Hindawi
Advances in High Energy Physics
Volume 2018, Article ID 2175818, 14 pages
https://doi.org/10.1155/2018/2175818

Research Article

λ Phase Transition in Horava Gravity

Wei Xu

School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China

Correspondence should be addressed to Wei Xu; xuwei@cug.edu.cn

Received 27 March 2018; Revised 5 July 2018; Accepted 22 July 2018; Published 9 August 2018

Academic Editor: Saibal Ray

We present another example of superfluid black hole containing λ phase transition in Horava gravity. After studying the extended thermodynamics of general dimensional Horava-Lifshitz AdS black holes, it is found that only the one with spherical horizon in four and five dimensions has λ phase transition, which is a line of (continuous) second-order phase transitions and was famous in the discussion of superfluidity of liquid ^{4}He. The "superfluid" black hole phase and "normal" black hole phase are also distinguished. Particularly, six-dimensional Horava-Lifshitz AdS black holes exhibit infinitely many critical points in $P-v$ plane and the divergent points for specific heat, for which they only contain the "normal" black hole phase and the "superfluid" black hole phase disappears due to the physical temperature constraint; therefore there is no similar phase transition. In more than six dimensions, there is no $P-v$ critical behavior. After choosing the appropriate ordering field, we study the critical phenomena in different planes of thermodynamical phase space. We also calculate the critical exponents, which are the same as the van der Waals fluid.

1. Introduction

Black hole thermodynamics always provides valuable insight into quantum properties of gravity, and it has been studied extensively for quite a long time, especially for the quantum and microscopic interpretation of black hole temperature and entropy (see [1, 2], for example). Besides, thermodynamics and phase transitions of AdS black holes have been of great interest since the Hawking-Page phase transition [3] between stable large black hole and thermal gas is explained as the confinement/deconfinement phase transition of gauge field [4] inspired by the AdS/CFT correspondence [5–7].

After treating the cosmological constant as a pressure with its conjugate quantity being the thermodynamic volume in thermodynamic phase space of charged AdS black holes [8–14], the small/large black hole phase transition is established in [15], which is exactly analogous to the liquid/gas phase transition of the van der Waals fluid. This kind of black hole phase transitions has attracted much attention (see the recent review papers [16, 17]). Besides, this semiclassical method of analogue is generalized to study the microscopic structure of black holes and sheds some light on the black hole molecules [18] and microscopic origin of the black hole reentrant phase transition [19]. The study is also extended to the quantum statistic viewpoint, as the superfluid black holes are reported recently [20]. In Lovelock gravity with conformally coupled scalar field, the authors present the first example of λ phase transition, which is a line of (continuous) second-order phase transitions and was famous for the successful quantum and microscopic interpretation of superfluidity of liquid ^{4}He.

In this paper, we present another example of black holes containing λ phase transition in Horava gravity, which is a candidate of quantum gravity in ultrahigh energy [21]. The Horava-Lifshitz (HL) black hole solutions, thermodynamics, and phase transitions have attracted a lot of attention [22–29] (see [30] for a review on the recent development of various areas). The general dimensional HL black hole solutions are also introduced [31]. We will consider the extended thermodynamics of general dimensional HL AdS black holes. It is shown that only the one with spherical horizon in four and five dimensions has λ phase transition. Note that the first example of "superfluid" black holes always has a hyperbolic horizon [20]. Particularly, six-dimensional HL AdS black holes exhibit infinitely many critical points in $P-v$ plane and the divergent points for specific heat, for which they only contain the "normal" black hole phase and the "superfluid" black hole phase disappears due to the physical temperature constraint; therefore there is no similar phase transition. In more than six dimensions, there is no $P-\nu$ critical behavior. After identifying parameter ϵ as the ordering field instead of pressure and temperature, we study the critical phenomena in different planes of thermodynamical phase space. We also obtain the critical exponents, which are the same as the van der Waals fluid.

The paper is structured as follows: in next section, we present the extended thermodynamics of generalized topological HL black holes. Then we study $P-V$ criticality in Section 3. We show the λ phase transition for four and five dimensions and the discussion for six dimensions in Sections 4 and 5, respectively. In Sections 6 and 7, we discuss the critical phenomena and calculate the critical exponents in different planes. In final section, some concluding remarks are given.

2. Extended Thermodynamics of Generalized HL Black Holes

In this section, we present the extended thermodynamics of generalized topological HL black holes in $(d+1)$ dimensions $(d \geq 3)$. We begin with the action of HL gravity at the $z=3$ UV fixed point, which can be reexpressed as [31]

$$\begin{aligned}
S &= \int \mathrm{d}t \left[L_0 + \left(1-\epsilon^2\right) L_1\right], \\
L_0 &\equiv \int \mathrm{d}^d x \sqrt{g} N \left[\frac{2}{\kappa^2}\left(K_{ij}K^{ij} - \lambda K^2\right)\right. \\
&\quad \left. + \frac{\kappa^2}{8\kappa_W^4}\frac{\Lambda_W}{1-d\lambda}\left((d-2)R - d\Lambda_W\right)\right], \\
L_1 &\equiv \int \mathrm{d}^d x \sqrt{g} N \frac{\kappa^2}{8\kappa_W^4}\frac{1}{1-d\lambda}\left[\left(1-\frac{d}{4}-\lambda\right)R^2\right. \\
&\quad \left. - (1-d\lambda) R^{ij} R_{ij}\right],
\end{aligned} \tag{1}$$

where the first two terms in the L_0 are the kinetic actions, while the residue corresponds to the potential actions. R_{ij} is the Ricci tensor, R is the Ricci scalar, and K_{ij} is defined by $K_{ij} = (1/2N)(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$, which are based on the ADM decomposition of the higher dimensional metric, i.e., $\mathrm{d}s^2_{d+1} = -N^2\mathrm{d}t^2 + g_{ij}(\mathrm{d}x^i - N^i\mathrm{d}t)(\mathrm{d}x^j - N^j\mathrm{d}t)$. Here the lapse, shift, and d-metric N, N^i, and g_{ij} are all functions of t and x^i, and a dot denotes a derivative with respect to t. There are five constant parameters in the action: Λ_W, λ, ϵ, κ, and κ_W. $\Lambda = (d/2(d-2))\Lambda_W$ is the cosmological constant. κ and κ_W have their origin as the Newton constant and the speed of light. λ represents a dynamical coupling constant which is susceptible to quantum corrections. We will fix $\lambda = 1$ in the following paper, only for which general relativity can be recovered in the large distance approximation. In addition, we will only consider the general values of ϵ in the region $0 \leq \epsilon^2 \leq 1$, as $\epsilon = 0$ corresponds to the so-called detailed-balance condition, and HL gravity with $\epsilon = 1$ returns back to general relativity.

Action Eq. (1) admits arbitrary dimensional topological AdS black holes with the metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{d-1,k} \tag{2}$$

and the horizon function [23, 31]

$$\begin{aligned}
f(r) &= k - \frac{2\Lambda_W}{(1-\epsilon^2)}\frac{r^2}{(d-1)(d-2)} \\
&\quad - r^{2-d/2}\sqrt{\frac{c_0}{(1-\epsilon^2)} + \frac{\epsilon^2}{(1-\epsilon^2)^2}\frac{4\Lambda_W^2 r^d}{(d-1)^2(d-2)^2}},
\end{aligned} \tag{3}$$

where $\mathrm{d}\Omega^2_{d-1,k}$ denotes the line element of a $(d-1)$-dimensional manifold with constant scalar curvature $(d-1)k$, and $k = 0, \pm 1$ indicates different topology of the spatial space. c_0 is integration constant, which is related to the black hole mass

$$M = -\frac{\Omega^2_{d-1,k}c^3}{16\pi G_N}\frac{1}{(d-2)\Lambda_W}c_0 = -\frac{c_0}{\Lambda_W}, \tag{4}$$

where we have chosen the natural units and $\Omega^2_{d-1,k} = 16(d-2)\pi$.

In AdS space-time, the cosmological constant is introduced as the thermodynamical pressure [15]:

$$P = -\frac{\Lambda}{8\pi} = -\frac{d}{16\pi(d-2)}\Lambda_W. \tag{5}$$

Then we can rewrite the horizon function as

$$\begin{aligned}
f(r) &= k + \frac{32\pi P r^2}{(1-\epsilon^2)\,d\,(d-1)} \\
&\quad - 4r^{2-d/2}\sqrt{\frac{(d-2)MP\pi}{d\left(1-\epsilon^2\right)} + \frac{64\epsilon^2P^2\pi^2r^d}{\left(1-\epsilon^2\right)^2 d^2 (d-1)^2}},
\end{aligned} \tag{6}$$

and the mass is

$$\begin{aligned}
M &= \frac{64P\pi r_+^d}{(d-1)^2(d-2)\,d} + \frac{\left(1-\epsilon^2\right)dk^2r_+^{d-4}}{16P(d-2)\pi} \\
&\quad + \frac{4kr_+^{d-2}}{(d-2)(d-2)},
\end{aligned} \tag{7}$$

where r_+ denotes the event horizon which is the largest positive root of $f(r) = 0$. The conjugate thermodynamic volume of pressure is

$$V = \frac{64\pi r_+^d}{(d-1)^2(d-2)\,d} - \frac{\left(1-\epsilon^2\right)dk^2r_+^{d-4}}{P^2(d-2)\pi}. \tag{8}$$

The entropy and temperature are presented in [31] with the following forms:

$$S = \begin{cases} 4\pi r_+^2 \left(1 + \dfrac{kd\left(1-\epsilon^2\right)\ln\left(r_+\right)}{8\left(d-2\right)P\pi r_+^2}\right) + S_0, & d = 3, \\ \dfrac{16\pi r_+^{d-1}}{\left(d-1\right)^2\left(d-2\right)}\left(1 + \dfrac{kd\left(d-1\right)^2\left(d-2\right)\left(1-\epsilon^2\right)}{32\left(d-2\right)\left(d-3\right)P\pi r_+^2}\right) + S_0, & d \geq 4, \end{cases} \tag{9}$$

$$T = \frac{1024P^2\pi^2 r_+^4 + 64k\left(d-1\right)\left(d-2\right)P\pi r_+^2 + k^2 d\left(d-1\right)^2\left(d-4\right)\left(1-\epsilon^2\right)}{8\left(d-1\right)\pi r_+\left(32\pi P r_+^2 + kd\left(d-1\right)\left(1-\epsilon^2\right)\right)}. \tag{10}$$

It is easy to check the first law of thermodynamics:

$$\mathrm{d}M = T\mathrm{d}S + V\mathrm{d}P, \tag{11}$$

while the Smarr relation always fails to exist. This can be easily found because of the existence of S_0 (and logarithmic term for $d = 3$) in black hole entropy, which is not fixed and may be calculated by invoking the quantum theory of gravity as argued in [23]. Note that ϵ scales as $[L]^0$, and there is no other dimensional quantity in extended thermodynamic phase space. In this meaning, the validity of the Smarr relation will bring another physical consideration on the parameter S_0.

In order to analyze the global thermodynamic stability and phase transition of the HL black hole, it is always to study the Gibbs free energy:

$$G = H - TS = M - TS, \tag{12}$$

as the black hole mass M should be considered as the enthalpy H in the extended thermodynamic phase space. For the local thermodynamic stability, one can turn to the specific heat of black hole

$$C = \frac{\mathrm{d}M}{\mathrm{d}T} = \frac{\mathrm{d}M/\mathrm{d}r_+}{\mathrm{d}T/\mathrm{d}r_+}. \tag{13}$$

We present the above two quantities in the Appendix, as their forms are very complicated.

3. $P - V$ Criticality

According to (10), we can obtain the equation of state (EOS):

$$\begin{aligned} &1024P^2\pi^2 r_+^4 + 64k\left(d-1\right)\left(d-2\right)P\pi r_+^2 \\ &\quad + k^2 d\left(d-1\right)^2\left(d-4\right)\left(1-\epsilon^2\right) \\ &\quad - 8\left(d-1\right)\pi r_+\left(32P\pi r_+^2 + kd\left(d-1\right)\left(1-\epsilon^2\right)\right)T \\ &\quad = 0, \end{aligned} \tag{14}$$

which reflects the double-valuedness of the pressure in the extended thermodynamic phase space. On the other hand, we can derive the pressure $P(T, r_+)$ as

$$\begin{aligned} P\left(T, r_+\right) = {} & \frac{\left(d-1\right)}{32\pi r_+^2}\left(-kd + 2\left(k + 2\pi r_+ T\right)\right. \\ & \left. + \sqrt{4\left(k + 2\pi r_+ T\right)^2 - kd\epsilon^2\left(4k - kd + 8\pi r_+ T\right)}\right), \end{aligned} \tag{15}$$

while another one is the negative pressure branch. After taking a series expansion, it leads to

$$P = \frac{\left(d-1\right)T}{4r_+} - \frac{k\left(d-1\right)\left(d\epsilon^2 + d - 4\right)}{32\pi r_+^2} + \mathcal{O}\left(r_+^{-3}\right). \tag{16}$$

Comparing the above equation with the Van der Waals equation

$$P = \frac{T}{v-b} - \frac{a}{v^2} \simeq \frac{T}{v} + \frac{bT}{v^2} - \frac{a}{v^2} + \mathcal{O}\left(v^{-3}\right), \tag{17}$$

one can easily find the specific volume $v \propto r+$. Therefore we will just use the horizon radius r_+ in EOS instead of the specific volume v and study the $P - r+$ behavior in the following paper.

To consider the $P - V$ criticality, we can focus on

$$\begin{aligned} \frac{\partial P}{\partial r_+} &= 0, \\ \frac{\partial^2 P}{\partial r_+^2} &= 0, \end{aligned} \tag{18}$$

to find the critical points. As the direct differentiation of P (see (15)) is too complicated, we prefer to employ the implicit differentiation on EOS (see (14)) and the above equations, and we can derive two simple conditions:

$$\begin{aligned} &512P^2\pi r_+^3 + 16\left(d-1\right)\left(k\left(d-2\right)r_+ - 6\pi r_+^2 T\right)P \\ &\quad - kd\left(d-1\right)^2\left(1-\epsilon^2\right)T = 0, \end{aligned} \tag{19}$$

$$96P\pi r_+^2 - 12\left(d-1\right)\pi r_+ T + \left(d-1\right)\left(d-2\right)k = 0. \tag{20}$$

Table 1: The critical points of HL AdS spherical black holes.

Dimensions	P_c	T_c	ϵ_c	critical relation $P_c r_+/T_c$
Four	$\frac{2\sqrt{3}-1}{48\pi r_+^2}$	$\frac{\sqrt{3}}{6\pi r_+}$	$\pm\frac{2}{9}\sqrt{9+6\sqrt{3}}$	$\frac{6-\sqrt{3}}{24}\approx 0.178$
Five	$\frac{1}{8\pi r_+^2}$	$\frac{1}{2\pi r_+}$	$\pm\frac{2}{3}\sqrt{2}$	$\frac{1}{4}$
Six	$\frac{1}{8\pi r_+^2}$	$\frac{1}{2\pi r_+}$	$\pm\frac{2}{5}\sqrt{5}$	$\frac{1}{4}$

They lead to the critical points:

$$P_c = \frac{k(d-1)(12\pi X + 2 - d)}{96\pi r_+^2}, \tag{21}$$

$$T_c = \frac{kX}{r_+}, \tag{22}$$

by which, the EOS (see (14)) is simplified as an equation of ϵ and results in the following condition:

$$\epsilon_c = \pm 2\sqrt{\frac{(2(d-5)X\pi + 2d - 7)}{3d(d-4)}}, \tag{23}$$

where

$$X = \frac{3(d-3) \pm \sqrt{-3(d-1)(d-5)}}{12\pi}. \tag{24}$$

It is interesting that there is no critical volume or horizon radius, but the critical parameter ϵ. To start the physical discussion, we should firstly calculate the physical critical points with real and positive pressure and temperature, which lead to the constraint

$$1 \le d \le 5. \tag{25}$$

Namely, there is $P-V$ criticality only in four, five, and six dimensions. Besides, the physical critical points are simplified as

$$\begin{aligned}
P_c &= \frac{k\left(2\sqrt{3}\gamma - 1\right)}{48\pi r_+^2},\\
T_c &= \frac{\sqrt{3}k\gamma}{6\pi r_+},\\
\epsilon_c &= \pm\frac{2}{9}\sqrt{9+6\sqrt{3}\gamma},\\
d &= 3;\\
P_c &= \frac{k}{8\pi r_+^2},\\
T_c &= \frac{k}{2\pi r_+},\\
\epsilon_c &= \pm\frac{2}{3}\sqrt{2},\\
d &= 4;\\
P_c &= \frac{k}{8\pi r_+^2},\\
T_c &= \frac{k}{2\pi r_+},\\
\epsilon_c &= \pm\frac{2}{5}\sqrt{5},\\
d &= 5,
\end{aligned} \tag{26}$$

where $\gamma = \pm 1$. It is easy to find that positive pressure and temperature require the conditions $(k = 1, \gamma = 1)$. In four dimensions $(d = 3)$, the critical behavior is studied in [29].

Totally, we conclude that only four-, five-, and six-dimensional HL AdS black holes with spherical horizon have physical critical points, as shown in Table 1.

In next sections, we will give the physical discussion about the critical phenomena, i.e., the infinitely critical points and continuous second-order phase transitions. Especially for four-dimensional case, it is reported in [29], which is called "peculiar critical phenomena". In Section 4, we will conclude that they are actually the famous λ phase transition after studying the specific heat C_P of HL black holes. We also present the λ phase transition in five dimensions. In Section 5, it is found that six-dimensional HL AdS black holes only contain "normal" black hole phase and thus no similar phase transition. In Sections 6 and 7, we discuss the critical phenomena and calculate the critical exponents in different planes, by identifying parameter ϵ as the ordering field instead of pressure and temperature, respectively.

4. λ Phase Transition in Four and Five Dimensions

In this section, we discuss phase transitions in four and five dimensions. Back to the critical points in Table 1, it is interesting that there is no critical volume or horizon radius, but the critical parameter ϵ. Moreover, the $P-r_+$ oscillatory behavior and the classical "swallow tail" characterizing the first-order phase transition are controlled by the parameter ϵ, instead of the temperature T, which is clearly shown in Figure 1. The $P-r_+$ diagrams are plotted at the same temperature T and $G-T$ diagrams are plotted at the same pressure P. When $\epsilon > \epsilon_c$, there exist the small/large black holes phase transition in both four and five dimensions, which is exactly the same as the liquid/gas phase transition

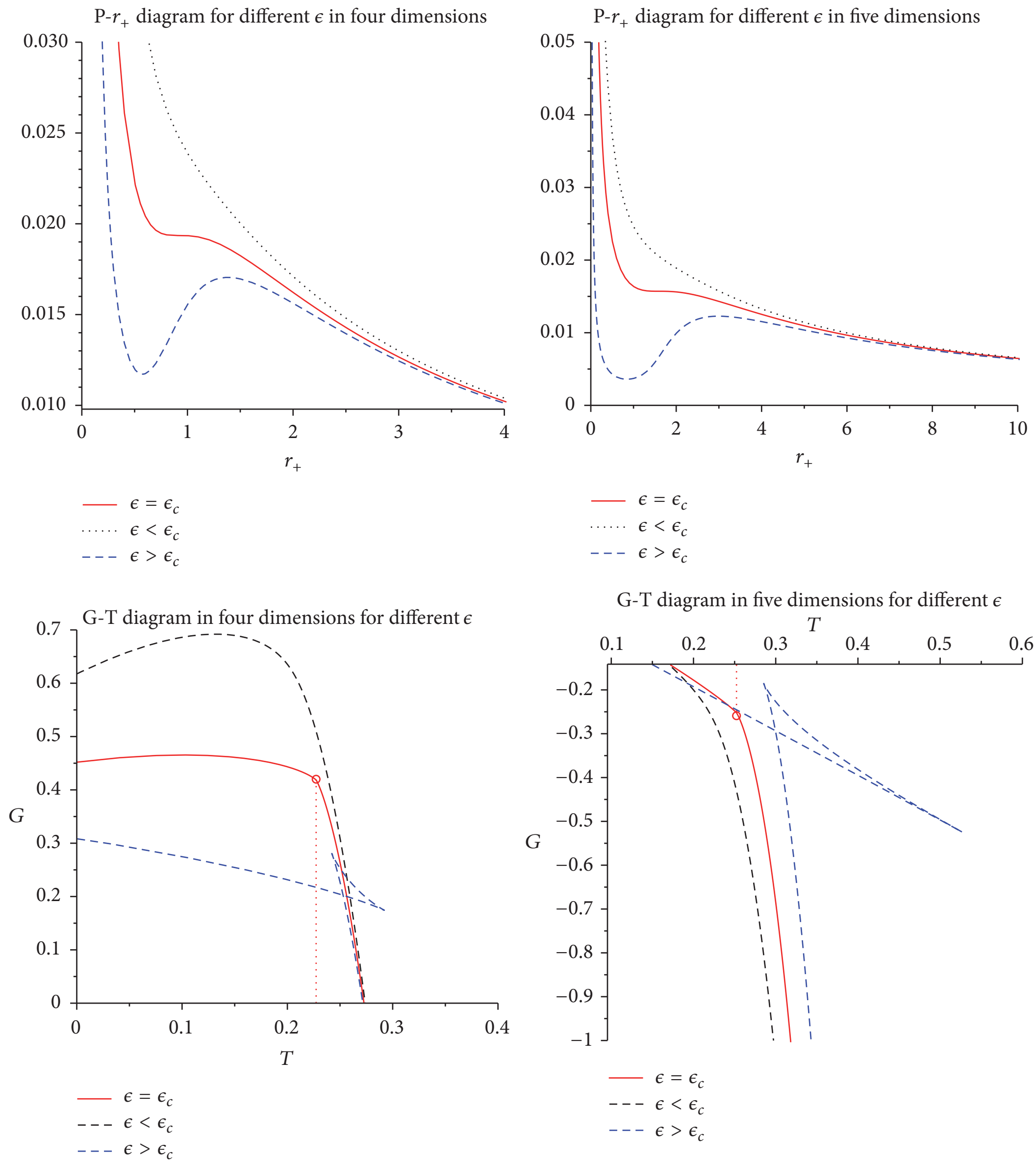


FIGURE 1: Curves of $P-r_+$ at the same temperature $T = 1/10$ and $G-T$ at the same pressure $P = 1/10$ in four and five dimensions for different ϵ. When $\epsilon > \epsilon_c$, there are the $P-r_+$ oscillatory behavior and the classical "swallow tail" characterizing the small/large black holes phase transition. When $\epsilon = \epsilon_c$, the second-order phase transition emerges and the dotted lines highlight the points where the second derivative of the Gibbs free energy G diverges.

of van der Waals fluid. Especially for $\epsilon = \epsilon_c$, the $P-r_+$ curves become a critical isotherm having an inflection point, and the second-order phase transition emerges.

On the other hand, black holes with spherical horizon and arbitrary mass M always have event horizon $r_+ > 0$, which is arbitrary as well. As a result, four and five dimensional HL AdS black holes with $\epsilon = \epsilon_c$ exhibit infinitely many critical points with arbitrary temperature T_c and horizon radius r_+ as

$$\begin{aligned} P_c &= \frac{\left(2\sqrt{3}-1\right)\pi}{4}T_c^2, \\ T_c &= \frac{\sqrt{3}}{6\pi r_+}, \\ d &= 3; \\ P_c &= \frac{\pi}{2}T_c^2, \\ T_c &= \frac{1}{2\pi r_+}, \\ d &= 4. \end{aligned} \tag{27}$$

Namely, every isotherm in $P-r_+$ diagrams is a critical isotherm, which has an inflection point at

$$r_+ = \begin{cases} \dfrac{\sqrt{3}}{6\pi T}, & d = 3; \\ \dfrac{1}{2\pi T}, & d = 4. \end{cases} \tag{28}$$

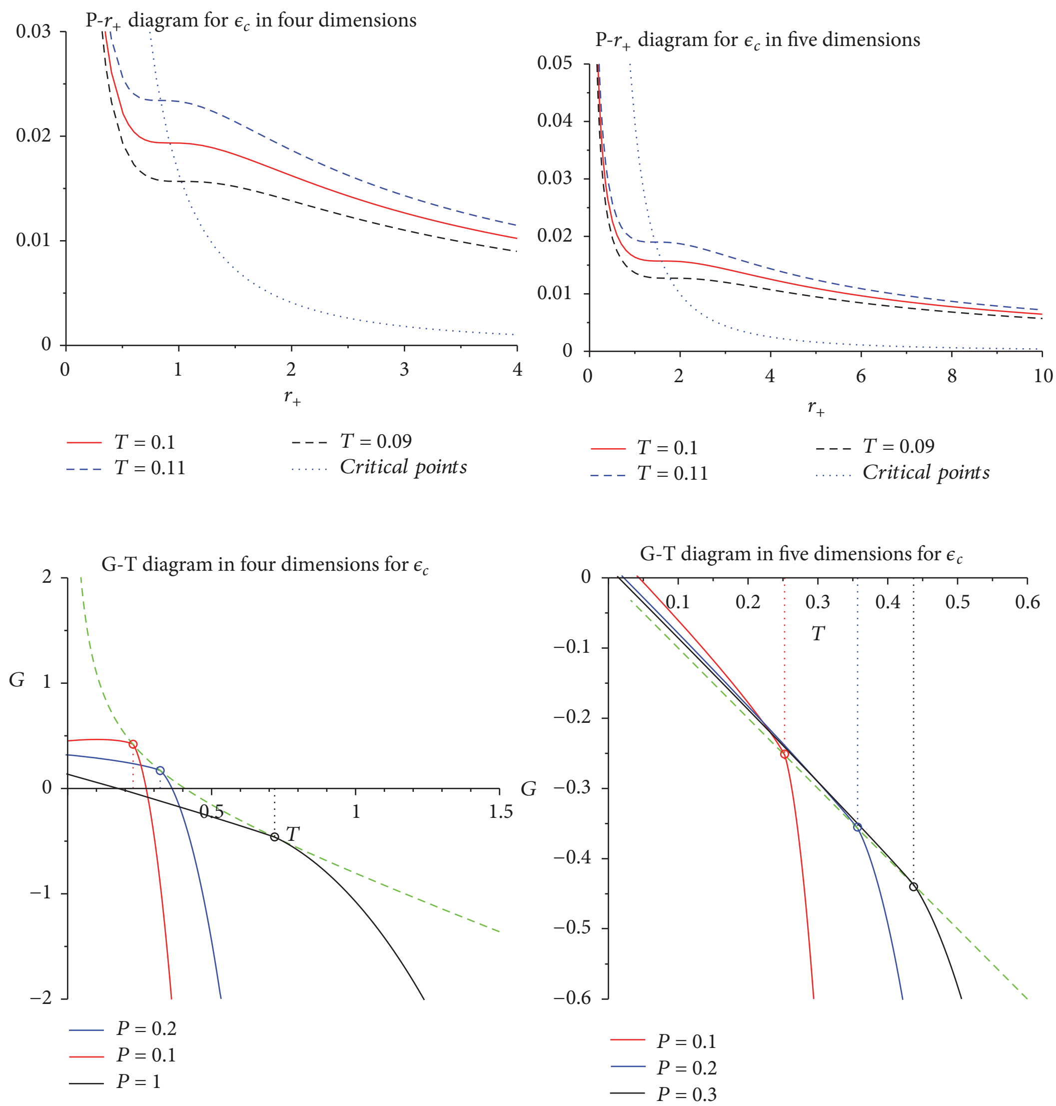


Figure 2: Curves of $P - r_+$ at the different temperature T and $G - T$ at the different pressure P in four and five dimensions for $\epsilon = \epsilon_c$. Every isotherm in $P - r_+$ diagrams is a critical isotherm, and the dotted lines describe the position of all critical points. A line of second-order phase transitions is shown in $G - T$ diagrams, and the dashed lines describe the position of all phase transition points.

One can find it easily in Figure 2, and the dotted lines in $P - r_+$ diagrams describe the position of all critical points (see (28)) at arbitrary temperature T. Besides, in $G - T$ diagrams, there is no first-order phase transition but rather a line of second-order phase transitions, for which the phase transition points are highlighted by the dotted lines in Figure 2. The dashed lines in $G - T$ diagrams describe the position of all phase transition points

$$T = \begin{cases} \sqrt{\dfrac{4P}{\left(2\sqrt{3} - 1\right)\pi}}, & d = 3; \\ \sqrt{\dfrac{2P}{\pi}}, & d = 4 \end{cases} \tag{29}$$

at arbitrary pressure P, where the second derivative of the Gibbs free energy G diverges.

To study the continuous second-order phase transitions, we focus on the specific heat C_P of black holes with $\epsilon = \epsilon_c$. From the $C_P - T$ diagrams in Figure 3, one can obtain that the specific heat C_P always diverges at the critical temperature (see (29)). This is the classical λ line, i.e., the line of second-order phase transitions, in $C_P - T$ diagrams, which was famous in the discussion of superfluidity of liquid ^{4}He. Similarly, in the $P - T$ parameter space (right two plots of Figure 3), a line of critical points, i.e., the λ line, separates the two phases of system, which are the "superfluid" black hole phase and "normal" black hole phase. To determine the two phases, we consider the $S - P$ diagrams, i.e., the middle two plots in Figure 3. The dashed lines highlight the critical pressure (see (27)). One can observe that the entropy of black holes with pressure smaller than the critical pressure is bigger, corresponding to the "normal" black hole phase. Another one is the "superfluid" black hole phase

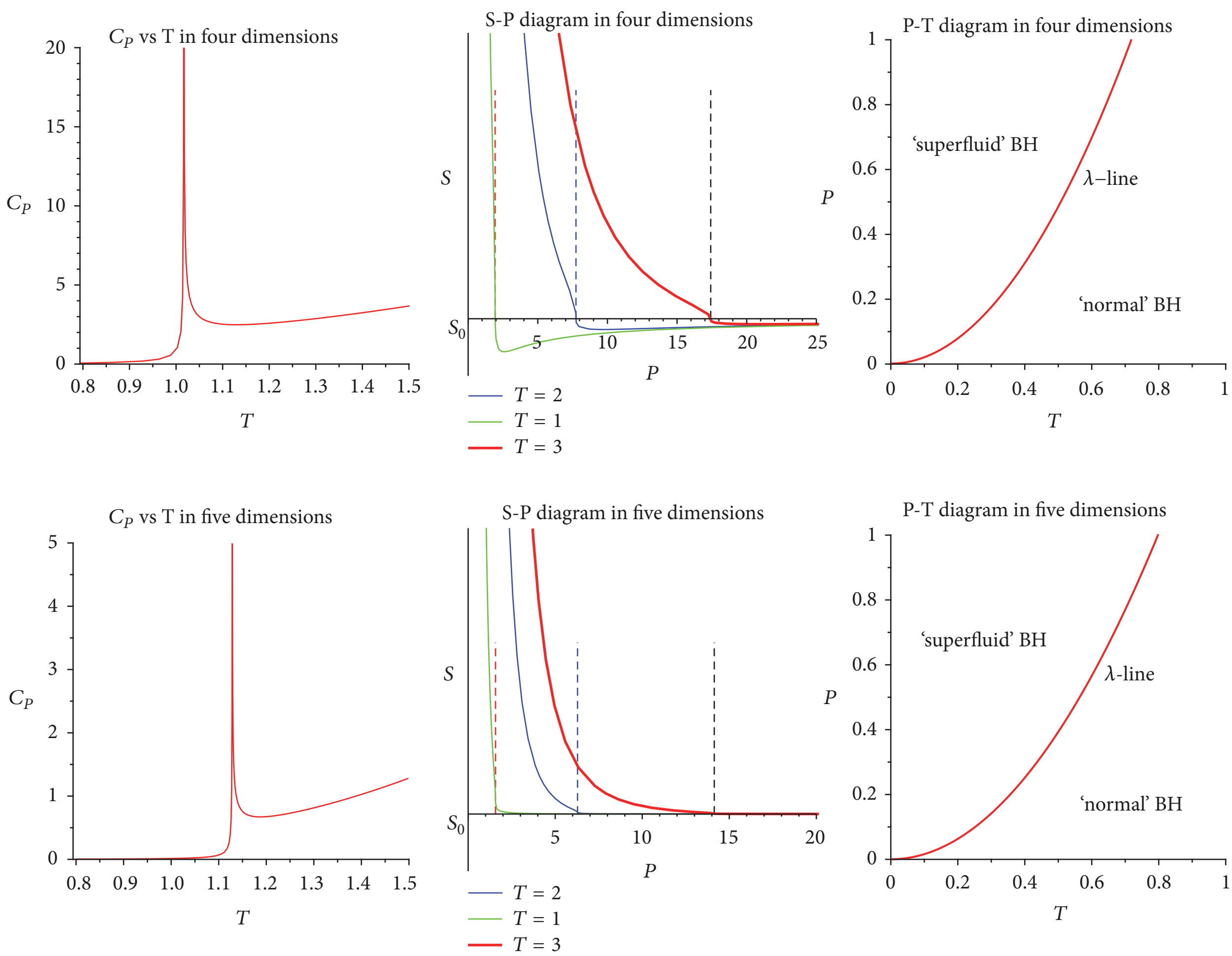


FIGURE 3: Curves of $C_P - T$ and $S - P$ and $P - T$ parameter space of black holes with $\epsilon = \epsilon_c$ in four and five dimensions. The λ line in $C_P - T$ and $P - T$ plots indicates the second-order phase transitions between "superfluid" black hole phase and "normal" black hole phase, which are distinguished in the $S - P$ plots by the dashed lines.

having smaller entropy and corresponding to black holes with pressure larger than the critical pressure. Particularly, the entropy of "superfluid" black hole phase is almost vanishing in five dimensions, while it seems to be negative in four dimensions. Note that the positivity of entropy depends on S_0, which is not clear and could be fixed by counting microscopic degrees of freedom in quantum theory of gravity as argued in [23].

Finally, we conclude that this continuous second-order phase transition between small/large black holes corresponds to the phase transition between "superfluid" black hole and "normal" black hole, which is firstly reported in Lovelock gravity with conformally coupled scalar field [20].

5. "Normal" Black Hole Phase in Six Dimensions

In six dimensions, there are critical points in $P - r_+$ plane as presented in Table 1. Actually, they describe infinitely many critical points with arbitrary temperature T_c and horizon radius r_+ for HL black holes with $\epsilon = \epsilon_c = \pm(2/5)\sqrt{5}$; i.e.,

$$P_c = \frac{\pi}{2} T_c^2, \qquad T_c = \frac{1}{2\pi r_+}, \tag{30}$$

which is exactly the same as the one in five dimensions. However, from Figure 4, it is strange that there is no $P - r_+$ oscillatory behavior and classical "swallow tail" for different ϵ; especially for $\epsilon = \epsilon_c$, one can never find the second-order phase transitions. This is different from the cases in four and five dimensions, and six-dimensional black holes seem to have no phase transition.

To interpret the strange critical point, we could study the specific heat C_P. One can look at the $C_P - T$ curve in the right plot of Figure 5. The specific heat C_P diverges at the critical point (phase transition point)

$$T = \sqrt{\frac{2P}{\pi}}, \tag{31}$$

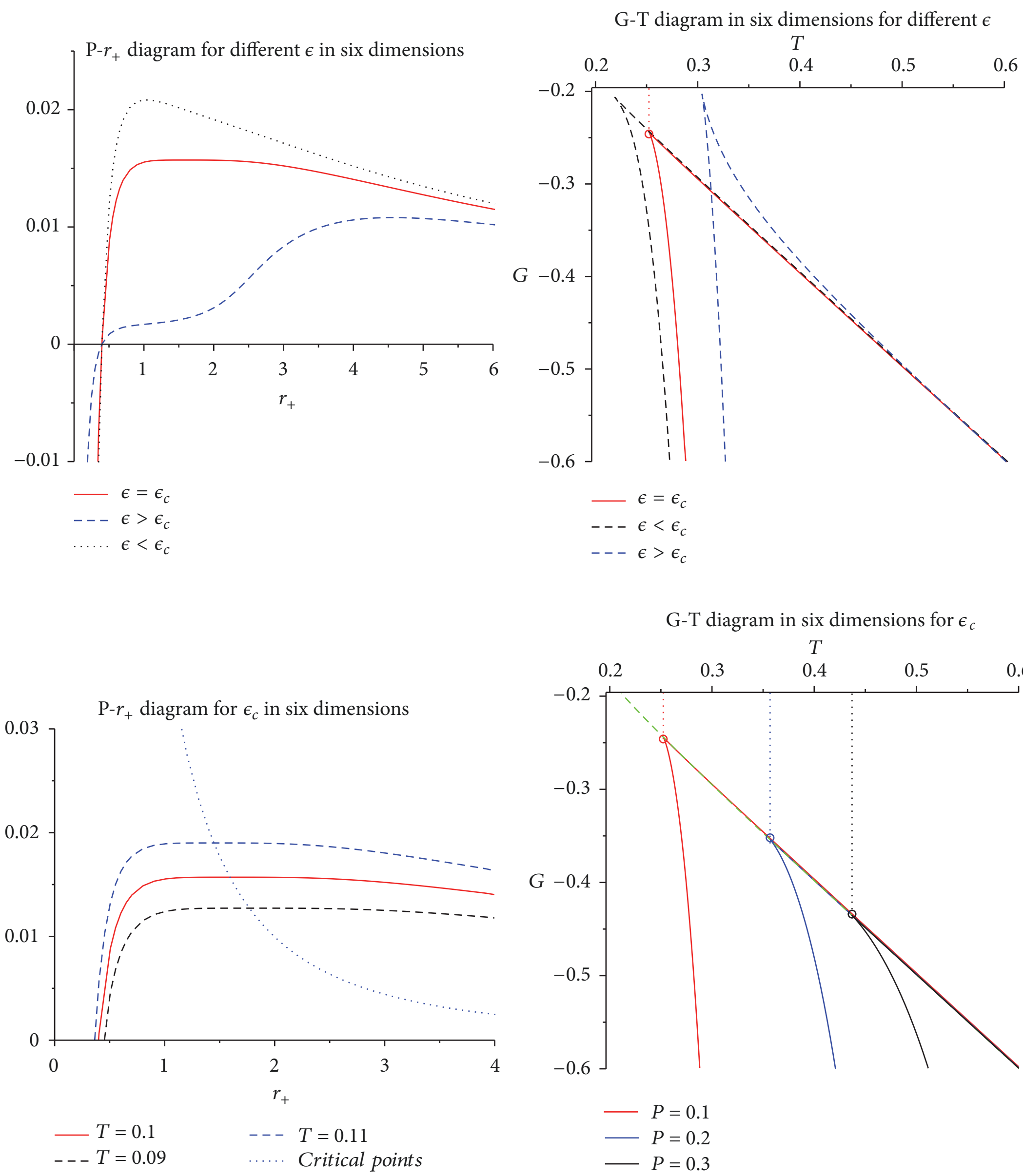


FIGURE 4: Curves of $P - r_+$ and $G - T$ in six dimensions for different ϵ and $\epsilon = \epsilon_c$. There is no $P - r_+$ oscillatory behavior and phase transition.

while the dashed line highlighting a lower bound of temperature destroys the λ line. It is easy to calculate the minimal temperature of six-dimensional HL AdS spherical black holes with $\epsilon = \epsilon_c$, for which the temperature is reduced to

$$T(r_+) = \frac{64P^2\pi^2 r_+^4 + 48P\pi r_+^2 + 1}{8\pi r_+ (1 + 8P\pi r_+^2)}. \tag{32}$$

Considering the first-order derivative

$$T' = \frac{(8P\pi r_+^2 - 1)^3}{8\pi r_+^2 (1 + 8P\pi r_+^2)^2}, \tag{33}$$

it leads to the minima at $r_+ = 1/\sqrt{8\pi P}$; i.e.,

$$T_{min} = \sqrt{\frac{2P}{\pi}}, \tag{34}$$

which is exactly the phase transition point (see (31)). This temperature bound could be treated as a physical temperature constraint, which cancels the "superfluid" black hole phase. This physical temperature constraint is equivalent to a upper bound of pressure, i.e., $P \le \pi T^2/2$, for arbitrary temperature. Then in the $S - P$ diagram in the middle one of Figure 5, one can also observe that there are no ("superfluid") black holes for $P > \pi T^2/2$, which correspond to the exact λ line in $P - T$ parameter space as shown in the right plot in Figure 5. Therefore, even six-dimensional HL AdS black holes exhibit infinitely many critical points in $P - v$ plane and the

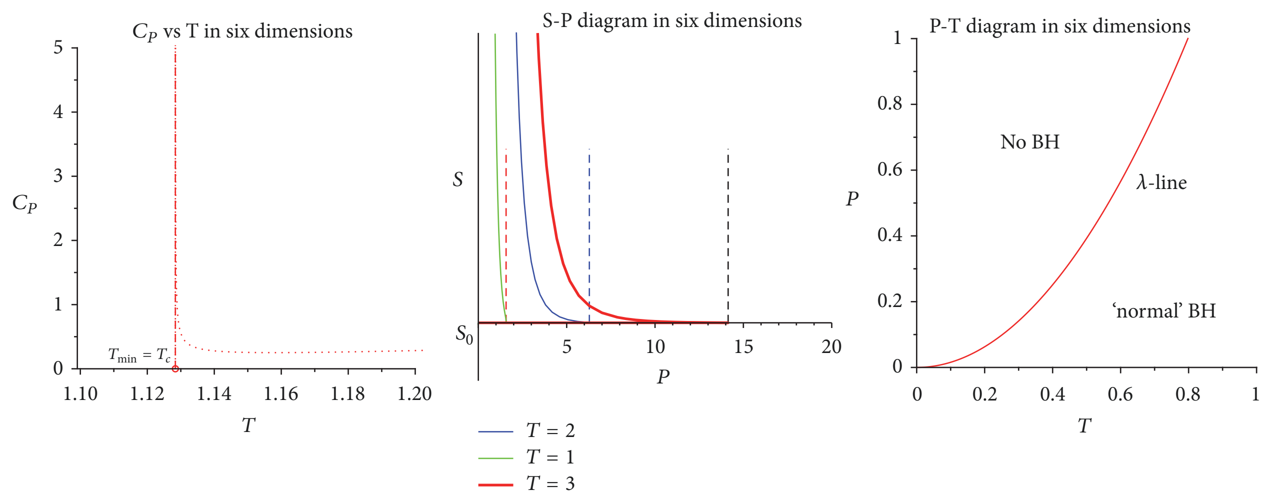


FIGURE 5: Curves of $C_P - T$ and $S - P$ and $P - T$ parameter space of black holes with $\epsilon = \epsilon_c$ in six dimensions. The dashed lines highlight the minimal temperature, which could be treated as a physical temperature constraint and cancel the "superfluid" black hole phase. Therefore there is no λ phase transition in six dimensions even when the specific heat diverges.

divergent points for specific heat; there does not exist λ phase transition, as they only contain the "normal" black hole phase and the "superfluid" black hole phase disappears in the $P - T$ parameter space due to the physical temperature constraint (see (34)).

6. Critical Phenomena in $\epsilon - r_+$ Plane

Because of the existence of infinite critical points, it is not able to calculate the critical exponents. Actually, to study the critical exponents for λ phase transitions of liquid ^{4}He, thermodynamic pressure is no longer the appropriate ordering field [32]. For the "superfluid" black hole, pressure should be instead of other parameters [20]. As for HL AdS black holes, there is only one option for the appropriate ordering field, i.e., parameter ϵ. Though ϵ is a dimensionless quantity, it does characterize the critical phenomena for four- and five-dimensional HL AdS black holes as shown later.

Firstly, we study the critical points in $\epsilon - r_+$ plane, for which the thermodynamic variables of EOS should be ϵ, r_+(i.e., v), T. Thus, we should firstly rewrite the EOS by rearranging the expression for temperature Eq. (10) for the chosen ordering field ϵ, which behaves as

$$\epsilon(T, r_+) = \pm\sqrt{\frac{\left(d(d-1)k + 32\pi P r_+^2\right)}{d(d-1)^2 k}\left((d-1) - \frac{32\pi P r_+^2}{8\pi r_+ T - (d-4)k}\right)}. \quad (35)$$

We follow the conditions

$$\frac{\partial \epsilon}{\partial r_+} = 0, \qquad \frac{\partial^2 \epsilon}{\partial r_+^2} = 0, \quad (36)$$

to find the critical points. However, the direct differentiation of ϵ is too complicated; we also prefer to employ the implicit differentiation on EOS (see (14)) and the above equations. After a careful calculation, we obtain the same two conditions (19) and (20). Thus, one can easily get the critical point:

$$\epsilon_c = \pm 2\sqrt{\frac{(2(d-5)X\pi + 2d - 7)}{3d(d-4)}}, \quad (37)$$

$$r_c = \frac{1}{4}\sqrt{\frac{k(d-1)(12\pi X + 2 - d)}{6\pi P}}, \quad (38)$$

$$T_c = 4kX\sqrt{\frac{6\pi P}{k(d-1)(12\pi X + 2 - d)}}, \quad (39)$$

where X has the value as (24) and P is a positive constant indicating AdS space-time.

Following the discussions in Section 3, it is shown that only four-, five-, and six-dimensional HL AdS black holes with spherical horizon have physical critical point, as shown in Table 2. Besides, in four and five dimensions, there exist the $\epsilon - r_+$ oscillatory behavior when $T > T_c$ as shown in Figure 6 and the classical "swallow tail" characterizing the small/large black holes phase transition when $\epsilon > \epsilon_c$ as shown in $G - T$ diagrams of Figure 1. In six dimensions, it is easy to check that the critical point leads to $\partial^3\epsilon/\partial r_+^3 = 0$. As a result, one can never find the $\epsilon - r_+$ oscillatory behavior as shown in Figure 6, and no first-order phase transition exists as shown in $G - T$ diagrams of Figure 4.

Finally, we consider the critical exponents in four and five dimensions. After introducing the dimensionless quantities

$$\Xi = \frac{\epsilon}{\epsilon_c}, \qquad \omega = \frac{r_+}{r_c} - 1, \qquad t = \frac{T}{T_c} - 1, \quad (40)$$

TABLE 2: The critical point of HL AdS spherical black holes in $\epsilon - r_+$ plane.

Dimensions	ϵ_c	r_c	T_c
Four	$\pm\frac{2}{9}\sqrt{9+6\sqrt{3}}$	$\sqrt{\frac{(2\sqrt{3}-1)}{48\pi P}}$	$\sqrt{\frac{4P}{(2\sqrt{3}-1)\pi}}$
Five	$\pm\frac{2}{3}\sqrt{2}$	$\sqrt{\frac{1}{8\pi P}}$	$\sqrt{\frac{2P}{\pi}}$
Six	$\pm\frac{2}{5}\sqrt{5}$	$\sqrt{\frac{1}{8\pi P}}$	$\sqrt{\frac{2P}{\pi}}$

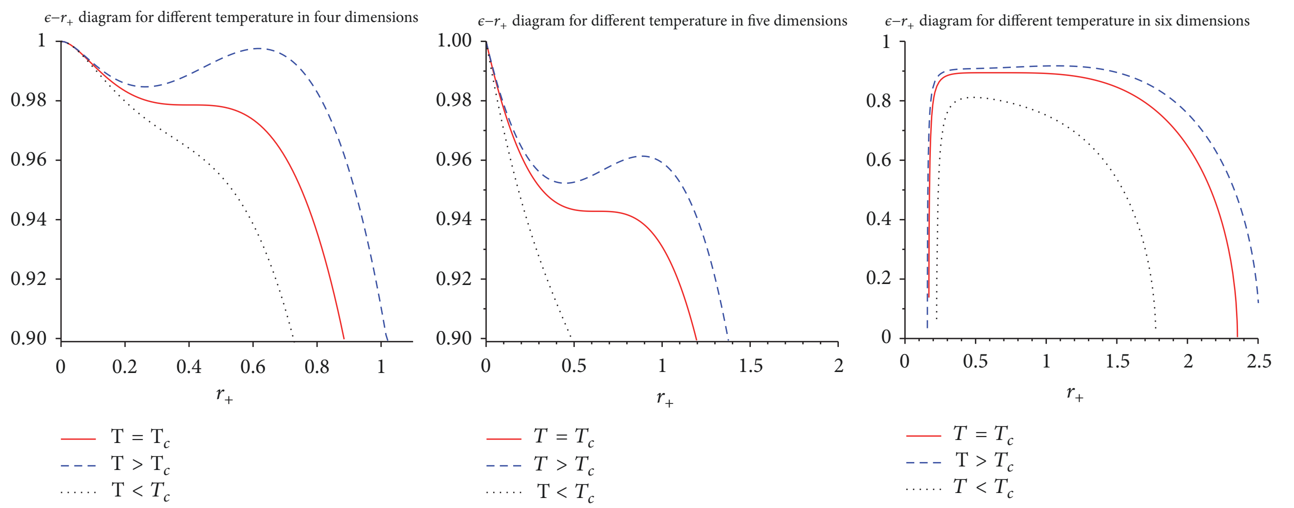


FIGURE 6: Curves of $\epsilon - r_+$ with $P = 1/10$ in four, five, and six dimensions for different temperature. In four and five dimensions, one can observe the $\epsilon - r_+$ oscillatory behavior when $T > T_c$, while no oscillatory behavior for six dimensions is observed.

the ordering field can be expanded near the critical point as

$$\Xi = \begin{cases} 1 + 0.115t + 0.234t\omega - 0.042\omega^3 + \mathcal{O}\left(t\omega^2, \omega^4\right), & d = 3, \\ 1 + \frac{1}{4}t + \frac{3}{8}t\omega - \frac{1}{16}\omega^3 + \mathcal{O}\left(t\omega^2, \omega^4\right), & d = 4, \end{cases} \quad (41)$$

where the coefficients for four-dimensional case all have complicated forms, and their approximate values are given. The above expansions both have the same form as that for the van der Waals fluid and the RN-AdS black hole [15]. As the system can be characterized by the critical exponents

$$\begin{aligned} C_\omega &\propto |t|^{-\alpha}, \\ \omega &\propto |t|^{\beta}, \\ \kappa_t &= -\omega^{-1}\left(\frac{\partial\omega}{\partial\Xi}\right)\bigg|_t \propto |t|^{-\gamma}, \\ \Xi &\propto |\omega|^{\delta}, \end{aligned} \quad (42)$$

one can obtain that

$$\begin{aligned} \alpha &= 0, \\ \beta &= \frac{1}{2}, \\ \gamma &= 1, \\ \delta &= 3. \end{aligned} \quad (43)$$

Particularly, $\omega \propto \sqrt{t}$ indicates that phase transition appears when $T > T_c$. Moreover, it is easy to check that they obey the scaling symmetry like the ordinary thermodynamic systems and in particular coincide with those for a superfluid [32].

7. Critical Phenomena in $P - r_+$ Plane with "Temperature" ϵ

It is interesting to find that the parameter ϵ controls the $P - r_+$ oscillatory behavior instead of the temperature T, other than the pressure P, as shown in $P - r_+$ diagrams of Figure 1. This indicates that there is still a critical phenomenon in $P - r_+$ plane. For this case, pressure could still be treated as the ordering field, while ϵ should be considered as "temperature". This is different from that for the first "superfluid" black holes [20].

Table 3: The critical point of HL AdS spherical black holes in $P - r_+$ plane with "temperature" ϵ.

Dimensions	P_c	r_c	ϵ_c
Four	$\dfrac{(2\sqrt{3}-1)\pi T^2}{4}$	$\dfrac{\sqrt{3}}{6\pi T}$	$\pm\dfrac{2}{9}\sqrt{9+6\sqrt{3}}$
Five	$\dfrac{\pi T^2}{2}$	$\dfrac{1}{2\pi T}$	$\pm\dfrac{2}{3}\sqrt{2}$
Six	$\dfrac{\pi T^2}{2}$	$\dfrac{1}{2\pi T}$	$\pm\dfrac{2}{5}\sqrt{5}$

To study the corresponding critical phenomena, we can still follow the procedure studying $P - r_+$ criticality in Section 3, which leads to the same three critical equations: (14), (19), and (20). As the thermodynamic variables of EOS for this case should be P, r_+ (i.e., ν), ϵ, we can find the following critical point:

$$P_c = \frac{(d-1)(12\pi X + 2 - d)}{96\pi k X^2} T^2, \tag{44}$$

$$r_c = \frac{kX}{T}, \tag{45}$$

with the critical "temperature"

$$\epsilon_c = \pm 2\sqrt{\frac{(2(d-5)X\pi + 2d - 7)}{3d(d-4)}}, \tag{46}$$

where X takes the value as (24) and T is a positive constant.

As similar as the discussions in Section 3, one can find that only four-, five-, and six-dimensional HL AdS black holes with spherical horizon have physical critical point, as shown in Table 3. In four and five dimensions, when "temperature" $\epsilon > \epsilon_c$, there exist the $P - r_+$ oscillatory behavior (Figure 1) and the classical "swallow tail" characterizing the small/large black holes phase transition as shown in $G - P$ diagrams of Figure 7. As for the case of six dimensions, it is easy to find that the critical point leads to $\partial^3 P/\partial r_+^3 = 0$. Therefore, one can not observe the $P - r_+$ oscillatory behavior as shown in Figure 4, and no first-order phase transition exists as shown in Figure 7.

Then, we can also calculate the critical exponents in four and five dimensions near the critical point. Similarly, we begin with the following dimensionless quantities:

$$p = \frac{P}{P_c},\qquad \omega = \frac{r_+}{r_c} - 1,\qquad \tau = \frac{\epsilon}{\epsilon_c} - 1. \tag{47}$$

The EOS can be reduced to the dimensionless case, for which its Taylor series expansion at the critical point takes the following forms:

$$p = \begin{cases} 1 - \dfrac{\left(56\sqrt{3} + 94\right)}{11}\tau + \left(10\sqrt{3} + 18\right)\tau\omega - \left(\sqrt{3} - 1\right)\omega^3 + \mathcal{O}\left(\tau\omega^2, \omega^4\right), & d = 3, \\ 1 - 8\tau + 12\tau\omega - \dfrac{1}{2}\omega^3 + \mathcal{O}\left(\tau\omega^2, \omega^4\right), & d = 4. \end{cases} \tag{48}$$

They still both have the same form as that for the van der Waals fluid and the RN-AdS black hole [15]. After introducing the critical exponents

$$C_\omega \propto |\tau|^{-\alpha},\qquad \omega \propto |\tau|^{\beta},\qquad \kappa_\tau = -\omega^{-1}\left(\frac{\partial\omega}{\partial p}\right)\Big|_\tau \propto |\tau|^{-\gamma},\qquad p \propto |\omega|^{\delta}, \tag{49}$$

it is easy to find

$$\alpha = 0,\qquad \beta = \frac{1}{2},\qquad \gamma = 1,\qquad \delta = 3, \tag{50}$$

which satisfy the scaling laws of the ordinary thermodynamic systems. Besides, $\omega \propto \sqrt{\tau}$ indicates that phase transition appears when $\epsilon > \epsilon_c$.

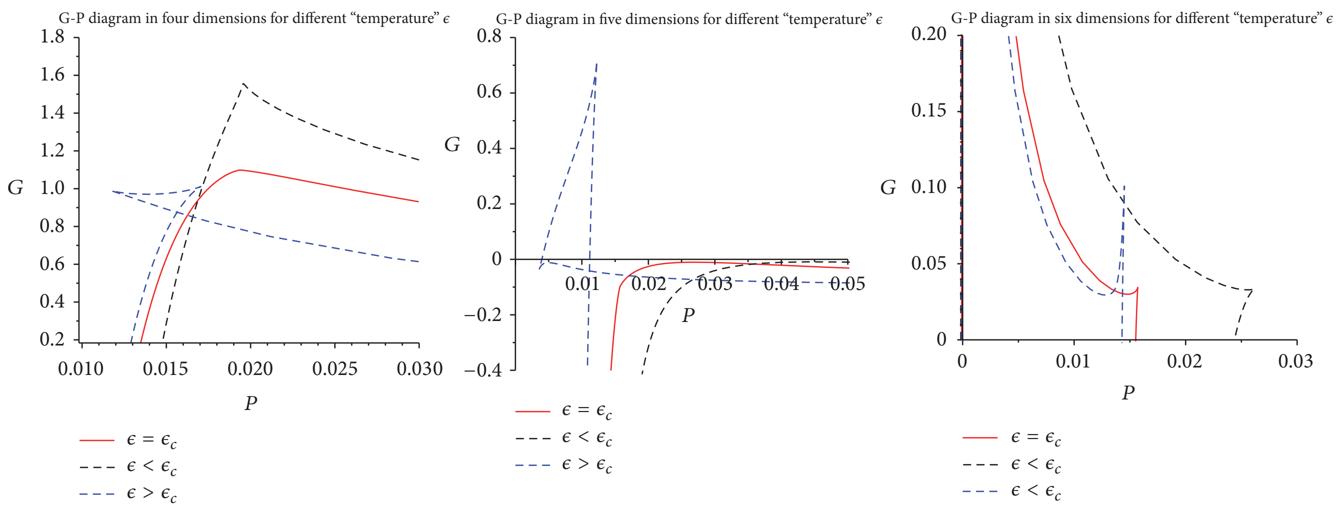


FIGURE 7: Curves of $G-P$ with $T = 1/10$ in four, five, and six dimensions for different "temperature" ϵ. In four and five dimensions, one can observe the classical "swallow tail" characterizing the small/large black holes phase transition when "temperature" $\epsilon > \epsilon_c$, while no "swallow tail" exists for six dimensions.

8. Discussion

In this paper, we study the extended thermodynamics of general dimensional HL AdS black holes and present another example of "superfluid" black holes. It is found that only four- and five-dimensional HL AdS black holes with spherical horizon have the λ phase transition, which correspond to the phase transition between "superfluid" black hole and "normal" black hole. After considering the behavior of entropy, the "superfluid" black hole phase and "normal" black hole phase are distinguished. Particularly, six-dimensional HL AdS black holes exhibit infinitely many critical points in $P-\nu$ plane and the divergent points for specific heat, for which they only contain the "normal" black hole phase and the "superfluid" black hole phase disappears due to the physical temperature constraint; therefore there is no similar phase transition. In more than six dimensions, there is no $P-V$ critical behavior. After identifying parameter ϵ as the ordering field instead of pressure and temperature, we study the critical phenomena in different planes of thermodynamical phase space. We also obtain the critical exponents in both planes, which are the same as the van der Waals fluid.

The "superfluid" black hole is firstly reported in Lovelock gravity with conformally coupled scalar field [20], which contains at least four free parameters in the action of theory. Comparing with it, the one in Horava gravity has only one free parameter ϵ. It is interesting to consider the necessary and sufficient conditions for general "superfluid" black hole; i.e., a black hole EOS must satisfy to display similar λ phase transition, which is still unclear. Besides, could the "superfluid" black holes appear in Einstein gravity? These are all interesting and left to be the future tasks.

Appendix

Gibbs Free Energy and Capacity

The Gibbs free energy is

$$G = \begin{cases} \dfrac{3k^2\left(1-\epsilon^2\right)}{16P\pi r_+} + 2kr_+ + \dfrac{16}{3}P\pi r_+^3 - \dfrac{\left(1024P^2\pi^2 r_+^4 + 128kP\pi r_+^2 - 12k^2\left(1-\epsilon^2\right)\right)}{16\pi r_+\left(32P\pi r_+^2 + 6k\left(1-\epsilon^2\right)\right)} \\ \quad \times \left(4\pi r_+^2\left(1+\dfrac{3k\left(1-\epsilon^2\right)\ln\left(r_+\right)}{8\pi r_+^2}\right) + S_0\right), & d = 3; \\ \dfrac{k^2 d r_+^{d-4}\left(1-\epsilon^2\right)}{16\left(d-2\right)P\pi} + \dfrac{4kr_+^{d-2}}{\left(d-1\right)\left(d-2\right)} + \dfrac{64P\pi r_+^6}{d\left(d-1\right)^2\left(d-2\right)} - \dfrac{1024P^2\pi^2 r_+^4 + 64k\left(d-1\right)\left(d-2\right)P\pi r_+^2 + k^2 d\left(d-1\right)^2\left(d-4\right)\left(1-\epsilon^2\right)}{8\left(d-1\right)\pi r_+\left(32P\pi r_+^2 + kd\left(d-1\right)\left(1-\epsilon^2\right)\right)} \\ \quad \times \left(\dfrac{16\pi r_+^{d-1}}{\left(d-1\right)^2\left(d-2\right)}\left(1+\dfrac{kd\left(d-1\right)^2\left(d-2\right)\left(1-\epsilon^2\right)}{32\left(d-2\right)\left(d-3\right)P\pi r_+^4}\right) + S_0\right), & d \geq 4. \end{cases} \quad \text{(A.1)}$$

The specific heat is

$$C_P = \frac{r_+^{d-3}}{2\left(d-1\right)\left(d-2\right)P}\left(32P\pi r_+^2 + kd\left(d-1\right)\left(1-\epsilon^2\right)\right)^2 \times \left(1024P^2\pi^2 r_+^4 + 64k\left(d-1\right)\left(d-2\right)P\pi r_+^2 + k^2 d\left(d-1\right)^2\left(d-4\right)\left(1-\epsilon^2\right)\right) \times \left(32768P^3\pi^3 r_+^6 - 1024k\left(d-1\right)\left(3d\epsilon^2 - d - 4\right)P^2\pi^2 r_+^4\right.$$

$$- 32k^2 d (d-1)^2 (d-8) P\pi r_+^2 \left(1-\epsilon^2\right)$$
$$- k^3 d^2 (d-1)^3 (d-4) \left(1-\epsilon^2\right)^2 \Big)^{-1}. \tag{A.2}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

Acknowledgments

Wei Xu was supported by the National Natural Science Foundation of China (NSFC), under Grants no. 11505065, no. 11374330, and no. 91636111, and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan).

References

[1] M. K. Parikh and F. Wilczek, "Hawking radiation as tunneling," *Physical Review Letters*, vol. 85, no. 24, pp. 5042–5045, 2000.

[2] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Physics Letters B*, vol. 379, no. 1–4, pp. 99–104, 1996.

[3] S. W. Hawking and D. N. Page, "Thermodynamics of black holes in anti-de Sitter space," *Communications in Mathematical Physics*, vol. 87, no. 4, pp. 577–588, 1982/83.

[4] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 3, pp. 505–532, 1998.

[5] J. Maldacena, "The large N limit of superconformal field theories and supergravity," *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 2, pp. 231–252, 1998.

[6] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Physics Letters B*, vol. 428, no. 1-2, pp. 105–114, 1998.

[7] E. Witten, "Anti de Sitter space and holography," *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 2, pp. 253–291, 1998.

[8] D. Kastor, S. Ray, and J. Traschen, "Enthalpy and the mechanics of AdS black holes," *Classical and Quantum Gravity*, vol. 26, no. 19, Article ID 195011, 2009.

[9] B. P. Dolan, "Pressure and volume in the first law of black hole thermodynamics," *Classical and Quantum Gravity*, vol. 28, no. 23, Article ID 235017, 2011.

[10] M. Cvetic, G. Gibbons, D. Kubiznak, and C. Pope, "Black hole enthalpy and an entropy inequality for the thermodynamic volume," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 84, Article ID 024037, 2011.

[11] B. P. Dolan, D. Kastor, D. Kubizňák, R. B. Mann, and J. Traschen, "Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 87, no. 10, Article ID 104017, 2013.

[12] D. Kastor, S. Ray, and J. Traschen, "Smarr formula and an extended first law for Lovelock gravity," *Classical and Quantum Gravity*, vol. 27, no. 23, Article ID 235014, 2010.

[13] A. Castro, N. Dehmami, G. Giribet, and D. Kastor, "On the universality of inner black hole mechanics and higher curvature gravity," *Journal of High Energy Physics*, vol. 2013, no. 07, 2013.

[14] B. M. El-Menoufi, B. Ett, D. Kastor, and J. Traschen, "Gravitational tension and thermodynamics of planar AdS spacetimes," *Classical and Quantum Gravity*, vol. 30, no. 15, Article ID 155003, p. 18, 2013.

[15] D. Kubiznak and R. B. Mann, "P-V criticality of charged AdS black holes," *Journal of High Energy Physics*, vol. 2012, no. 7, article 033, 2012.

[16] N. Altamirano, D. Kubiznak, R. B. Mann, and Z. Sherkatghanad, "Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume," *Galaxies*, vol. 2, 2014.

[17] D. Kubiznak, R. B. Mann, and M. Teo, "Black hole chemistry: thermodynamics with lambda," *Classical and Quantum Gravity*, vol. 34, no. 6, 2017.

[18] S. Wei and Y. Liu, "Insight into the microscopic structure of an AdS black hole from a thermodynamical phase transition," *Physical Review Letters*, vol. 115, no. 11, 2015, Erratum: vol. 116, no. 16, 2016.

[19] M. K. Zangeneh, A. Dehyadegari, A. Sheykhi, and R. Mann, "Microscopic origin of black hole reentrant phase transitions," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 97, no. 8, 2018.

[20] R. A. Hennigar, R. B. Mann, and E. Tjoa, "Superfluid black holes," *Physical Review Letters*, vol. 118, no. 2, Article ID 021301, 2017.

[21] P. Hořava, "Quantum gravity at a Lifshitz point," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 79, no. 8, Article ID 084008, 2009.

[22] H. Lü, J. Mei, and C. N. Pope, "Solutions to Hořava gravity," *Physical Review Letters*, vol. 103, no. 9, Article ID 091301, 2009.

[23] R. G. Cai, L. M. Caoand, and N. Ohta, "Topological black holes in Horava-Lifshitz gravity," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 80, no. 2, Article ID 024003, 2009.

[24] R.-G. Cai, L.-M. Cao, and N. Ohta, "Thermodynamics of black holes in Hořava–Lifshitz gravity," *Physics Letters B*, vol. 679, no. 5, pp. 504–509, 2009.

[25] Y. S. Myung, "Thermodynamics of black holes in the deformed Horava-Lifshitz gravity," *Physics Letters B: Particle Physics, Nuclear Physics and Cosmology*, vol. 678, no. 1, pp. 127–130, 2009.

[26] Q. Cao, Y. Chen, and K. Shao, "Black hole phase transitions in Hořava-Lifshitz gravity," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 83, no. 6, 2011.

[27] B. R. Majhi and D. Roychowdhury, "Phase transition and scaling behavior of topological charged black holes in Horava-Lifshitz gravity," *Classical and Quantum Gravity*, vol. 29, no. 24, Article ID 245012, 2012.

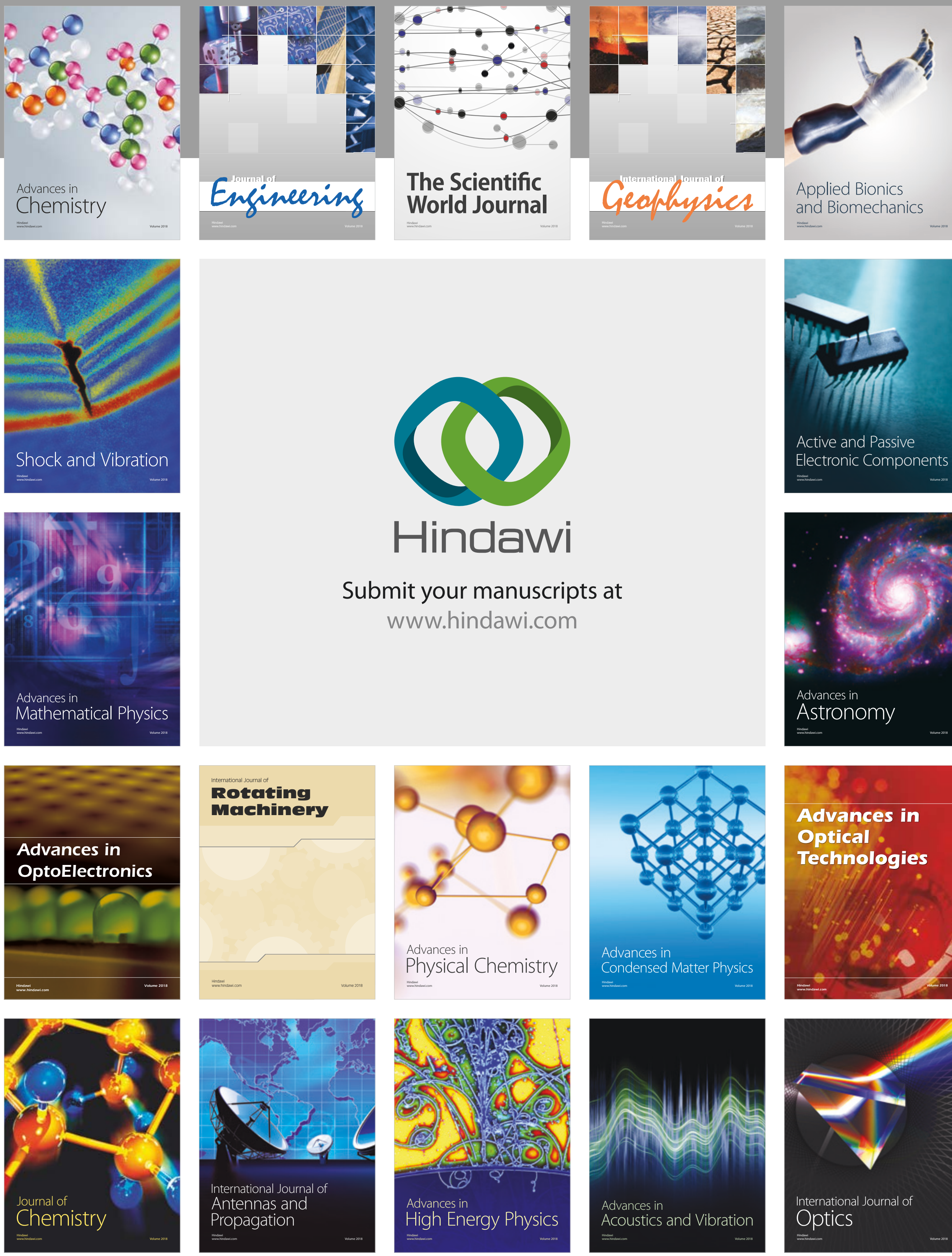
[28] J.-X. Mo, X.-X. Zeng, G.-Q. Li, X. Jiang, and W.-B. Liu, "A unified phase transition picture of the charged topological black hole in Hořava-Lifshitz gravity," *Journal of High Energy Physics*, vol. 2013, no. 56, 2013.

[29] M.-S. Ma and R.-H. Wang, "Peculiar P-V criticality of topological Hořava-Lifshitz black holes," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 96, no. 2, Article ID 024052, 2017.

[30] A. Wang, "Horava gravity at a Lifshitz point: a progress report," *International Journal of Modern Physics D: Gravitation, Astrophysics, Cosmology*, vol. 26, no. 7, Article ID 1730014, 2017.

[31] T. Li, Y. Qi, Y. Wu, and Y. Zhang, "Topological charged black holes in generalized Hořava-Lifshitz gravity," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 90, no. 12, 2014.

[32] P. B. Weichman, A. W. Harter, and D. L. Goodstein, " Criticality and superfluidity in liquid ," *Reviews of Modern Physics*, vol. 73, no. 1, pp. 1–15, 2001.


Advances in Chemistry
Journal of Engineering
The Scientific World Journal
International Journal of Geophysics
Applied Bionics and Biomechanics
Shock and Vibration
Hindawi
Submit your manuscripts at
www.hindawi.com
Active and Passive Electronic Components
Advances in Mathematical Physics
Advances in Astronomy
Advances in OptoElectronics
International Journal of Rotating Machinery
Advances in Physical Chemistry
Advances in Condensed Matter Physics
Advances in Optical Technologies
Journal of Chemistry
International Journal of Antennas and Propagation
Advances in High Energy Physics
Advances in Acoustics and Vibration
International Journal of Optics