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Research Article

Integrable and Superintegrable Systems with Higher Order Integrals of Motion: Master Function Formalism

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We construct two-dimensional integrable and superintegrable systems in terms of the master function formalism and relate them to *Mielnik's* and *Marquette's* construction in supersymmetric quantum mechanics. For two different cases of the master functions, we obtain two different two-dimensional superintegrable systems with higher order integrals of motion.

1. Introduction

It is known from classical and quantum mechanics that a system with N degrees of freedom is called completely integrable if it allows N functionally independent constants of the motion [1]. From the mathematical and physical point of view, these systems play a fundamental role in description of physical systems due to their many interesting properties. A system is superintegrable if one could obtain more than N constants of the motion and if there exist $2N-1$ constants of the motion, the system is maximally superintegrable or just superintegrable [2–5]. Recently the study of superintegrable systems has been considered for different potentials and many researches have been studied for calculating the spectrum of these systems by different methods. In [6, 7], the spectrum of these systems has been calculated by an algebraic method using the realization of some Lie groups.

For a two-dimensional quantum integrable system with Hamiltonian H, there is always one operator like A_1 which commutes with Hamiltonian of the system, that is, $[H, A_1] = 0$. For a quantum superintegrable system, one should define another operator such as A_2 which commutates with the Hamiltonian of system, that is, $[H, A_2] = 0$, but $[A_1, A_2] \neq 0$. In other words, for a two-dimensional superintegrable system, there are two integrals of the motion (A_1, A_2) in addition to the Hamiltonian. The superintegrability with the second- and third-order integrals was the object of a series of articles [8–11]. The systems studied have second- and third-order integrals. Although superintegrability and supersymmetric quantum mechanics (SUSYQM) are two separated fields, many quantum systems, such as the harmonic oscillator, the Hydrogen atom, and the Smorodinsky-Winternitz potential, have both supersymmetry and superintegrable conditions [12–16]. These articles show that superintegrability is accurately connected with supersymmetry. For example, in [17], Marquette used the results obtained by Mielnik [18] and generated new superintegrable systems. Mielnik has shown that the factorization of second-order operators is not essentially unique. He has considered the Hamiltonian of the harmonic oscillator in one dimension as the simplest case:

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2, \quad (1)$$

where it can be factorized by two types of the first-order operators of creation and annihilation as follows:

$$a_{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + x\right),$$
$$b_{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + \beta(x)\right). \quad (2)$$

For two superpartner Hamiltonians H_1 and H_2 where $a_+a_- = H - 1/2 = H_1$ and $a_-a_+ = H + 1/2 = H_2$, he has demanded that $H_2 = b_- b_+$ and obtained the inverted product $b_+ b_-$ as a certain new Hamiltonian:

$$H' = b_+ b_- = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} - \varphi'(x), \quad (3)$$

where $\varphi(x)$ is a function obtained from the general solution of Riccati equation considering $\beta = x + \varphi(x)$. The creation and annihilation operators of the third order for H' are described by expressions $s_+ = b_+ a_+ b_-$, $s_- = b_+ a_- b_-$, where a_+ and a_- are the creation and annihilation operators for H_2. Marquette [17] has taken the Hamiltonian H_2 in the x-axis and its superpartner H' given by (3) in the y-axis. Hence, he has obtained a two-dimensional superintegrable system as $H_s = H_x + H_y$, which can be separated in Cartesian coordinates with creation and annihilation operators $a_+(x)$, $a_-(x)$, $s_+(y)$, and $s_-(y)$. Also, he has shown that the Hamiltonian H_s possesses the following integrals of motion:

$$\begin{aligned} \mathcal{K} &= H_x - H_y, \\ \mathcal{A}_1 &= a_+(x)\, s_-(y) - a_-(x)\, s_+(y), \\ \mathcal{A}_2 &= a_-(x)\, s_+(y) + a_+(x)\, s_-(y), \end{aligned} \quad (4)$$

where these integrals are of order 2, 3, and 4 for shape invariant potentials [17].

On the other hand, in [19, 20], the authors have shown that the second-order differential equations and their associated differential equations in mathematical physics have the shape invariant property of supersymmetry quantum mechanics. They have shown that by using a polynomial of a degree not exceeding two, called the master function, the associated differential equations can be factorized into the product of rising and lowering operators. The master function formalism has been used in relativistic quantum mechanics for solving the Dirac equation [21, 22].

As *Mielnik's* and *Marquette's* method for generating superintegrable systems can be applied to other systems obtained in the context of supersymmetric quantum mechanics, hence, in this paper, we show that the supersymmetry method for obtaining the integrable and superintegrable systems can be related to master function formalism. In fact, we use the master function approach for 1-dimensional shape invariant potentials and generate 2-dimensional integrable systems. Also for a particular class of shape invariant systems, we generate 2-dimensional superintegrable systems. This class contains the harmonic oscillator, the singular harmonic oscillator, and their supersymmetric isospectral deformations.

The paper is presented as follows: in Section 2, we review how one can generate integrals of motion for two-dimensional superintegrable system from the creation and annihilation operators. In Section 3, we consider a particular quantum system for applying the *Mielnik's* and *Marquette's* method and obtain a superintegrable potential separable in Cartesian coordinates. In Section 4, we briefly review the master function formalism and then in Section 5, we use this approach to obtain integrable systems and particular cases of the superintegrable systems that satisfy the oscillator-like (Heisenberg) algebra with higher order integrals of motion in terms of the master function and weight function. In Section 6, we give two examples to show how this method works in constructing oscillator-like two-dimensional superintegrable systems. Paper ends with a brief conclusion in Section 7.

2. Two-Dimensional Superintegrable System and Its Integrals of Motion

According to [17, 23, 24], for a two-dimensional Hamiltonian separable in Cartesian coordinates as

$$H(x, y, p_x, p_y) = H_x(x, p_x) + H_y(y, p_y), \quad (5)$$

where the creation and annihilation operators (polynomial in momenta) $A_+(x)$, $A_-(x)$, $A_+(y)$, and $A_-(y)$ satisfy the following equations:

$$\begin{aligned} [H_x, A_-(x)] &= -\lambda_x A_-(x), \\ [H_y, A_-(y)] &= -\lambda_y A_-(y), \\ [H_x, A_+(x)] &= \lambda_x A_+(x), \\ [H_y, A_+(y)] &= \lambda_y A_+(y), \end{aligned} \quad (6)$$

one can show that the operators $f_1 = A_+^m(x) A_-^n(y)$ and $f_2 = A_-^m(x) A_+^n(y)$ commute with the Hamiltonian H: that is,

$$[H, f_1] = [H, f_2] = 0, \quad (7)$$

if

$$m\lambda_x - n\lambda_y = 0, \quad m, n \in Z^+. \quad (8)$$

Also the following sums of f_1 and f_2 commute with the Hamiltonian

$$\begin{aligned} I_1 &= A_+^m(x)\, A_-^n(y) - A_-^m(x)\, A_+^n(y), \\ I_2 &= A_+^m(x)\, A_-^n(y) + A_-^m(x)\, A_+^n(y); \end{aligned} \quad (9)$$

that is, I_1 and I_2 are the integrals of motion. The order of these integrals of motion depends on the order of the creation and annihilation operators. On the other hand, the Hamiltonian H possesses a second-order integral as $K = H_x - H_y$, such that the integral I_2 is the commutator of I_1 and K. Thus, the Hamiltonian H is a superintegrable system and H, I_1, and K are its integrals of motion.

3. Mielnik-Marquette Method and Superintegrable Model Obtained from Shifted Oscillator Hamiltonian

In this section, for reviewing the Mielnik-Marquette method, we consider shifted oscillator Hamiltonian as

$$H = -\frac{d^2}{dx^2} + \frac{1}{4}\omega^2\left(x - \frac{2b}{\omega}\right)^2 - \frac{\omega}{2}. \quad (10)$$

We introduce the following first-order operators:

$$a_- = \frac{d}{dx} + \frac{1}{2}\omega x - b,$$
$$a_+ = -\frac{d}{dx} + \frac{1}{2}\omega x - b, \quad (11)$$

where the supersymmetric partner Hamiltonians are calculated as

$$H_1 = a_- a_+ = H + \omega,$$
$$H_2 = a_+ a_- = H. \quad (12)$$

It is obvious that H_1 and H_2 have the shape invariant properties. Now, according to (2), we define the new operators b_- and b_+ such that

$$H_1 = H + \omega = b_- b_+. \quad (13)$$

The above equation gives the following Riccati equation:

$$\beta^2 + \beta' = \frac{1}{4}\omega^2 x^2 - b\omega x + b^2 + \frac{\omega}{2}, \quad (14)$$

where a particular solution is

$$\beta(x) = \beta_0(x) = \frac{1}{2}\omega x - b. \quad (15)$$

Now, if we consider

$$\beta(x) = \beta_0(x) + \varphi(x), \quad (16)$$

then we can obtain the following first-order linear inhomogeneous equation:

$$z' + (-2\beta_0) z = 1, \quad (17)$$

where $z = 1/\varphi(x)$. After solving the above equation, we get

$$\varphi(x) = \frac{1}{z(x)} = \frac{e^{-(\omega/2)x^2 + 2bx}}{\sqrt{\pi/2\omega}e^{2b^2/\omega}\operatorname{erf}\left(\sqrt{\omega/2}(x - 2b/\omega)\right) + C}, \quad (18)$$

where C is the constant of integration. Using the function $\varphi(x)$, we obtain

$$H' = b_+ b_- = H_1 - \varphi'(x), \quad (19)$$

where its creation and annihilation operators are given by the following expressions:

$$s_+ = b_+ a_+ b_-,$$
$$s_- = b_+ a_- b_-. \quad (20)$$

According to Marquette method, we take the x-axis for Hamiltonian H_1 and the y-axis for its superpartner H' and we have the following two-dimensional superintegrable system:

$$\begin{aligned} H_s &= H_x + H_y = H_1 + H' \\ &= -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + \frac{1}{4}\omega^2\left(x - \frac{2b}{\omega}\right)^2 \\ &\quad + \frac{1}{4}\omega^2\left(y - \frac{2b}{\omega}\right)^2 - \omega - \frac{d\varphi}{dy}. \end{aligned} \quad (21)$$

This Hamiltonian possesses the integral of motion given by (4), which are of order 2, 3, and 4.

4. The Master Function Formalism

According to [19, 20], the general form of the differential equation in master function approach is written as

$$\begin{aligned} &A(x)\Phi_n'' + \frac{(A(x)w(x))'}{w(x)}\Phi_n'(x) \\ &- \left(n\left(\frac{(A(x)w(x))'}{w(x)}\right)' + \frac{n(n-1)}{2}A''(x)\right)\Phi_n(x) \\ &= 0, \end{aligned} \quad (22)$$

where $A(x)$ as master function is at most a second-order polynomial and $w(x)$ is the nonnegative weight function in interval (a, b). By differentiating (22) m times and then multiplying it by $(-1)^m A^{m/2}(x)$, we get the following associated second-order differential equation in terms of the master function and weight function:

$$\begin{aligned} &A(x)\Phi_{n,m}'' + \frac{(A(x)w(x))'}{w(x)}\Phi_{n,m}'(x) \\ &+ \left[-\frac{1}{2}\left(n^2 + n - m^2\right)A'' \right. \\ &+ (m-n)\left(\frac{A(x)w'(x)}{w(x)}\right)' - \frac{m^2}{4}\frac{\left(A'(x)\right)^2}{A(x)} \\ &\left. - \frac{m}{2}\frac{A'(x)w'(x)}{w(x)}\right]\Phi_{n,m}(x) = 0, \end{aligned} \quad (23)$$

where

$$\Phi_{n,m}(x) = (-1)^m A^{m/2}\left(\frac{d}{dx}\right)^m \Phi_n(x). \quad (24)$$

Changing the variable $dx/dr = \sqrt{A(x)}$ and defining the new function $\Psi_n^m(r) = A^{1/4}(x)w^{1/2}(x)\Phi_{n,m}(x)$, one can obtain the Schrodinger equation as

$$-\frac{d^2}{dr^2}\Psi_n^m(r) + v_m(x(r))\Psi_n^m(r) = E(n,m)\Psi_n^m(r), \quad m = 0, 1, 2, \ldots, n, \quad (25)$$

where the most general shape invariant potential is

$$v_m(x(r)) = -\frac{1}{2}\left(\frac{A(x)w'(x)}{w(x)}\right)' - \frac{2m-1}{4}A''(x) + \frac{1}{4A(x)}\left(\frac{A(x)w'(x)}{w(x)}\right)^2 + \frac{m}{2}\frac{A'(x)w'(x)}{w(x)} + \frac{4m^2-1}{16}\frac{A'^2(x)}{A(x)}, \quad (26)$$

and the energy spectrum $E(n,m)$ is as

$$E(n,m) = -(n-m+1) \cdot \left[\left(\frac{A(x)w'(x)}{w(x)}\right)' + \frac{1}{2}(n+m)A''(x)\right]. \quad (27)$$

According to [19, 20] the first-order deferential operators are written as

$$A_\pm = \mp\frac{d}{dr} + W_m(x(r)), \quad (28)$$

where the superpotential $W_m(x(r))$ is expressed in terms of the master function $A(x)$ and weigh function $w(x)$ as

$$W_m(x(r)) = -\frac{A(x)w'(x)/2w(x) + ((2m-1)/4)A'(x)}{\sqrt{A(x)}}. \quad (29)$$

The Hamiltonian H_1 and H_2 called the superpartner Hamiltonians are written as

$$\begin{aligned} H_1 &= A_-A_+ = -\frac{d^2}{dr^2} + W_m^2(r) + W_m'(r) \\ &= -\frac{d^2}{dr^2} + v_1(r), \\ H_2 &= A_+A_- = -\frac{d^2}{dr^2} + W_m^2(r) - W_m'(r) \\ &= -\frac{d^2}{dr^2} + v_2(r), \end{aligned} \quad (30)$$

where $v_1(r)$ and $v_2(r)$ are called the partner potentials in the concept of supersymmetry in nonrelativistic quantum mechanics. Furthermore, if the partner potentials have the same shape and differ only in parameters, then potentials $v_1(r)$ and $v_2(r)$ are called the shape invariant potentials that satisfy

$$v_1(r,a_0) = v_2(r,a_1) + R(a_1), \quad (31)$$

where $R(a_1)$ is independent of any dynamical variable and a_1 is a function of a_0. Potentials which satisfy in this condition are exactly solvable, although shape invariance is not the most general integrability or superintegrability condition.

5. Integrable and Superintegrable Systems Obtained from the Master Function Formalism

In this section, we try to relate the Mielnik-Marquette method to the master function approach. Hence, we define the following new operators:

$$B_\pm = \mp\frac{d}{dr} + \omega(r), \quad (32)$$

where $\omega(r)$ as the new superpotential must be related to the general form of the master function superpotential $W_m(x(r))$. Their product yields to Hamiltonians as

$$\begin{aligned} B_-B_+ &= -\frac{d^2}{dr^2} + \omega^2(r) + \omega'(r), \\ B_+B_- &= -\frac{d^2}{dr^2} + \omega^2(r) - \omega'(r). \end{aligned} \quad (33)$$

Now if we demand $A_-A_+ = B_-B_+$ then we can obtain the following Riccati equation in terms of master function:

$$\omega^2(r) + \omega'(r) = W_m^2(r) + W_m'(r), \quad (34)$$

where a particular solution is $\omega(r) = W_m(r)$. The general solution can be obtained like

$$\omega(r) = W_m(r) + \lambda(r), \quad (35)$$

which yields

$$\lambda^2(r) + 2W_m(r)\lambda(r) + \lambda'(r) = 0. \quad (36)$$

We consider the transformation $f(r) = 1/\lambda(r)$ and obtain a first-order linear inhomogeneous differential equation as

$$f'(r) - 2W_m(r)f(r) = 1, \quad (37)$$

where the general solution is

$$f(r) = \exp\left[2\int W_m(r)\,dr\right] \cdot \left(C + \int \exp\left[2\int W_m(r')\,dr'\right]dr\right), \quad (38)$$

where C is constant. Hence,

$$\omega(r) = W_m(r) + \frac{e^{-\int 2W_m(r)dr}}{C + \int e^{\int 2W_m(r')dr'}dr}. \quad (39)$$

Using the function $f(r)$ given by (38), the superpartner Hamiltonian is given by

$$\begin{aligned} H' &= H_2 - \lambda'(r) \\ &= -\frac{d^2}{dr^2} + W_m^2(r) - W_m'(r) - \lambda'(r), \end{aligned} \quad (40)$$

which is the general form of Hamiltonian in terms of master function. Now if we catch $H_r = H_2$ and $H_{r'} = H'$

(the Hamiltonian H' is thus given in terms of the variable r' vertical to r), then we obtain a new two-dimensional integrable Hamiltonian as

$$\begin{aligned} H_s &= H_r + H_{r'} \\ &= -\frac{d^2}{dr^2} - \frac{d^2}{dr'^2} + W_m^2(r) + W_m^2(r') - W_m'(r) \\ &\quad - W_m'(r') - \lambda'(r'). \end{aligned} \tag{41}$$

Therefore, we have obtained the general form of the 2-dimensional integrable Hamiltonian in terms of master function which can be separated in radial coordinates. This separation of variable implies the existence of a second-order integral as $K = H_r - H_{r'}$. Hence, H_s is an integrable system. Now, for generating superintegrable systems, we can obtain the creation and annihilation operators for H' from H_2 as

$$\begin{aligned} S_+ &= B_+ A_+ B_-, \\ S_- &= B_+ A_- B_-, \end{aligned} \tag{42}$$

where $A_\pm$ and $B_\pm$ are given in (28) and (32). It is necessary to mention that these ladder operators satisfy (6) only for a particular class of shape invariant systems and in general, the 2-dimensional system H_s, obtained from a given master function, is not a superintegrable system. In other words, we cannot obtain 2-dimensional superintegrable system for all of the shape invariant cases given in [19, 20]. This particular class contains the harmonic oscillator, the singular harmonic oscillator, and their supersymmetric isospectral deformations.

Hence, if relation (6) exists, according to (9) we can obtain the integrals of motion for Hamiltonian (41) as

$$\begin{aligned} K &= H_r - H_{r'}, \\ A_1 &= A_+^m(r) S_-^n(r') - A_-^m(r) S_+^n(r'), \\ A_2 &= A_+^m(r) S_-^n(r') + A_-^m(r) S_+^n(r'). \end{aligned} \tag{43}$$

In the next section, we apply this formalism for some particular cases of shape invariant potentials in terms of master function.

6. Examples of Two-Dimensional Superintegrable Systems as a Result of Master Function Approach

In this section, we would apply the master function formalism of the previous section for two examples and show how these results allow us to obtain 2-dimensional superintegrable systems with higher order integrals.

Example 1. Let $A(x) = 1$; then according to [19], $w(x) = e^{-(\beta/2)x^2}$ that $x = r - 2\alpha/\beta$, $\beta > 0$ and the interval is $(-\infty, +\infty)$. Using (29), we obtain the superpotential as

$$W_m(r) = -\frac{\beta}{2}\left(r - \frac{2\alpha}{\beta}\right). \tag{44}$$

According to (27), the energy spectrum is as

$$E = n - m + 1, \tag{45}$$

and also the ladder operators given by (28) related to (44) satisfy a Heisenberg algebras (6). Now, substituting expression $W_m(r)$ in (38) yields the following relation in terms of the error function:

$$\lambda(r) = \frac{e^{-\beta(r^2/2)+2\alpha r}}{C + \sqrt{\pi/2\beta}e^{2\alpha^2/\beta}\operatorname{erf}\left(\sqrt{\beta/2}\,(r - 2\alpha/\beta)\right)}, \tag{46}$$

and so

$$\begin{aligned} \omega(r) &= W_m(r) + \lambda(r) \\ &= \frac{\beta}{2}\left(r - \frac{2\alpha}{\beta}\right) \\ &\quad + \frac{e^{\beta(r^2/2)+\alpha r}}{C + \sqrt{\pi/2\beta}e^{2\alpha^2/\beta}\operatorname{erf}\left(\sqrt{\beta/2}\,(r - 2\alpha/\beta)\right)}. \end{aligned} \tag{47}$$

Substituting this expression in (40) and (41) yields the family of superpartner H' and a two-dimensional superintegrable Hamiltonian H_s, respectively, as

$$\begin{aligned} H' &= H_2 - \lambda'(r), \\ H_s &= -\frac{d^2}{dr^2} - \frac{d^2}{dr'^2} \\ &\quad + \frac{\beta^2}{4}\left\{\left(r - \frac{2\alpha}{\beta}\right)^2 + \left(r' - \frac{2\alpha}{\beta}\right)^2\right\} - \beta \\ &\quad - \lambda'(r'), \end{aligned} \tag{48}$$

where

$$\begin{aligned} &\lambda'(r') \\ &= \frac{(\beta r' + \alpha)\, e^{(\beta/2)r'^2+\alpha r'}}{C + \sqrt{\pi/2\beta}e^{2\alpha^2/\beta}\operatorname{erf}\left(\sqrt{\beta/2}\,(r' - 2\alpha/\beta)\right)} \\ &\quad - \frac{\left(e^{(\beta/2)r'^2+\alpha r'}\right)\left(e^{2\alpha^2/\beta}e^{-(1/2)\beta(r'-2\alpha/\beta)^2}\right)}{\left[C + \sqrt{\pi/2\beta}e^{2\alpha^2/\beta}\operatorname{erf}\left(\sqrt{\beta/2}\,(r' - 2\alpha/\beta)\right)\right]^2}, \end{aligned} \tag{49}$$

$$H_2 = -\frac{d^2}{dr^2} + \frac{\beta^2}{4}\left(r - \frac{2\alpha}{\beta}\right)^2 - \frac{\beta}{2}. \tag{50}$$

It is seen that 2-dimensional superintegrable Hamiltonian (48) is the same as (21). We can find the general form of the operators S_+ and S_- in terms of the master function for this oscillator-like potential as follows:

$$\begin{aligned} S_+ &= -\frac{d^3}{dr^3} - W_m\frac{d^2}{dr^2} + \left(-2\omega' - W_m' + \omega^2\right)\frac{d}{dr} \\ &\quad + \left(-\omega'' - W_m'\omega - W_m\omega' + \omega\omega' + W_m\omega^2\right), \end{aligned}$$

$$S_- = \frac{d^3}{dr^3} - W_m \frac{d^2}{dr^2} + \left(-2\omega' - W_m' - \omega^2\right)\frac{d}{dr} + \left(\omega'' - W_m'\omega - W_m\omega' + \omega\omega' + W_m\omega^2\right). \tag{51}$$

Thus, we have obtained a 2-dimensional superintegrable system with integrals given by (43) as

$$\begin{aligned} K &= H_r - H_{r'}, \\ A_1 &= A_+(r) S_-(r') - A_-(r) S_+(r'), \\ A_2 &= A_+(r) S_-(r') + A_-(r) S_+(r'), \end{aligned} \tag{52}$$

where

$$\begin{aligned} K &= -\frac{d^2}{dr^2} + \frac{d^2}{dr'^2} + W_m^2(r) - W_m^2(r') - W_m'(r) + W_m'(r') + \lambda'(r'), \\ A_1 &= 2W_m(r)\frac{d^3}{dr'^3} + 2W_m(r')\frac{d^3}{dr\,dr'^2} - 2\left(-2\omega'(r') - W_m'(r')\right)\frac{d^2}{dr\,dr'} - 2\left(-W_m'(r')\omega(r') - W_m(r')\omega'(r') + \omega(r')\omega'(r') + W_m(r')\omega^2(r')\right)\frac{d}{dr} \\ &\quad - 2W_m(r)\omega^2(r')\frac{d}{dr'} + 2W_m(r)\omega''(r'), \\ A_2 &= -2\frac{d^4}{dr\,dr'^3} + 2\omega^2(r')\frac{d^2}{dr\,dr'} - 2W_m(r)\cdot W_m(r')\frac{d^2}{dr'^2} - 2\omega''(r')\frac{d}{dr} + 2W_m(r)\cdot\left(-2\omega'(r') - W_m'(r')\right)\frac{d}{dr'} + 2W_m(r) \\ &\quad \cdot\left(-W_m'(r')\omega(r') - W_m(r')\omega'(r') + \omega(r')\omega'(r') + W_m(r')\omega^2(r')\right). \end{aligned} \tag{53}$$

These integrals are of order 2, 3, and 4.

Example 2. According to [19] for $A(x) = x$, we have $w(x) = x^\alpha e^{-\beta x}$, $x = r^2/4$, $\alpha > -1$, $\beta > 0$ and the interval is $[0, +\infty)$. Now, using (29), the superpotential and the energy spectrum are as

$$\begin{aligned} W_m(r) &= -\frac{1}{r}\left(\alpha + m - \frac{1}{2}\right) + \frac{\beta}{4}r, \\ E &= \beta(n - m + 1). \end{aligned} \tag{54}$$

This system has also the ladder operators that satisfy the form of (6); hence, substituting expression $W_m(r)$ in (38) yields the following relation in terms of Whittaker function:

$$\begin{aligned} f(r) &= \frac{\beta^{-(\alpha+m)}}{(\alpha+m)}\left(\frac{r}{2}\right)^{-(2\alpha+2m-1)} e^{(1/4)\beta r^2}\Bigg(C + e^{-(1/8)\beta r^2}\Bigg[\frac{1}{(\alpha+m+1)}\left(\frac{\beta r^2}{4}\right)^{\alpha/2+m/2} \\ &\quad \cdot M_{(1/2)\alpha+(1/2)m,(1/2)\alpha+(1/2)m+1/2}\left(\frac{1}{4}\beta r^2\right) + \left(\frac{\beta r^2}{4}\right)^{\alpha/2+m/2-1} \\ &\quad \cdot M_{(1/2)\alpha+(1/2)m+1,(1/2)\alpha+(1/2)m+1/2}\left(\frac{1}{4}\beta r^2\right)\Bigg]\Bigg), \end{aligned} \tag{55}$$

where the Whittaker function $M_{\mu,\nu}(z)$ is the solution of the following differential equation:

$$y'' + \left(-\frac{1}{4} + \frac{\mu}{z} + \frac{1/4 - \nu^2}{z^2}\right)y = 0. \tag{56}$$

It can be also defined in terms of the confluent hypergeometric function as

$$M_{\mu,\nu}(z) = e^{(-(1/2)z)} z^{(1/2+\nu)}\, {}_1F_1\left(\frac{1}{2} + \nu - \mu, 1 + 2\nu, z\right). \tag{57}$$

The family of superpartner Hamiltonians H' and the two-dimensional superintegrable Hamiltonian H_s are thus calculated by (40) and (41), respectively. The creation and annihilation operators for the Hamiltonian H_2 are as

$$\begin{aligned} M_+(r) &= A_+^2(r) A_-(r), \\ M_-(r) &= A_+(r) A_-^2(r), \end{aligned} \tag{58}$$

where $A_\pm(r)$ is given in (28) and from (42), we have the creation and annihilation operators of the Hamiltonian H' as

$$\begin{aligned} R_+(r') &= B_+(r') M_+(r') B_-(r'), \\ R_-(r') &= B_+(r') M_-(r') B_-(r'), \end{aligned} \tag{59}$$

where $B_\pm(r)$ is given by (32) and (39). We can also find the integrals of motion of the Hamiltonian H_s from (43) as

$$\begin{aligned} K &= H_r - H_{r'}, \\ A_1 &= M_+(r) R_-(r') - M_-(r) R_+(r'), \\ A_2 &= M_+(r) R_-(r') + M_-(r) R_+(r'), \end{aligned} \tag{60}$$

that are of the order 2, 7, and 8.

7. Conclusion

In this article, we have used the results obtained by Mielnik in the concept of SUSYQM and related it to master function formalism for constructing two-dimensional integrable and superintegrable systems with higher order integrals of motion. From this procedure, we have generated the superintegrable systems for two different cases of master functions $A(x) = 1$ and $A(x) = x$ and have shown that the higher integrals of motion are in order 2, 3, 4 and 2, 7, 8, respectively.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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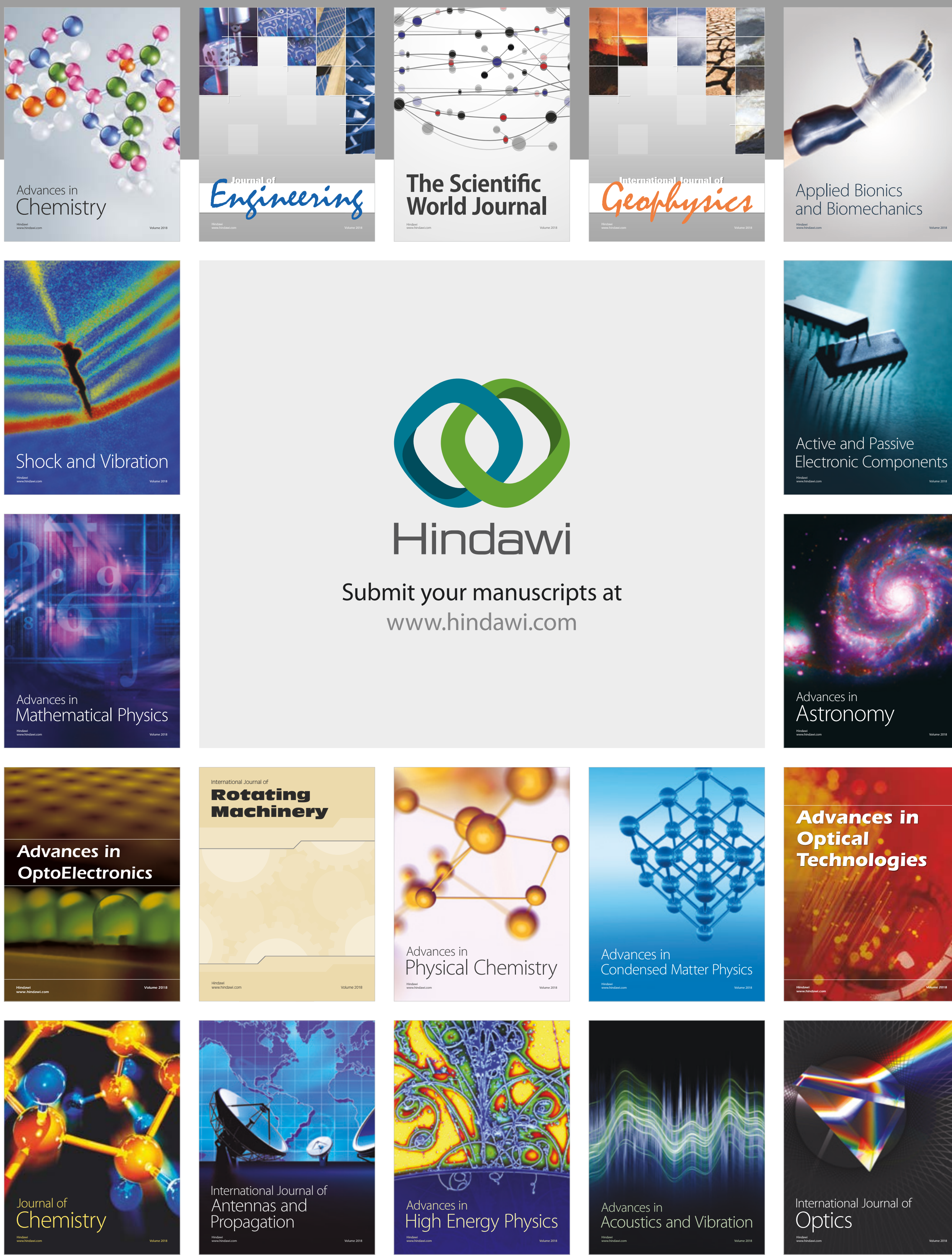
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