution to Einstein-hydrodynamic equations into a magnetized metric was presented, by (nonlinearly) adding the magnetic field. In spherical coordinates, this has the general form

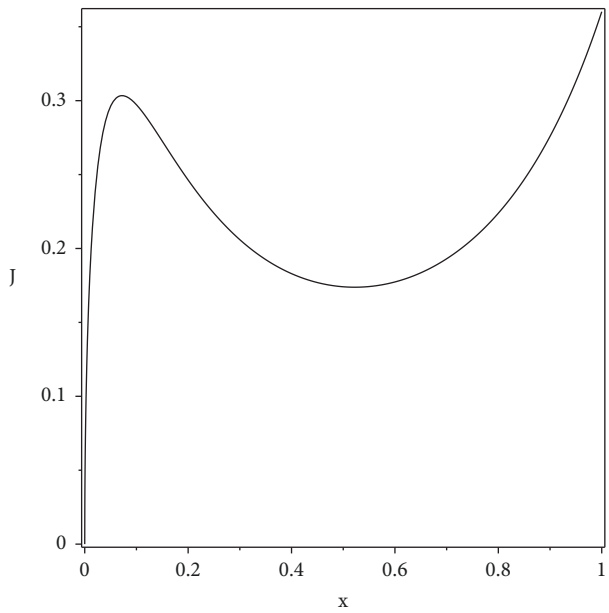

FIGURE 1: The radial current (50) in terms of the variable $x \approx \sqrt{2}MB_*^2\rho^2/(8\omega)$.

$$ds^2 = \Lambda^2 g_{11}\,(dr)^2 + \Lambda^2 r^2\,(d\theta)^2 + \frac{r^2\sin^2\theta}{\Lambda^2}\,(d\varphi)^2 - \Lambda^2 g_{00}\,(dt)^2\,, \tag{1}$$

with the metric functions g_{11} and g_{00} depending only on r and

$$\Lambda = 1 + \frac{B_*^2}{4} r^2 \sin^2\theta, \tag{2}$$

where, for the moment, B_* is a parameter related to the magnetic field intensity.

In the pseudo-orthonormal Cartan frame corresponding to the metric (1),

$$\begin{aligned} \omega^1 &= \Lambda\sqrt{g_{11}}dr, \\ \omega^2 &= \Lambda r d\theta, \\ \omega^3 &= \frac{r\sin\theta}{\Lambda}d\varphi, \\ \omega^4 &= \Lambda\sqrt{g_{00}}dt, \end{aligned} \tag{3}$$

for the potential

$$A_2 = \frac{B_0 r\sin\theta}{2}, \tag{4}$$

where B_0 is the strength of the magnetic field on the axis, and the Maxwell tensor components, corresponding to a poloidal magnetic field with B_ρ and B_θ, are given by the relations

$$\begin{aligned} F_{23} &= \frac{B_0\cos\theta}{\Lambda^2}, \\ F_{13} &= \frac{B_0\sin\theta}{\Lambda^2\sqrt{g_{11}}}, \end{aligned} \tag{5}$$

pointing out a prolate (in shape) star.

Once we assume $g_{00} = g_{11} = 1$, we can switch to cylindrical coordinates $\{\rho, \varphi, z, t\}$, by

$$\begin{aligned} \rho &= r\sin\theta, \\ z &= r\cos\theta, \end{aligned} \tag{6}$$

so that the magnetized metric (1) turns into the simple Melvin expression

$$ds^2 = \Lambda^2\,(d\rho)^2 + \frac{\rho^2}{\Lambda^2}\,(d\varphi)^2 + \Lambda^2\,(dz)^2 - \Lambda^2\,(dt)^2\,, \tag{7}$$

with

$$\Lambda = 1 + \frac{B_*^2\rho^2}{4}. \tag{8}$$

Within an $SO(3,1)$-gauge covariant formulation, we introduce the pseudo-orthonormal frame

$$\begin{aligned} e_1 &= \frac{1}{\Lambda}\partial_\rho, \\ e_2 &= \frac{\Lambda}{\rho}\partial_\varphi, \\ e_3 &= \frac{1}{\Lambda}\partial_z, \\ e_4 &= \frac{1}{\Lambda}\partial_z, \end{aligned} \tag{9}$$

whose corresponding dual base is

$$\begin{aligned}\omega^1 &= \Lambda d\rho,\\ \omega^2 &= \frac{\rho}{\Lambda}d\varphi,\\ \omega^3 &= \Lambda dz,\\ \omega^4 &= \Lambda dt,\end{aligned} \tag{10}$$

so that the metric (7) gets the Minkowskian form $ds^2 = \eta_{ab}\omega^a\omega^b$, with $\eta_{ab} = [1, 1, 1, -1]$. The first Cartan equation,

$$d\omega^a = \Gamma^a_{.[bc]}\omega^b \wedge \omega^c, \tag{11}$$

with $1 \le b < c \le 4$ and $\Gamma^a_{.[bc]} = \Gamma^a_{.bc} - \Gamma^a_{.cb}$, leads to the following connection one-form:

$$\begin{aligned}\Gamma_{12} &= \left(\frac{\Lambda'}{\Lambda^2} - \frac{1}{\rho\Lambda}\right)\omega^2,\\ \Gamma_{13} &= -\frac{\Lambda'}{\Lambda^2}\omega^3,\\ \Gamma_{14} &= \frac{\Lambda'}{\Lambda^2}\omega^4,\end{aligned} \tag{12}$$

where Λ' is the derivative of Λ with respect to ρ.

Employing the second Cartan equation

$$\mathscr{R}_{ab} = d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^c_{.b}, \tag{13}$$

one derives the curvature two-forms $\mathscr{R}_{ab} = R_{abcd}\omega^c \wedge \omega^d$, with $1 \le c < d \le 4$, leading to the curvature components

$$\begin{aligned}R_{1212} &= \frac{2B_*^2}{\Lambda^4}\left[1 - \frac{B_*^2\rho^2}{8}\right],\\ R_{3434} &= \frac{B_*^4\rho^2}{4\Lambda^4},\\ R_{1313} &= R_{2323} = -\frac{B_*^2}{2\Lambda^4}\left[1 - \frac{B_*^2\rho^2}{4}\right] = -R_{1414}\\ &= -R_{2424},\end{aligned} \tag{14}$$

pointing out the special radius value $\rho_* = 2/B_*$, for which only the components $R_{1212} = B_*^2/\Lambda^4 = R_{3434}$ are surviving and the Weyl tensor vanishes.

Since the scalar curvature is zero, the Einstein tensor components are given by the Ricci tensor components, as

$$G_{11} = G_{22} = -G_{33} = G_{44} = \frac{B_*^2}{\Lambda^4}. \tag{15}$$

In the pseudo-orthonormal frame whose dual bases are (10), it turns out that the potential (4), generating the magnetic induction along Oz, gets the familiar expression

$$A_2 = \frac{B_0\rho}{2}, \tag{16}$$

and the essential component of the Maxwell tensor reads

$$F_{12} = A_{2|1} + \Gamma_{212}A_2 = \frac{B_0}{\Lambda^2}, \tag{17}$$

where $f_{|1} = e_1(f)$.

Using the energy-momentum tensor components

$$T_{11} = T_{22} = -T_{33} = T_{44} = \frac{1}{2}F_{12}^2 = \frac{1}{2}\frac{B_0^2}{\Lambda^4}, \tag{18}$$

in the Einstein equations $G_{ab} = \kappa_0 T_{ab}$, one gets the following relation between the parameters B_* and B_0:

$$B_*^2 = \frac{\kappa_0 B_0^2}{2}, \tag{19}$$

with $\kappa_0 = 8\pi G/c^4$.

3. Exactly Solvable Klein–Gordon Equation

In this section, we are going to construct the wave function of the charged bosons, considered as test particles evolving in the crust of a relativistic magnetar. The complex scalar field of mass μ, minimally coupled to gravity, is described by the $SO(3,1) \times U(1)$ gauge covariant Klein–Gordon equation

$$\eta^{ab}\Phi_{|ab} - \eta^{ab}\Phi_{|c}\Gamma^c{}_{ab} = \mu^2\Phi + 2iqA_2\Phi_{|2} + q^2A_2^2\Phi, \tag{20}$$

which, in the pseudo-orthonormal frame with the dual bases (10), reads

$$\begin{aligned}&\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Phi}{\partial\rho}\right] + \frac{\Lambda^4}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{\partial^2\Phi}{\partial t^2}\\ &\quad= \left[\mu^2\Lambda^2 + iqB_0\Lambda^3\frac{\partial\Phi}{\partial\varphi} + \left(\frac{qB_0\rho\Lambda}{2}\right)^2\right]\Phi.\end{aligned} \tag{21}$$

The above form suggests the variables separation

$$\Phi = \phi(\rho)\, e^{im\varphi}e^{ip_z z}e^{-i\omega t}, \tag{22}$$

which leads to the following differential equation for the unknown function ϕ,

$$\begin{aligned}&\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\phi}{\partial\rho}\right] + \left[\omega^2 - p_z^2 - \frac{m^2}{\rho^2}\Lambda^4 - \mu^2\Lambda^2 + mqB_0\Lambda^3\right.\\ &\quad\left. - \left(\frac{qB_0\rho\Lambda}{2}\right)^2\right]\phi = 0,\end{aligned} \tag{23}$$

with Λ defined in (8).

This can be exactly integrated, its solution being expressed in terms of the Heun biconfluent function as [9, 10]

$$\phi(y) \sim \exp\left[-\frac{y^2}{2} - \frac{\beta y}{2}\right] y^{\alpha/2}\mathrm{HeunB}\left[\alpha, \beta, \gamma, \delta, y\right], \tag{24}$$

where the variable and the parameters are, respectively, given by

$$\begin{aligned} y &= \frac{\sqrt{b}B_* \rho^2}{4}, \\ b &= qB_0 - \frac{mB_*^2}{2} \approx qB_0, \\ \alpha &= \pm m, \\ \beta &\approx 2\frac{\sqrt{qB_0}}{B_*}, \\ \gamma &\approx m - \frac{\mu^2}{qB_0}, \\ \delta &\approx -\frac{2\left[\omega^2 - p_z^2 - \mu^2 + mqB_0\right]}{B_* \sqrt{qB_0}}. \end{aligned} \tag{25}$$

Let us point out that the Heun biconfluent equation has one regular singularity at the origin and one irregular at ∞ and can be obtained, from the Heun general equation, by a process of successive confluences [10].

Regarding the asymptotic behavior of the function (24), solution to (23), that has a singularity in $\rho \to 0$, due to the exponential term, this is vanishing for large y-values. On the other hand, for a regular solution at the origin $y = 0$ (where HeunB(0) = 1), one has to choose the plus sign of α in (26).

4. The $SO(3,1) \times U(1)$-Gauge Covariant Dirac Equation

For relativistic fermions of mass M, coupled to the external magnetic field generated by (16), the Dirac equation has the $SO(3,1) \times U(1)$-gauge covariant expression

$$\gamma^a \Psi_{;a} + M\Psi = 0, \tag{27}$$

where ";" stands for the covariant derivative

$$\Psi_{;a} = e_a \Psi + \frac{1}{4}\Gamma_{bca}\gamma^b\gamma^c\Psi - iqA_a\Psi. \tag{28}$$

In view of the relations (12), the term expressing the Ricci spin-connection in (27) reads

$$\frac{1}{4}\Gamma_{bca}\gamma^a\gamma^b\gamma^c = \frac{f}{\Lambda}\gamma^1, \tag{29}$$

where we have introduced the function

$$f = \frac{1}{2}\left[\frac{1}{\rho} + \frac{\Lambda'}{\Lambda}\right]. \tag{30}$$

With the explicit form of the Dirac equation (27) being

$$\begin{aligned} \frac{1}{\Lambda}\Bigg[\gamma^1\left(\partial_\rho + f\right) + \frac{\Lambda^2}{\rho}\gamma^2\partial_\varphi + \gamma^3\partial_z + \gamma^4\partial_t + M\Lambda \\ - iq\gamma^2\Lambda A_2\Bigg]\Psi = 0, \end{aligned} \tag{31}$$

one may use the variables separation

$$\Psi = e^{i(m\varphi + p_z z - \omega t)}\psi(\rho), \tag{32}$$

to derive the differential equation satisfied by the part depending on ρ; i.e.,

$$\begin{aligned} &\gamma^1\left[\psi' + f\psi\right] \\ &\quad + i\left\{\gamma^2\Lambda\left[\frac{m\Lambda}{\rho} - \frac{qB_0\rho}{2}\right] + p_z\gamma^3 - \omega\gamma^4 - iM\Lambda\right\}\psi \\ &\quad = 0. \end{aligned} \tag{33}$$

With the following function substitution

$$\psi = \frac{1}{2\sqrt{\rho\Lambda}}\chi, \tag{34}$$

the above equation becomes

$$\gamma^1\chi' + i\left\{\gamma^2 F + p_z\gamma^3 - \omega\gamma^4 - iM\Lambda\right\}\chi = 0, \tag{35}$$

where

$$F(\rho) = \Lambda\left[\frac{m\Lambda}{\rho} - \frac{qB_0\rho}{2}\right], \tag{36}$$

and we are going to use the Dirac representation for the γ^i matrices,

$$\begin{aligned} \gamma^\mu &= -i\beta\alpha^\mu, \\ \gamma^4 &= -i\beta, \\ \mu &= \overline{1,3}, \end{aligned} \tag{37}$$

with

$$\begin{aligned} \beta &= \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix}, \\ \alpha^\mu &= \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \end{aligned} \tag{38}$$

where σ^μ denotes the usual Pauli matrices.

In the following, we are assuming that the particle is not moving along the magnetic field direction, i.e., $p_z = 0$, and the bispinor χ is of the form

$$\chi(\rho) = \begin{bmatrix} \zeta(\rho) \\ \eta(\rho) \end{bmatrix}, \tag{39}$$

so that (35) decouples in two equations for the (two-component) spinors ζ and η; i.e.,

$$\begin{aligned} \sigma^1\zeta' + iF\sigma^2\zeta &= i(\omega + M\Lambda)\eta, \\ \sigma^1\eta' + iF\sigma^2\eta &= i(\omega - M\Lambda)\zeta. \end{aligned} \tag{40}$$

Applying the usual procedure, one gets the following differential equations:

$$\begin{aligned}
&\zeta_A'' - \frac{M\Lambda'}{\omega + M\Lambda}\zeta_A' \\
&\quad + \left\{\omega^2 - M^2\Lambda^2 - F^2 \mp \left[F' - \frac{MF\Lambda'}{\omega + M\Lambda}\right]\right\}\zeta_A = 0, \\
&\eta_A'' + \frac{M\Lambda'}{\omega - M\Lambda}\eta_A' \\
&\quad + \left\{\omega^2 - M^2\Lambda - F^2 \mp \left[F' + \frac{MF\Lambda'}{\omega - M\Lambda}\right]\right\}\eta_A = 0,
\end{aligned} \tag{41}$$

which cannot be analytically solved. However, by imposing the condition $B_*^2 \ll qB_0$ and neglecting the powers of ρ larger than 3, (41) get the simpler forms:

$$\begin{aligned}
&\zeta_A'' - \frac{MB_*^2\rho}{2(\omega + M)}\left[1 - \frac{MB_*^2\rho^2}{4(\omega + M)}\right]\zeta_A' + \left\{\omega^2 - M^2\right. \\
&\quad \left. + \left(m \pm \frac{1}{2}\right)qB_0 - \frac{m(m \mp 1)}{\rho^2} - \left(\frac{qB_0\rho}{2}\right)^2\right\}\zeta_A = 0, \\
&\eta_A'' + \frac{MB_*^2\rho}{2(\omega - M)}\left[1 + \frac{MB_*^2\rho^2}{4(\omega - M)}\right]\eta_A' + \left\{\omega^2 - M^2\right. \\
&\quad \left. + \left(m \pm \frac{1}{2}\right)qB_0 - \frac{m(m \mp 1)}{\rho^2} - \left(\frac{qB_0\rho}{2}\right)^2\right\}\eta_A = 0.
\end{aligned} \tag{42}$$

The corresponding solutions, i.e.,

$$\begin{aligned}
\zeta_1 &= \left\{\sqrt{\rho}, \rho^m\right\}u_1, \\
\zeta_2 &= \left\{\sqrt{\rho}, \rho^{m+1}\right\}u_2, \\
\eta_1 &= \left\{\sqrt{\rho}, \rho^m\right\}v_1, \\
\eta_2 &= \left\{\sqrt{\rho}, \rho^{m+1}\right\}v_2,
\end{aligned} \tag{43}$$

are expressed in terms of Heun's biconfluent functions [8–10]

$$\begin{aligned}
u_1 &= \mathrm{HeunB}\left[\alpha_1, \beta, \gamma^+, \delta_1^+, x_u\right], \\
u_2 &= \mathrm{HeunB}\left[\alpha_2, \beta, \gamma^+, \delta_2^+, x_u\right]; \\
v_1 &= \mathrm{HeunB}\left[\alpha_1, \beta, \gamma^-, \delta_1^-, x_v\right]; \\
v_2 &= \mathrm{HeunB}\left[\alpha_2, \beta, \gamma^-, \delta_2^-, x_v\right],
\end{aligned} \tag{44}$$

of variables

$$\begin{aligned}
x_u &= -\frac{\sqrt{2}MB_*^2\rho^2}{8(\omega + M)}, \\
x_v &= \frac{\sqrt{2}MB_*^2\rho^2}{8(\omega - M)}
\end{aligned} \tag{45}$$

and parameters

$$\begin{aligned}
\alpha_1 &= m - \frac{1}{2}, \\
\alpha_2 &= m + \frac{1}{2}, \\
\beta &= \sqrt{2}, \\
\gamma^\pm &= -2\left[\frac{qB_0(\omega \pm M)}{MB_*^2}\right]^2, \\
\delta_1^\pm &= \pm\frac{2\sqrt{2}}{MB_*^2}(\omega \pm M)\left[\omega^2 - M^2 + \left(m + \frac{1}{2}\right)qB_0\right], \\
\delta_2^\pm &= \pm\frac{2\sqrt{2}}{MB_*^2}(\omega \pm M)\left[\omega^2 - M^2 + \left(m - \frac{1}{2}\right)qB_0\right],
\end{aligned} \tag{46}$$

and therefore the components of the bispinor ψ in (34) are given by

$$\begin{aligned}
\psi_1 &= \frac{1}{2\sqrt{\Lambda}}\left\{1, \rho^{m-1/2}\right\}u_1, \\
\psi_2 &= \frac{1}{2\sqrt{\Lambda}}\left\{1, \rho^{m+1/2}\right\}u_2, \\
\psi_3 &= \frac{1}{2\sqrt{\Lambda}}\left\{1, \rho^{m-1/2}\right\}v_1, \\
\psi_4 &= \frac{1}{2\sqrt{\Lambda}}\left\{1, \rho^{m+1/2}\right\}v_2.
\end{aligned} \tag{47}$$

Using the expressions (47) in (32), one may compute the radial current density, meaning particles per unit time and per unit covariant 2-surface

$$d\Sigma_1 = \omega^2 \wedge \omega^3 = \rho d\varphi \wedge dz, \tag{48}$$

as

$$j^1 = i\overline{\Psi}\gamma^1\Psi = \Psi^\dagger\alpha^1\Psi = \frac{1}{2\Lambda}\left[u_1v_2 + u_2v_1\right] \tag{49}$$

and the corresponding (radial) current,

$$I(\rho) = \int_{-L_z/2}^{L_z/2}\int_0^{2\pi} e_a^1 j^a d\Sigma_1 = \frac{\pi L_z\rho}{\Lambda^2}\left[u_1v_2 + u_2v_1\right], \tag{50}$$

represented in Figure 1, as a function of

$$x \approx \frac{\sqrt{2}MB_*^2\rho^2}{8\omega}. \tag{51}$$

One may notice that, for $x \ll 1$, the current is suddenly increasing from zero to a maximum value, which depends on the ratio M/ω and on the magnetic field intensity.

The case corresponding to massless fermions is significantly less complicated. Thus, the equation

$$\begin{aligned}
&\frac{1}{\Lambda}\left[\gamma^1\left(\partial_\rho + f\right) + \frac{\Lambda^2}{\rho}\gamma^2\partial_\varphi + \gamma^3\partial_z + \gamma^4\partial_t\right]\Psi_0 \\
&\quad - iq\gamma^2A_2\Psi_0 = 0,
\end{aligned} \tag{52}$$

with the variables separation (32), leads to the following differential equation satisfied by the part depending on ρ; i.e.,

$$\gamma^1\left[\psi_0' + f\psi_0\right] + i\left\{\gamma^2\Lambda\left[\frac{m\Lambda}{\rho} - \frac{qB_0\rho}{2}\right] + p\gamma^3 - \omega\gamma^4\right\}\psi_0 = 0. \tag{53}$$

As customary for massless fermions, we are going to use the Weyl representation for the γ matrices,

$$\gamma^\mu = -i\beta\alpha^\mu, \quad \gamma^4 = -i\beta, \quad \mu = \overline{1,3}, \tag{54}$$

with

$$\alpha^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & -\sigma^\mu \end{pmatrix}, \tag{55}$$

and the bispinor ψ_0 will be taken as

$$\psi_0(\rho) = \begin{bmatrix} \zeta(\rho) \\ \eta(\rho) \end{bmatrix}. \tag{56}$$

Once (53) decouples in two equations for ζ and η, one gets, for the *up* spinor's components, the second-order differential equations

$$\zeta_A'' + 2f\zeta_A' + \left[\omega^2 - p_z^2 + f^2 - F^2 + \partial_\rho\left(f \mp F\right)\right]\zeta_A = 0, \tag{57}$$

and similarly for η_A. Within the same approximation $B_*^2 \ll qB_0$ and neglecting the powers of ρ larger than 2, (57) turns into the simpler forms

$$\frac{d^2\zeta_A}{d\rho^2} + \left[\frac{1}{\rho} + \frac{B_*^2\rho}{2}\right]\frac{d\zeta_A}{d\rho} + \left[\omega^2 - p_z^2 + qB_0\left(m \pm \frac{1}{2}\right) - \left(m \mp \frac{1}{2}\right)^2\frac{1}{\rho^2} - \left(\frac{qB_0\rho}{2}\right)^2\right]\zeta_A = 0, \tag{58}$$

whose solutions can be expressed either in terms of confluent hypergeometric functions, as [13]

$$\zeta_1 = x^{(1/2)(m-1/2)}e^{-x/2}U\left[-\frac{\omega^2 - p_z^2}{2qB_0} + \frac{B_*^2}{4qB_0}, m + \frac{1}{2}, x\right];$$
$$\zeta_2 = x^{-(1/2)(m+1/2)}e^{-x/2}U\left[-\frac{\omega^2 - p_z^2}{2qB_0} + \frac{B_*^2}{4qB_0} - \left(m - \frac{1}{2}\right), -\left(m - \frac{1}{2}\right), x\right], \tag{59}$$

or in terms of Whittaker functions [13], as

$$\zeta_1 = \frac{1}{\sqrt{x}}\mathrm{W}_{\lambda_1,\mu_1}(x), \quad \zeta_2 = \frac{1}{\sqrt{x}}\mathrm{W}_{\lambda_2,\mu_2}(x), \tag{60}$$

with the dimensionless variable

$$x = \frac{B_*^2\rho^2}{4}\sqrt{1 + \left(\frac{2qB_0}{B_*^2}\right)^2} \approx \frac{qB_0\rho^2}{2} \tag{61}$$

and parameters

$$\lambda_1 \approx \frac{\omega^2 - p_z^2}{2qB_0} - \frac{B_*^2}{4qB_0} + \frac{1}{2}\left(m + \frac{1}{2}\right), \quad \mu_1 = \frac{1}{2}\left(m - \frac{1}{2}\right);$$
$$\lambda_2 \approx \frac{\omega^2 - p_z^2}{2qB_0} - \frac{B_*^2}{4qB_0} + \frac{1}{2}\left(m - \frac{1}{2}\right), \quad \mu_1 = \frac{1}{2}\left(m + \frac{1}{2}\right). \tag{62}$$

5. Conclusions

Within the framework of the gauge-invariant geometry, based on the semi-direct product of the local groups $SO(3,1)$ and $U(1)$, the present paper is focusing on the Klein–Gordon and Dirac equations describing particles evolving in a background endowed with Melvin's metric.

By making use of Cartan's formalism, we have derived the corresponding Einstein–Melvin equations leading to the essential relation between the model's parameters (19). As a remark, switching to the canonical bases, the third covariant induction component is given by the expression

$$B_z = \sqrt{|g|}F_{(c)}^{12} = \sqrt{|g|}e_a^1e_b^2F^{ab} \equiv B_0. \tag{63}$$

In case of bosons, the Klein–Gordon equation can be integrated exactly, its solution being given by the Heun biconfluent functions of parameters (26).

Equations (41) coming from the Dirac equation (31) have rather complicated expressions, containing several additional terms which were neglected in [6]. In the assumption $B_*^2 \ll qB_0$, we have been able to find solutions for (42), expressed in terms of Heun's biconfluent functions (44). The corresponding radial particle-current (50), represented in Figure 1, is starting from zero, at the origin $\rho = 0$, and exhibits a rather nontrivial behavior, characterized by a sudden growth to a maximum value. This one is followed by a local minimum which might signal, in the approximation we have used, the presence of a plateau. The seemingly far away unlimited increasing is a result of the violation of the approximation holding condition which demands $x < 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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