

## Research Article

# Nonrelativistic Arbitrary $l$ -States of Quarkonium through Asymptotic Iteration Method

Hakan Ciftci <sup>1</sup> and Hasan Fatih Kisoglu <sup>2</sup>

<sup>1</sup>Gazi Üniversitesi, Fen-Edebiyat Fakültesi, Fizik Bölümü, 06500 Teknikokullar-Ankara, Turkey

<sup>2</sup>Department of Basic Sciences, Faculty of Maritime, Mersin University, Mersin, Turkey

Correspondence should be addressed to Hakan Ciftci; [hciftci@gazi.edu.tr](mailto:hciftci@gazi.edu.tr) and Hasan Fatih Kisoglu; [hasanfatihk@mersin.edu.tr](mailto:hasanfatihk@mersin.edu.tr)

Received 15 February 2018; Accepted 27 March 2018; Published 29 May 2018

Academic Editor: Chun-Sheng Jia

Copyright © 2018 Hakan Ciftci and Hasan Fatih Kisoglu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. The publication of this article was funded by SCOAP<sup>3</sup>.

The energy eigenvalues with any  $l \neq 0$  states and mass of heavy quark-antiquark system (quarkonium) are obtained by using Asymptotic Iteration Method in the view of nonrelativistic quantum chromodynamics, in which the quarks are considered as spinless for easiness and are bounded by Cornell potential. A semianalytical formula for energy eigenvalues and mass is achieved via the method in scope of the perturbation theory. The accuracy of this formula is checked by comparing the eigenvalues with the ones numerically obtained in this study and with exact ones in literature. Furthermore, semianalytical formula is applied to  $c\bar{c}$ ,  $b\bar{b}$ , and  $c\bar{b}$  meson systems for comparing the masses with the experimental data.

## 1. Introduction

Investigation of an atomic or subatomic system is done by achieving an energy spectrum of the system. This is carried out, generally, for the events in which the system is bounded by a potential function. Besides, the scattering states or resonance states can also be observed in the investigation of the system. The eigenvalues (or eigenenergies) of Hamiltonian of this system is obtained for a given potential function. In order to do this, various mathematical methods are used in quantum mechanics. One of these, named Asymptotic Iteration Method (AIM), has been commonly used since 2003 [1]. AIM can be used for analytically as well as numerically (or approximately) solvable problems [2–4]. Moreover, it can be used for obtaining the perturbative energy eigenvalues of the system without any need of the unperturbative eigenstate [5, 6].

As a subatomic system, a quarkonium that is composed of a heavy quark-antiquark ( $q\bar{q}$ ) pair has attracted attention of particle physicists since the first half of 1970, and [7–11] are just a few studies of them. In most of these studies, for easiness, the system is examined via Schrödinger equation in nonrelativistic quantum chromodynamics (NRQCD), assuming that the quarks are spinless [12–15]. Cornell potential is one of the potential functions that represent

interactions between the quarks in such a  $q\bar{q}$  system. It is used for obtaining the mass and energy spectrum of the quarkonium and obtaining the hadron decay widths [7–9, 16]. Cornell potential is given as

$$V(r) = -\frac{A}{r} + B^2 r \quad (1)$$

where  $A$  and  $B$  are positive constants ( $B$  is in energy dimension). As is seen in (1), Cornell potential has two parts: one is the Coulombic term and the other is the linear part. For obtaining the energy levels and mass of the quarkonium,  $A$  and  $B$  may be fitted to the first-few states. Therefore, the full spectrum of the quarkonium can be constructed through these potential parameters.

In literature, it is possible to find many studies in which the solutions of Schrödinger equation for Cornell potential have been obtained. For example, in [17], Hall has found an approximate energy formula to construct an energy spectrum of Schrödinger equation for Cornell potential, under some conditions. Jacobs et al. [13] have compared the eigenvalues of Schrödinger and spinless Salpeter equations in the cases of Cornell potential and Wisconsin potential [18]. Vega and friends have obtained, for  $l=0$  states, the energy spectrum,

mass, and wavefunctions at the origin for  $c\bar{c}$ ,  $b\bar{b}$ , and  $b\bar{c}$  mesons by using the usual variation method in the scope of supersymmetric quantum mechanics (SUSYQM) [19, 20], in [12]. They have also compared their results with the exact ones in literature and with the experimental data.

In this study, we attempted to get the energy eigenvalues (for any  $l \neq 0$  states) and masses of heavy mesons by using Asymptotic Iteration Method in the view of NRQCD, in which the quarks are considered as spinless for easiness and are bounded by Cornell potential. We achieved a semianalytical formula for constructing the energy spectrum and obtaining the masses of the mesons, using the method in scope of the perturbation theory. The accuracy of this formula was cross-checked by comparing the eigenvalues with the ones numerically obtained in this study and with the exact ones in literature. Furthermore, semianalytical formula was applied to  $c\bar{c}$ ,  $b\bar{b}$ , and  $b\bar{c}$  heavy mesons for comparing the masses with the experimental data.

AIM has been firstly applied to Schrödinger equation for Cornell potential by Hall and Saad in [21]. They have used Airy function as an asymptotic form of the wavefunction and have got highly-accurate numerical results in their study. Alternatively, we obtained a semianalytical mass-energy formula for quarkonium by having differential equation which gives polynomial solutions for asymptotic forms of the wavefunction of the system.

This paper is organized as follows: we give a short summary of AIM in Section 2, while Section 3 includes the main problem. In Section 4, we give numerical results for the eigenenergies and obtain semianalytical energy formula by applying perturbation theory to our problem in the view of AIM. Furthermore, in Section 4, we compare our energy spectrum and masses with the exact ones in literature and with the experimental data. Finally, Section 5 includes some comments about our results.

## 2. The Asymptotic Iteration Method (AIM)

According to the organization of the paper, we summed up AIM in this section, while it is comprehensively introduced in [1]. The AIM is used to solve second-order homogeneous linear differential equations in the following form:

$$y''(x) = \lambda_0(x) y'(x) + s_0(x) y(x) \quad (2)$$

where  $\lambda_0(x)$  and  $s_0(x)$  have continuous derivatives in the defined interval of the  $x$  independent variable. If there is an asymptotic condition such as

$$\frac{s_n}{s_{n-1}} = \frac{\lambda_n}{\lambda_{n-1}} \equiv \alpha \quad (3)$$

for  $n \in \mathbb{Z}^+$ , where  $n$  is large enough, the general solution of (2) is obtained as

$$y(x) = \exp\left(-\int^x \alpha(t) dt\right) \cdot \left[ C_2 + C_1 \int^x \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau)) d\tau\right) dt \right] \quad (4)$$

with the functions

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1} \\ s_n &= s'_{n-1} + s_0 \lambda_{n-1} \end{aligned} \quad (5)$$

As a field of application, AIM can be used to deal with Schrödinger equation (or energy eigenvalue problem) in mathematical physics. The eigenvalues can be obtained through the following quantization condition:

$$\begin{aligned} \delta_n(x, E) &= s_n(x, E) \lambda_{n-1}(x, E) - \lambda_n(x, E) s_{n-1}(x, E) \\ &= 0 \end{aligned} \quad (6)$$

If the energy eigenvalues ( $E$ ) can be obtained from (6), independently from the  $x$  variable, the problem is exactly solvable. In this case, the eigenvalue and eigenfunction of  $n$ th energy level can be derived in explicit algebraic form via  $n$  iterations. However, there are limited numbers of suitable potentials for this case.

As for the approximately (or numerically) solvable problems,  $\delta_n$  depends on both  $x$  and  $E$ . In this case, an appropriate value,  $x \equiv x_0$ , should be determined to solve  $\delta_n(x, E) = 0$  with respect to  $E$  [2, 6]. The energy eigenvalue of an  $n$ th level is obtained through  $q$  iterations where  $q \geq n$ .

## 3. Formulation of the Problem

Consider the following Cornell potential:

$$V(r) = -\frac{A}{r} + B^2 r \quad (7)$$

where  $A, B$  are real and positive constants ( $B$  is in energy dimension) and  $r \in (0, \infty)$ . If we substitute  $V(r)$  into Schrödinger equation in three dimensions, we have

$$\left\{ \frac{d^2}{dr^2} + \varepsilon - \left[ -\frac{\alpha}{r} + \rho r + \frac{l(l+1)}{r^2} \right] \right\} \Psi(r) = 0 \quad (8)$$

where  $\varepsilon = 2\mu E_n$ ,  $\alpha = 2\mu A$ , and  $\rho = 2\mu B^2$ .  $E_n$  and  $\mu = m_1 m_2 / (m_1 + m_2)$  are energy eigenvalue of  $n$ th level and reduced mass of the  $q\bar{q}$  system, respectively ( $m_1$  and  $m_2$  are quark masses). Besides, we study in natural units (i.e.,  $\hbar, c = 1$ ) for the system. After changing the variable, in (8), as  $r = u^2$ , then substituting  $\Psi(u) = u^{1/2} g(u)$ , we get

$$\begin{aligned} g''(u) &+ \left[ 4\varepsilon u^2 + 4\alpha - 4\rho u^4 - \frac{4l(l+1) + 3/4}{u^2} \right] g(u) \\ &= 0 \end{aligned} \quad (9)$$

If one puts  $g(z) = z^{\gamma+1} e^{-z^3/3} f(z)$  into (9), in accordance with the domain of the problem, we have

$$\begin{aligned} f''(z) &= 2 \left[ z^2 - \frac{\gamma+1}{z} \right] f'(z) \\ &+ [2(\gamma+2)z - \sigma z^2 - \omega] f(z) \end{aligned} \quad (10)$$

TABLE 1: Comparisons of the perturbative energy eigenvalues ( $E_{pert}$ ) (in GeV) with those obtained by direct application of AIM ( $E_{nl}$ ) for the potential parameters  $A = 1$  and  $B = 1$  GeV, and for the reduced mass  $\mu = 1/2$  GeV.

$n$	$E_{n0}$	$E_{pert}$	$l$	$E_{0l}$	$E_{pert}$
0	1.39788	1.41015	0	1.39788	1.41015
1	3.47509	3.47509	1	2.82565	2.82690
2	5.03291	5.03224	2	3.85058	3.85089
3	6.37015	6.36948	3	4.72675	4.72687
4	7.57493	-	4	5.51698	-
5	8.68791	-	5	6.24840	-

where  $\omega = 4\alpha/(4\rho)^{1/3}$ ,  $\sigma = 4\epsilon/(4\rho)^{2/3}$ ,  $\gamma = 2l + 1/2$ , and  $z = (4\rho)^{1/6}u$ . The final equation is suitable for applying AIM. After this point, we can apply AIM to the problem in two different ways: one is direct application (i.e., approximate solution) to get the numerical results and the other is usage of the method in scope of perturbation theory to obtain perturbative energies through a perturbation expansion as follows:

$$\sigma = \sigma_0 + \omega\sigma_1 + \omega^2\sigma_2 + \dots \quad (11)$$

where  $\sigma_0, \sigma_1, \sigma_2, \dots$  are perturbation expansion coefficients. These can be obtained independently from the potential parameters. Thus, we can get a semianalytical formula for the energy eigenvalues. One can also achieve the mass-energy of the system by using this formula, as given in Section 4.

**3.1. Numerical Results.** In this section, we directly apply AIM to (10) to get the energy eigenvalues for different potential parameters, and we compare our results with the perturbative energies, for which (28) in the next section has been used.

$$f''(z) = 2 \left[ z^2 - \frac{\gamma+1}{z} \right] f'(z) + [2(\gamma+2)z - \sigma z^2 - \omega] f(z) \quad (12)$$

From (12), it is easily seen that  $\lambda_0(z) = 2[z^2 - (\gamma+1)/z]$  and  $s_0(z) = 2(\gamma+2)z - \sigma z^2 - \omega$  according to (2). We tabulate the results of direct application of AIM in Tables 1, 2, and 3. For simplicity, in the calculations, the reduced mass has been considered  $\mu = 1/2$  GeV. In Table 1 the potential parameters have been chosen as  $A = 1$  and  $B = 1$  GeV while  $A = 1$ ,  $B = 0.1$  GeV in Table 2, and  $A = 1$ ,  $B = 10$  GeV in Table 3.  $E_{pert}$ , seen in the tables, is for the comparison and has been obtained by using (28).

As can be seen from Tables 1–3, the perturbative energy eigenvalues are in very good agreement with the numerically obtained ones, even for small values of the parameter  $B$ . Furthermore, they are in accordance with each other for  $B \geq 1$  GeV, while  $A = 1$  (see in Tables 1 and 3). Additionally, this agreement is much better for higher quantum states. The perturbative eigenvalues are a little bit different from that obtained as numerically, for  $B < 1$  GeV,  $A = 1$ , and the lower quantum states (see in Table 2). However, they are in agreement for the higher levels.

TABLE 2: Comparisons of the perturbative energy eigenvalues ( $E_{pert}$ ) (in GeV) with those obtained by direct application of AIM ( $E_{nl}$ ) for the potential parameters,  $A = 1$  and  $B = 0.1$  GeV, and for the reduced mass  $\mu = 1/2$  GeV.

$n$	$E_{n0}$	$E_{pert}$	$l$	$E_{0l}$	$E_{pert}$
0	-0.221031	-0.164433	0	-0.221031	-0.164433
1	0.0347222	0.033627	1	0.0174006	0.023501
2	0.141913	0.138477	2	0.102472	0.104008
3	0.220287	0.217229	3	0.159831	0.160406
4	0.286111	-	4	0.206238	-
5	0.344602	-	5	0.246681	-

TABLE 3: Comparisons of the perturbative energy eigenvalues ( $E_{pert}$ ) (in GeV) with those obtained by direct application of AIM ( $E_{nl}$ ) for the potential parameters,  $A = 1$  and  $B = 10$  GeV, and for the reduced mass  $\mu = 1/2$  GeV.

$n$	$E_{n0}$	$E_{pert}$	$l$	$E_{0l}$	$E_{pert}$
0	46.4022	46.4047	0	46.4022	46.4047
1	85.3393	85.3394	1	70.0161	70.0165
2	116.729	116.729	2	89.7154	89.7154
3	144.315	144.315	3	107.334	107.334
4	169.461	-	4	123.562	-
5	192.851	-	5	138.761	-

## 4. Perturbation Theory

Although the usage of perturbation method in the frame of AIM is comprehensively introduced in [5], we give a summary about the methodology in this section, assuming that the potential of a system is written as

$$V(x) = V_0(x) + \eta V_p(x) \quad (13)$$

where  $V_0(x)$  is solvable (unperturbed Hamiltonian) potential.  $V_p(x)$  and  $\eta$  are potential of the perturbed Hamiltonian and perturbation expansion parameter, respectively. The Schrödinger equation then reads

$$\left( -\frac{d^2}{dx^2} + V_0(x) + \eta V_p(x) \right) \Psi(x) = E\Psi(x) \quad (14)$$

where  $E_n$  eigenvalues are written as a series expansion of  $j$ th-order correction  $E_n^{(j)}$  as follows:

$$E_n = E_n^{(0)} + \eta E_n^{(1)} + \eta^2 E_n^{(2)} + \dots = \sum_{j=0}^{\infty} \eta^j E_n^{(j)} \quad (15)$$

After substituting  $\psi(x) = \psi_0(x)f(x)$  in (14), one can obtain the following equation for  $f(x)$ :

$$f''(x) = \lambda_0(x, \eta, E) f'(x) + s_0(x, \eta, E) f(x) \quad (16)$$

and the termination condition in this case can be written as

$$\begin{aligned} \delta_n(x, \eta, E) &= s_n(x, \eta, E) \lambda_{n-1}(x, \eta, E) \\ &\quad - \lambda_n(x, \eta, E) s_{n-1}(x, \eta, E) = 0 \end{aligned} \quad (17)$$

Once  $\delta_n(x, \eta, E)$  is expanded about  $\eta = 0$ , we obtain

$$\begin{aligned} \delta_n(x, \eta, E) &= \delta_n(x, 0, E) + \frac{\eta}{1!} \frac{\partial \delta_n(x, \eta, E)}{\partial \eta} \Big|_{\eta=0} \\ &+ \frac{\eta^2}{2!} \frac{\partial^2 \delta_n(x, \eta, E)}{\partial \eta^2} \Big|_{\eta=0} + \dots \\ &= \sum_{k=0}^{\infty} \eta^k \delta_n^{(k)}(x, E) = 0 \end{aligned} \quad (18)$$

where  $\delta_n^{(k)}(x, E) = (1/k!) (\partial^k \delta_n(x, \eta, E) / \partial \eta^k) |_{\eta=0}$ .

According to perturbation method in the framework of AIM, solving the equation  $\delta_n(x, 0, E) = 0$  with respect to (unknown)  $E$  gives  $E_n^{(0)}$  (eigenvalues of unperturbed Hamiltonian), equation  $\delta_n^{(1)}(x, E) = 0$  gives  $E_n^{(1)}$  (first-order correction to  $E_n$ ),  $\delta_n^{(2)}(x, E)$  gives  $E_n^{(2)}$  (second-order correction to  $E_n$ ), and so on. Besides, the perturbative eigenfunctions can be achieved in the same vein with the eigenvalues. This is an alluring feature of the AIM usage in the perturbation theory for obtaining the eigenfunctions  $f_n(x)$  given as follows:

$$f_n(x) = \exp\left(-\int^x \alpha_n(t, \eta) dt\right) \quad (19)$$

where  $\alpha_n(t, \eta) \equiv s_n(t, \eta) / \lambda_n(t, \eta)$ .  $\alpha_n(t, \eta)$  is expanded about  $\eta = 0$  in a similar manner, done for obtaining the eigenvalues. So,

$$\alpha_n(t, \eta) = \sum_{k=0}^{\infty} \eta^k \alpha_n^{(k)}(t) \quad (20)$$

where  $\alpha_n^{(k)} = (1/k!) (\partial^k \alpha_n(x, \eta) / \partial \eta^k) |_{\eta=0}$ . Thus, perturbation expansion of  $f_n(x)$  is written as follows:

$$\begin{aligned} f_n(x) &= \exp\left[\sum_{k=0}^{\infty} \eta^k \left(-\int^x \alpha_n^{(k)}(t) dt\right)\right] \\ &= \prod_{k=0}^{\infty} f_n^{(k)}(x) \end{aligned} \quad (21)$$

where  $k$ th-order correction  $f_n^{(k)}(x)$  to  $f_n(x)$  is

$$f_n^{(k)}(x) = \eta^k \left(-\int^x \alpha_n^{(k)}(t) dt\right) \quad (22)$$

**4.1. Perturbation Theory for the Cornell Potential.** For our problem, we may apply the perturbation expansion which has been elucidated in previous section to the following differential equation:

$$\begin{aligned} f''(z) &= 2 \left[ z^2 - \frac{\gamma+1}{z} \right] f'(z) \\ &+ [2(\gamma+2)z - \sigma z^2 - \omega] f(z) \end{aligned} \quad (23)$$

Suppose that  $\sigma$  is written as follows:

$$\sigma(n, l) = \sigma_0(n, l) + \sigma_1(n, l) \omega + \sigma_2(n, l) \omega^2 + \dots \quad (24)$$

where  $\omega$  is the perturbation expansion parameter. So, the energy eigenvalue is yielded as

$$E_{pert} = \left( \frac{(4\rho)^{2/3}}{8\mu} \right)^{2/3} \sigma(n, l) \quad (25)$$

and more clearly

$$\begin{aligned} E_{pert} &= \frac{(4\rho)^{2/3}}{8\mu} \sigma_0(n, l) + \frac{(4\rho)^{1/3}}{2\mu} \alpha \sigma_1(n, l) \\ &+ \frac{2\alpha^2}{\mu} \sigma_2(n, l) + \dots \end{aligned} \quad (26)$$

In the above expansion, the general form of the zeroth-order correction  $\sigma_0$  is obtained via

$$\delta^{(0)}(z, 0, \sigma_0) = 0 \quad (27)$$

The first-order correction,  $\sigma_1$ , is obtained by using the equation  $\delta^{(1)}(z, 0, \sigma_1) = 0$  in the same manner with  $\sigma_0$ , while  $\delta^{(2)}(z, 0, \sigma_2) = 0$  is used for  $\sigma_2$ . Numerical results of  $\sigma_0$ ,  $\sigma_1$ , and  $\sigma_2$  coefficients, obtained by AIM, are reported in Table 4 for some energy levels. Besides, for  $\mu = 1/2$  GeV, comparisons of the perturbative energy eigenvalues with the ones obtained by direct application of AIM have been given in Tables 1, 2, and 3, in previous section. We emphasize, in Table 4, that corrections to the perturbation expansion do not depend on the potential parameters.

As a practice, we have applied our perturbation expansion formula (up to second-order correction) to get the ground-state energies of quarkonium in Table 5, for various values of the parameter  $A$ , while  $B = 1$  GeV and  $\mu = 1/2$  GeV. In Table 5, we also report comparisons of the perturbative energy eigenvalues with the ones of s-wave heavy quarkonium from [10, 21].

As is seen from Table 5, the results for which our perturbation expansion (up to second-order correction) has been used are in very good agreement with [10, 21] for small values of  $A$ . However, our analytical results are a little bit different from the exact ones as  $A$  gets larger values. It seems that the perturbation expansion, which includes third-order correction, may give more accurate results. The more correction term we add to the perturbative expansion, the more compatible results we get. Nevertheless, we can say that (26) can be used as an eigenvalue formula of the Schrödinger equation in case of Cornell potential, for practical purposes. So, one can use the following formula:

$$\begin{aligned} E_{pert} &= \frac{(4\rho)^{2/3}}{8\mu} \sigma_0(n, l) + \frac{(4\rho)^{1/3}}{2\mu} \alpha \sigma_1(n, l) \\ &+ \frac{2\alpha^2}{\mu} \sigma_2(n, l) \end{aligned} \quad (28)$$

for obtaining the eigenvalues and mass of the quarkonium for Cornell potential. Besides, it can be fit to mass formula of experimental values for determining the potential parameters  $A$  and  $B$ . The advantage of (28) is that the coefficients  $\sigma_0$ ,  $\sigma_1$ , and  $\sigma_2$  are independent of the potential parameters.

TABLE 4: Perturbation coefficients of the expansion given as (24) and (26). Notice that corrections to the perturbation expansion do not depend on the potential parameters.

$l$	$n$	$\sigma_0(n,l)$	$\sigma_1(n,l)$	$\sigma_2(n,l)$
0	0	3.71151	-0.525933	-0.0232729
	1	6.48922	-0.366743	-0.00767365
	2	8.76334	-0.297538	-0.00400191
	3	10.7732	-0.256486	-0.00251618
1	0	5.33566	-0.322683	-0.00554189
	1	7.75358	-0.258925	-0.00282569
	2	9.85399	-0.222298	-0.00176295
	3	11.7558	-0.197751	-0.00122526
2	0	6.74357	-0.244191	-0.00241586
	1	8.93661	-0.208300	-0.00148846
	2	10.9037	-0.184664	-0.00102765
	3	12.7146	-0.167585	-0.000761053
3	0	8.01784	-0.200753	-0.00134507
	1	10.0516	-0.177251	-0.000921458
	2	11.9129	-0.160449	-0.000679139
	3	13.6471	-0.147666	-0.000525832

TABLE 5: Comparisons of energy eigenvalues (in GeV) obtained by using the perturbation expansion formula in (26) ( $E_{pert}$ ) with the ones of s-wave heavy quarkonium from [10, 21]. The potential parameter  $B$  is taken as  $B = 1$  GeV, while the reduced mass is  $\mu = 1/2$  GeV in this case. The eigenvalues of [10, 21] are exact results.

$A$	$E_{00}$ (Ref. [10])	$E_{00}$ (Ref. [21])	$E_{pert}$	$A$	$E_{00}$ (Ref. [10])	$E_{00}$ (Ref. [21])	$E_{pert}$
0.2	2.16732	2.16732	2.16741	0.1	2.25368	2.25368	2.25369
0.4	1.98850	1.98850	1.98923	0.3	2.07895	2.07895	2.07927
0.6	1.80107	1.80107	1.80367	0.5	1.89590	1.89590	1.89740
0.8	1.60441	1.60441	1.61063	0.7	1.70394	1.70393	1.70808
1	1.39788	1.39788	1.41015	0.9	1.50242	1.50242	1.51132
1.2	1.18084	1.18083	1.20221	1.1	1.29071	1.29071	1.30711
1.4	0.95264	0.95264	0.98683	1.3	1.06817	1.06817	1.09545
1.6	0.71266	0.71266	0.76400	1.5	0.83416	0.83416	0.87635
1.8	0.46027	0.46026	0.53373	1.7	0.58805	0.58805	0.64980

**4.2. Energy Eigenvalues and Mass Spectrum for Heavy Quarkonium.** In this section, we tested our formula through cross-checking with the exact results in literature and with the experimental data. For comparing our energy eigenvalues with the exact ones, the parameters of Cornell potential have been considered  $A = 0.52$  and  $B = 0.43$  GeV. Besides, we have chosen the quark masses as  $m_c = 1.84$  GeV and  $m_b = 5.18$  GeV, in this case [12].

Also, we tested our formula by comparing our results, for the masses of heavy mesons, with the experimental data. For doing this, we have taken the quark masses as  $m_c = 1.44$  GeV and  $m_b = 4.87$  GeV and the potential parameters as  $A = 0.64$  and  $B = 0.39$  GeV. All these values have been obtained by fitting our formula to the experimental data in [22].

In Table 6, we compared our energy eigenvalues calculated by using (28) with the ones of [12]. Furthermore, in Table 7, we gave our results for the masses of the mesons obtained by the same equation. Table 7 also includes the experimental data got from [22].

It can be seen from Table 6 that the energy eigenvalues of the mesons  $c\bar{c}$ ,  $b\bar{b}$ , and  $b\bar{c}$ , obtained by (28), are more compatible with the exact ones, than those of [12]. The difference between AIM and [12] becomes clearer as the energy level increases. Similar things can be said for the masses in Table 7: the results obtained via AIM are closer to the experimental data than those of [12].

## 5. Conclusion

We have used AIM to obtain both, the eigenvalues of Schrödinger equation and mass of  $q\bar{q}$  system for Cornell potential, in three dimensions. AIM has some advantages such as being used for either exactly or numerically (or approximately) solvable problems. Furthermore, one can use AIM in the frame of perturbation theory. Once it is performed to obtain perturbative solutions, the wavefunction of unperturbed Hamiltonian is not needed to get the corrections to the perturbation expansion.



TABLE 6: Comparisons of the energy eigenvalues (in GeV) of the mesons  $c\bar{c}$ ,  $b\bar{b}$ , and  $b\bar{c}$  calculated by using (28) with the exact ones of [12]. The parameters of Cornell potential are  $A = 0.52$  and  $B = 0.43$  GeV, while the quark masses are  $m_c = 1.84$  GeV and  $m_b = 5.18$  GeV.

$E_n$	$c\bar{c}$			$b\bar{b}$			$b\bar{c}$		
	Exact [12]	Ref. [12]	AIM	Exact [12]	Ref. [12]	AIM	Exact [12]	Ref. [12]	AIM
1s	0.2575	0.2578	0.2660	-0.1704	-0.1702	-0.1216	0.1110	0.1113	0.1269
2s	0.8482	0.8096	0.8481	0.4214	0.3579	0.4203	0.6813	0.6324	0.6803
3s	1.2720	1.1427	1.2715	0.7665	0.5612	0.7635	1.0686	0.9065	1.0668

TABLE 7: Comparisons of the masses (in GeV), obtained via AIM, of the heavy mesons  $c\bar{c}$ ,  $b\bar{b}$ , and  $b\bar{c}$  with the ones of [12], and with the experimental data from [22]. In this case, we have taken the quark masses as  $m_c = 1.44$  GeV and  $m_b = 4.87$  GeV, and the potential parameters as  $A = 0.64$  and  $B = 0.39$  GeV, for our calculations. All these parameters have been obtained by fitting our formula, given in (28), to the experimental data.

$M_n$	$c\bar{c}$			$b\bar{b}$			$b\bar{c}$		
	Exp.	Ref. [12]	AIM	Exp.	Ref. [12]	AIM	Exp.	Ref. [12]	AIM
1s	3.097	3.097	3.096	9.460	9.350	9.462	6.275	6.291	6.362
2s	3.686	3.649	3.672	10.023	9.878	10.027	6.842	6.812	6.911
3s	4.039	3.963	4.085	10.355	10.081	10.361	-	7.087	7.284
4s	-	-	4.433	10.579	-	10.624	-	-	7.593
1p	3.511	-	3.521	9.899	-	9.963	-	-	6.792
2p	3.927	-	3.951	10.260	-	10.299	-	-	7.178
3p	-	-	4.310	10.512	-	10.564	-	-	7.494
1d	-	-	3.800	10.164	-	10.209	-	-	7.051

In the present study, the energy eigenvalues in the case of Cornell potential have been achieved by direct application of the method. Besides, we have performed perturbation theory in the view of AIM for the problem and found a semianalytical formula for energy eigenvalues. Numerical results obtained by using this formula, for the reduced mass  $\mu = 1/2$  GeV, conform with the exact results of [10, 21], in a wide spectrum of the potential parameters  $A$  and  $B$  (especially for  $B > A$ ). Furthermore, the results are compatible with the ones obtained directly, in Section 3. It is also possible to see from the results that the perturbative eigenvalues fit in with the exact ones for higher quantum states, even for the large values of  $A$ . For any values of  $A$  and  $B$ , the higher quantum states are more consonant with the exact ones than the lower states. The perturbation expansion, which includes third-order correction, may give more accurate results. The more correction terms we add to the perturbative expansion, the more compatible results we may get.

We have also tested our semianalytical formula, by cross-checking it with the exact results in literature and with the experimental data. It can be seen, from Table 6, that our energy eigenvalues calculated by using (28) are more compatible with the exact ones than those of [12]. Furthermore, the difference between our results and [12] becomes clearer as the energy level increases. By using AIM, we have also obtained mass results which are closer to the experimental data than [12].

As a consequence, semianalytical formula achieved for energy eigenvalues and mass of quarkonium can be used for practical purposes in the case of Cornell potential. If our formula is fitted to the experimental data, the potential

parameters (and masses of the quarks, if it is needed) can also be obtained.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

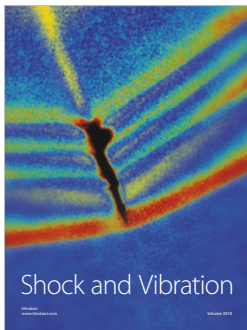
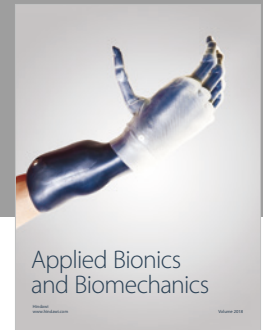
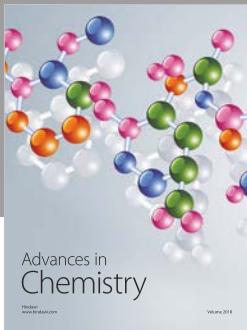
## Acknowledgments

This academic work was supported by the Mersin Technology Transfer Office Academic Writing Center of Mersin University.

## References

- [1] H. Çiftçi, R. L. Hall, and N. Saad, "Asymptotic iteration method for eigenvalue problems," *Journal of Physics A: Mathematical and General*, vol. 36, no. 47, pp. 11807–11816, 2003.
- [2] N. Saad, R. L. Hall, and N. Saad, "Exact and approximate solutions of Schrödinger's equation for a class of trigonometric potentials," *Central European Journal of Physics*, vol. 11, no. 1, pp. 37–48, 2013.
- [3] E. Olğar, R. Koç, and H. Tütüncüler, "The exact solution of the s-wave Klein-Gordon equation for the generalized Hulthén potential by the asymptotic iteration method," *Physica Scripta*, vol. 78, no. 1, Article ID 015011, 2008.

- [4] F. M. Fernández, “On an iteration method for eigenvalue problems,” *Journal of Physics A: Mathematical and General*, vol. 37, no. 23, pp. 6173–6180, 2004.
- [5] H. Ciftci, R. L. Hall, and N. Saad, “Perturbation theory in a framework of iteration methods,” *Physics Letters A*, vol. 340, no. 5-6, pp. 388–396, 2005.
- [6] H. Ciftci and H. F. Kisoglu, “Accidental Degeneracies in N dimensions for Potential Class  $\alpha r^{2d-2} - \beta r^{d-2}$  via Asymptotic Iteration Method (AIM),” *Communications in Theoretical Physics*, vol. 67, p. 350, 2017.
- [7] E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T.-M. Yan, “Spectrum of charmed quark-antiquark bound states,” *Physical Review Letters*, vol. 34, no. 6, pp. 369–372, 1975.
- [8] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. -. Yan, “Charmonium: The model,” *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 17, no. 11, pp. 3090–3117, 1978.
- [9] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, “Charmonium: comparison with experiment,” *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 21, no. 1, pp. 203–233, 1980.
- [10] H. Chung S and J. Lee, “Cornell potential parameters for S-wave heavy quarkonia,” *Journal of the Korean Physical Society*, vol. 52, p. 1151, 2008.
- [11] M. Hamzavi and A. A. Rajabi, “Scalar–vector–pseudoscalar Cornell potential for a spin-1/2 particle under spin and pseudospin symmetries: 1 + 1 dimensions,” *Annals of Physics*, vol. 334, pp. 316–320, 2013.
- [12] A. Vega and J. Flores, “Heavy quarkonium properties from Cornell potential using variational method and supersymmetric quantum mechanics,” *Pramana – Journal of Physics*, vol. 87, 2016, Article ID 73.
- [13] S. Jacobs, M. G. Olsson, and C. Suchyta, “Comparing the Schrödinger and spinless Salpeter equations for heavy-quark bound states,” *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 33, no. 11, pp. 3338–3348, 1986.
- [14] B. Grinstein, “A modern introduction to quarkonium theory,” *International Journal of Modern Physics A*, vol. 15, p. 461, 2000.
- [15] W. Lucha, F. Schoberl, and D. Gromes, “Bound states of quarks,” *Physics Reports*, vol. 200, no. 4, pp. 127–240, 1991.
- [16] P. Evans, C. Allton, and J. Skullerud, *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 89, no. 7, 2014.
- [17] R. L. Hall, “Simple eigenvalue formula for the Coulomb-plus-linear potential,” *Physical Review D*, vol. 30, p. 433, 1984.
- [18] K. Hagiwara J and S. Jacobs, “Probing QCD with heavy quark bound states,” *Physics Letters B*, vol. 131, pp. 455–460, 1983.
- [19] E. Gozzi, M. Reuter, and W. Thacker D, “Variational methods via supersymmetric techniques,” *Physics Letters A*, vol. 183, p. 29, 1993.
- [20] F. Cooper, A. Khare, and U. Sukhatme, “Supersymmetry and quantum mechanics,” *Physics Reports*, vol. 251, no. 5-6, pp. 267–385, 1995.
- [21] L. R. Hall and S. Nasser, “Schrödinger spectrum generated by the Cornell potential,” *Open Physics*, vol. 13, p. 83, 2015.
- [22] C. Patrignani and etal. (Particle Data Group), “Review of Particle Physics,” *Chinese Physics C*, vol. 40, 2016, 100001.



Hindawi

Submit your manuscripts at  
[www.hindawi.com](http://www.hindawi.com)

