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Research Article

Thermodynamic Volume Product in Spherically Symmetric and Axisymmetric Spacetime

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We have examined the thermodynamic volume products for spherically symmetric and axisymmetric spacetime in the framework of *extended phase space*. Such volume products are usually formulated in terms of the outer horizon ($\mathscr{H}^+$) and the inner horizon ($\mathscr{H}^-$) of black hole (BH) spacetime. Besides volume product, the other thermodynamic formulations like *volume sum, volume minus, and volume division* are considered for a wide variety of spherically symmetric spacetime and axisymmetric spacetime. Like area (or entropy) product of multihorizons, the mass-independent (universal) features of volume products sometimes also *fail*. In particular, for a spherically symmetric AdS spacetime, the simple thermodynamic volume product of $\mathscr{H}^\pm$ is not mass-independent. In this case, more complicated combinations of outer and inner horizon volume products are indeed mass-independent. For a particular class of spherically symmetric cases, i.e., Reissner Nordström BH of Einstein gravity and Kehagias-Sfetsos BH of Hořava Lifshitz gravity, the thermodynamic volume products of $\mathscr{H}^\pm$ are indeed *universal*. For axisymmetric class of BH spacetime in Einstein gravity, all the combinations are *mass-dependent*. There has been no chance to formulate any combinations of volume product relation to be mass-independent. Interestingly, *only the rotating BTZ black hole* in 3D provides that the volume product formula is mass-independent, i.e., *universal*, and hence it is quantized.

1. Introduction

It has been examined by a number of researchers that the area (or entropy) product of various spherically symmetric and axisymmetric BHs are mass-independent (universal) [1–9]. For instance, Ansorg and Hennig [1] demonstrated that for a stationary and axisymmetric class of Einstein-Maxwell gravity the area product formula satisfied the universal relation as

$$\mathscr{A}_h\mathscr{A}_c = (8\pi J)^2 + \left(4\pi Q^2\right)^2. \tag{1}$$

$\mathscr{A}_h$ and $\mathscr{A}_c$ are area of outer horizon (OH) or event horizon (EH) and inner horizon (IH) or Cauchy horizon (CH). The parameters, J and Q, are denoted as the angular momentum and charge of the black hole (BH), respectively.

On the other hand, Cvetič et al. [2] extended this work for a higher dimensions spacetime and showed that for multihorizon BHs the area product formula should be quantized by satisfying the following relation:

$$\mathscr{A}_h\mathscr{A}_c = \left(8\pi\ell_{pl}^2\right)^2 N, \quad N \in \mathbb{N}. \tag{2}$$

ℓ_{pl} is the Planck length. This relation indicates that the product relation is indeed universal in nature. This is a very fascinating topic of research since 2009.

Aspects of BH thermodynamic properties have started by the seminal work of Hawking and Page [10]; they first proposed that certain type of phase transition occurs between small and large BHs in case of Schwarzschild-AdS BH. This phase transition is now called the famous Hawking-Page phase transition. For a charged AdS BH, the study of thermodynamic properties is initiated by Chamblin et al. [11, 12], where the authors demonstrated the critical behaviour of Van der Waal like liquid-gas phase transitions. This has been brought into a new form by Kubizňák and Mann [13] by examining the thermodynamic properties, i.e., $P-V$ criticality of Reissner Nordström AdS BH in the extended phase space. They determined the BH equation of state and computed the critical exponent by using the mean field theory and also computed the other thermodynamic features.

Motivated by the above-mentioned work and our previous investigation [14] in which we have considered the *extended phase space* framework for a wide variety of spherically symmetric AdS spacetime. In the present work, we would like to extend our study for various classes of spherically symmetric BHs and axisymmetric BHs. In the extended phase formalism, the cosmological constant is treated as thermodynamic pressure P and its conjugate variable as thermodynamic volume $\mathscr{V}$ [13, 15–17]. They are defined as

$$P=-\frac{\Lambda}{8\pi}=\frac{3}{8\pi\ell^2}. \tag{3}$$

and

$$\mathscr{V}=\left(\frac{\partial M}{\partial P}\right)_{S,Q,J} \tag{4}$$

The extended phase space is more meaningful than conventional phase space due to the following reasons. The conventional phase space allows the physical parameters like temperature, entropy, charge, and potential, whereas the extended phase space allows the parameters like pressure, volume, and enthalpy (rather than internal energy). In addition to that, the mass parameter should be considered there as enthalpy of the system, which is useful to study the critical behaviour of the thermodynamic system. The BH equation of state could be used to study for comparisons with the classical thermodynamic equation of state (Van der-Waal equation). Once the BH thermodynamic equation of state is in hand, then one may compute different thermodynamic quantities like isothermal compressibility, specific heat at constant pressure, and so forth.

This thermodynamic volume (there are different types of definitions regarding the volume of a BH in the literature; the idea regarding the BH volume was first introduced by Parikh [18]; for other types of definition like dynamical volume and vector volume, see [19–21]; here we are particularly interested regarding the thermodynamic volume [22]) of a spherically symmetric BH and for OH should read

$$\mathscr{V}_h=\frac{4}{3}\pi r_h^3=\frac{\mathscr{A}_h r_h}{3}. \tag{5}$$

r_h is OH radius. Similarly, this volume for IH should be

$$\mathscr{V}_c=\frac{4}{3}\pi r_c^3=\frac{\mathscr{A}_c r_c}{3}. \tag{6}$$

It should be noted that the thermodynamic volume of CH can be obtained by using the symmetric properties [14] of OH radius r_h and IH radius r_c, i.e.,

$$\mathscr{V}_c=V_h\big|_{r_h\leftrightarrow r_c}. \tag{7}$$

Another important point in the extended phase space is that the ADM mass should be treated as the total enthalpy of the thermodynamic system, i.e., $M=H=U+P\mathscr{V}$, where U is thermal energy of the system [15]. Therefore the first law of BH thermodynamics in this phase space for any spherically symmetric spacetime and for OH should be

$$dH=T_h dS_h+\mathscr{V}_h dP+\Phi_h dQ. \tag{8}$$

The quantities T_h, S_h, and Φ_h are denoted as the BH temperature, entropy, and electric potential of OH. The parameter Q is denoted as the charge of a BH.

Analogously, the first law of BH mechanics for IH should be

$$dH=-T_c dS_c+\mathscr{V}_c dP+\Phi_c dQ. \tag{9}$$

The quantities T_c, S_c, and Φ_c are denoted as the corresponding BH temperature, entropy, and electric potential which could be defined on the IH.

When we add the rotation parameter, the first law of BH thermodynamics in the extended phase space (for axisymmetric spacetime and for OH) becomes

$$dH=T_h dS_h+\mathscr{V}_h dP+\Phi_h dQ+\Omega_h dJ. \tag{10}$$

Ω_h and J are the angular velocity defined on the OH and the angular momentum of BH. For IH, the first law becomes

$$dH=-T_c dS_c+\mathscr{V}_c dP+\Phi_c dQ+\Omega_c dJ. \tag{11}$$

Ω_c is the angular velocity defined on the IH. Using symmetric features of r_h and r_c, one can determine the following thermodynamic relations for IH:

$$\begin{aligned}\mathscr{A}_c&=\mathscr{A}_h\big|_{r_h\leftrightarrow r_c},\\ \mathscr{S}_c&=\mathscr{S}_h\big|_{r_h\leftrightarrow r_c},\\ \Omega_c&=\Omega_h\big|_{r_h\leftrightarrow r_c},\\ \Phi_c&=\Phi_h\big|_{r_h\leftrightarrow r_c}\\ T_c&=-T_h\big|_{r_h\leftrightarrow r_c},\\ \mathscr{V}_c&=\mathscr{V}_h\big|_{r_h\leftrightarrow r_c}.\end{aligned} \tag{12}$$

However in this work, we wish to extend our study by computing the volume product, volume sum, volume minus, and volume division in the *extended phase space* for various spherically symmetric BHs and axisymmetric BHs (including the various AdS spacetime). By evaluating these quantities we prove that for a spherically symmetric AdS spacetime the simple volume product is *not* mass-independent. In this case, somewhat complicated combination of volume functional relations of OH and IH are indeed mass-independent. For instance, we have derived the mass-independence volume functional relation for RN-AdS BH as

$$f\left(\mathscr{V}_h,\mathscr{V}_c\right)=\ell^2, \tag{13}$$

where

$$f(\mathcal{V}_h, \mathcal{V}_c) = \left(\frac{3}{32\pi}\right)^{1/3} \frac{\left(8\pi\ell^2 Q^2/3\right)}{\left(\mathcal{V}_h \mathcal{V}_c\right)^{1/3}} - \left(\frac{3}{4\pi}\right)^{2/3} \left[\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3} + \left(\mathcal{V}_h \mathcal{V}_c\right)^{1/3}\right]. \quad (14)$$

For simple Reissner Nordström BH (which is a spherically symmetric solution of Einstein equation) of Einstein gravity and Kehagias-Sfetsos BH of Hořava Lifshitz gravity, the thermodynamic volume products of $\mathcal{H}^{\pm}$ are mass-independent. Therefore they behave as a universal character by its own features. Moreover, we have derived the thermodynamic volume functional relation for Hořava Lifshitz-AdS BH and phantom AdS BH. The phantom fields are exotic because they were produced via negative energy density. Furthermore we have derived volume functional relation for regular BH. Regular BH is a kind of BH which is free from a curvature singularity.

Whereas for axisymmetric class of BHs including AdS spacetime there has been no chance to formulate any possible combinations of thermodynamic volume product to be mass-independent, it should be noted that, for a KN-AdS BH, there may be a possibility of formulating the area (or entropy) product relations to be mass-independent. The reason is that, for a simple Kerr BH, the area (or entropy) product is universal, i.e., mass-independent, while the *volume product* is not! This is because the thermodynamic volume is proportional to the spin parameter. That is why *there has been no chance to produce any combinations of volume product* of $\mathcal{H}^{\pm}$ to be mass-independent. Therefore the axisymmetric BHs show *no* universal behaviour for volume products. Interestingly, only rotating BTZ BH shows the mass-independent feature. Thus *only* axisymmetric BHs in 3D provided the universal character of thermodynamic volume product.

In our previous investigation [8, 9], we computed the BH area (or entropy) products, BH temperature products, Komar energy products, and specific heat products for various classes of BHs. Besides the area (or entropy) product, it should be important to study whether the thermodynamic *volume product, volume sum, volume minus, and volume division* for all the horizons are universal or not and whether they should be quantized or not. This is the main motivation behind this work.

The structure of the paper is as follows. In Section 2, we shall compute the various thermodynamic volume products for spherically symmetric BHs and conclude that the product is mass-independent. In Section 3, we compute various thermodynamic volume products for axisymmetric spacetime and conclude that the product is mass-dependent. Interestingly, for the spinning BTZ BH, the said volume product is *mass-independent*.

2. Spherically Symmetric BH

In this section, we would consider various spherically symmetric BHs.

2.1. Reissner Nordström BH. We begin with charged BH with zero cosmological constant which is a solution of Einstein equation. The metric form is given by

$$ds^2 = -\mathcal{Z}(r)\, dt^2 + \frac{dr^2}{\mathcal{Z}(r)} + r^2 d\Omega_2^2, \quad (15)$$

where

$$\mathcal{Z}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (16)$$

and $d\Omega_2^2$ is metric on the unit sphere in two dimensions.

The OH (there are several definitions of horizons for a static spherically symmetric spacetime; we have used Killing horizons for computations of thermodynamic volume) radius and IH radius read

$$r_h = M + \sqrt{M^2 - Q^2} \quad (17)$$

$$r_c = M - \sqrt{M^2 - Q^2}. \quad (18)$$

M and Q denote the mass and charge of BH, respectively. When $M^2 > Q^2$, it describes a BH; otherwise it has a naked singularity. The thermodynamic volume for OH and IH should read

$$\mathcal{V}_h = \frac{4}{3}\pi r_h^3 \quad (19)$$

$$\mathcal{V}_c = \frac{4}{3}\pi r_c^3. \quad (20)$$

The thermodynamic volume (in the limit $Q = 0$, one obtains the thermodynamic volume for Schwarzschild BH; since in this case the BH has only OH located at $r_h = 2M$, therefore the volume should be $\mathcal{V}_h = (32/3)\pi M^3$; thus for an isolated Schwarzschild BH, the thermodynamic volume should be mass-dependent; therefore it is not universal and not quantized in nature by its own character) product for OH and IH should be

$$\mathcal{V}_h \mathcal{V}_c = \frac{16}{9}\pi^2 Q^6. \quad (21)$$

It is indeed mass-independent; thus it is universal in character and it is also quantized.

The volume sum for OH and IH is calculated to be

$$\mathcal{V}_h + \mathcal{V}_c = \frac{32}{3}\pi M^3 \left(1 - \frac{3}{4}\frac{Q^2}{M^2}\right). \quad (22)$$

Similarly, one can compute the volume minus for OH and IH as

$$\mathcal{V}_h - \mathcal{V}_c = \frac{32}{3}\pi M^2 \sqrt{M^2 - Q^2} \left(1 - \frac{Q^2}{4M^2}\right), \quad (23)$$

and the volume division should be

$$\frac{\mathcal{V}_h}{\mathcal{V}_c} = \left(\frac{M + \sqrt{M^2 - Q^2}}{M - \sqrt{M^2 - Q^2}}\right)^3. \quad (24)$$

It follows from the calculation that all these quantities are mass-dependent so they are not universal in nature by its own right. From (23) and (24), we can easily see that, in the extremal limit $M^2 = Q^2$, one obtains $\mathscr{V}_h = \mathscr{V}_c$. This is a new condition of extreme limit in spherically symmetric cases.

2.2. Hořava Lifshitz BH. In this section, we would briefly review the UV complete theory of gravity which is a nonrelativistic renormalizable theory of gravity known as Hořava Lifshitz [23–25] gravity. It reduces to Einstein's gravity at large scales for the value of dynamical coupling constant $\lambda = 1$. Using ADM formalism, one could write the metric as

$$ds^2 = -N^2 dt^2 + g_{ij}\left(dx^i - N^i dt\right)\left(dx^j - N^j dt\right). \quad (25)$$

In addition for a spacelike hypersurface with a fixed time the extrinsic curvature K_{ij} is given by

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right). \quad (26)$$

A dot represents a derivative with respect to t. The generalized action for Hořava Lifshitz could be written as

$$\begin{aligned} S = \int dt d^3x \sqrt{g} N \Bigg[& \frac{2}{\kappa^2}\left(K_{ij}K^{ij} - \lambda K^2\right) \\ & + \frac{\kappa^2\mu^2\left(\Lambda_w R - 3\Lambda_w^2\right)}{8(1-3\lambda)} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 \\ & - \frac{\kappa^2}{2w^4}\left(C_{ij} - \frac{\mu w^2}{2}R_{ij}\right)\left(C^{ij} - \frac{\mu w^2}{2}R^{ij}\right) + \mu^4 R \Bigg]. \end{aligned} \quad (27)$$

Here $\kappa^2, \lambda, \mu, w$, and Λ are the constant parameters and the cotton tensor, C_{ij}, is defined to be

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R_l^j - \frac{1}{4}\epsilon^{ikj}\partial_k R\right). \quad (28)$$

As compared with Einstein's general relativity, one could obtain the speed of light, Newtonian constant, and the cosmological constant as

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_w}{1-3\lambda}} \quad (29)$$

$$G = \frac{\kappa^2}{32\pi c} \quad (30)$$

$$\Lambda = \frac{3}{2}\Lambda_w, \quad (31)$$

respectively. It should be mentioned here that when $\lambda = 1$, the first three terms in (27) reduce to that one obtains as in Einstein's gravity. It must also be noted that λ is a dynamic coupling constant and for $\lambda > 1/3$, the cosmological constant should be a negative one. However, it could be made a positive one if one could give a following transformation like $\mu \to i\mu$ and $w^2 \to -iw^2$. Here we restrict ourselves that the BH solution is in the limit of $\Lambda_w \to 0$. That is why, we have to set $N^i = 0$ and to get the spherically symmetric solution we have to choose the metric ansatz as

$$ds^2 = -N^2(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (32)$$

In order to get the spherically symmetric solution, substitute the metric ansatz 32 into the action and one obtains reduced Lagrangian as

$$\begin{aligned} \mathscr{L} = \frac{\kappa^2\mu^2 N}{8(1-3\lambda)\sqrt{g}} \Bigg[& (2\lambda - 1)\frac{(g-1)^2}{r^2} - 2\lambda\frac{g-1}{r}g' \\ & + \frac{g-1}{2}g'^2 - 2\omega\left(1 - g - rg'\right)\Bigg], \end{aligned} \quad (33)$$

where $\omega = 8\mu^2(3\lambda - 1)/\kappa^2$. Here we are interested to investigate the situation $\lambda = 1$, i.e., $\omega = 16\mu^2/\kappa^2$. Then one finds the solution of the metric [26] as

$$N^2(r) = g = 1 - \sqrt{4M\omega r + \omega^2 r^4} + \omega r^2, \quad (34)$$

where M is an integration constant related to the mass parameter. Thus the static, spherically symmetric solution is given by

$$ds^2 = -g(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (35)$$

For $r \gg (M/\omega)^{1/3}$, one gets the usual behaviour of a Schwarzschild BH. The BH horizons correspond to $g(r) = 0$. The OH radius and IH radius should read

$$r_h = M + \sqrt{M^2 - \frac{1}{2\omega}} \quad (36)$$

$$r_c = M - \sqrt{M^2 - \frac{1}{2\omega}}, \quad (37)$$

where M and ω denote the mass and coupling constant of BH, respectively. When $M^2 > 1/2\omega$, it describes a BH and when $M^2 < 1/2\omega$, it describes a naked singularity.

The thermodynamic volume product for KS BH should be

$$\mathscr{V}_h\mathscr{V}_c = \frac{2\pi^2}{9\omega^3}. \quad (38)$$

It indicates that it is mass-independent; therefore it is universal in nature and it also be quantized. We do not calculate other possible combinations because it is clear that these combinations are surely mass-dependent as we have seen in case of RN BH. It should be mentioned that the Smarr formula is satisfied in case of Einstein-Aether theory and some variants of infrared HL gravity [27]. It would be interesting if one could examine what the status of HL gravity is when the extended phase space formalism is applied. It could be found elsewhere.

2.3. Nonrotating BTZ BH. The nonrotating BTZ BH is a solution of Einstein-Maxwell gravity in three spacetime dimensions. The metric form is given by

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right)dt^2 + \frac{dr^2}{(r^2/\ell^2 - M)} + r^2 d\phi^2. \quad (39)$$

M is the ADM mass of the BH and $-\Lambda = 1/\ell^2 = 8\pi P G_3$ denotes the cosmological constant. Here we have set $c = \hbar = k = 1$. The BH OH is located at $r_h = \sqrt{8G_3 M}\ell$ (we have already mentioned that in the extended phase space $\ell = \sqrt{3/8\pi P}$; in the subsequent expression, we have to put this condition to obtain the results in terms of thermodynamic pressure). G_3 is 3D Newtonian constant. Interestingly, the thermodynamic volume for 3D static BTZ BH is computed in [17]

$$\mathcal{V}_h = \pi r_h^2 = 8\pi G_3 M \ell^2 \quad (40)$$

This is an isolated case and the thermodynamic volume is mass-dependent; thus it is not quantized as well as it is not universal. $\Lambda = -1/\ell^2$ is cosmological constant.

2.4. Schwarzschild-AdS BH. This BH is a solution of Einstein equation. The form of the metric function is given by

$$\mathcal{Z}(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2}, \quad (41)$$

where $\Lambda = -3/\ell^2$ is cosmological constant. The horizon radii could be calculated from the following equation:

$$r^3 + \ell^2 r - 2M\ell^2 = 0. \quad (42)$$

Among the three roots, only one root is real. Therefore the BH possesses only one physical horizon which is located at

$$r_h = \left(\frac{\ell}{3}\right)^{2/3}\left(9M + \sqrt{3}\sqrt{\ell^2 + 27M^2}\right)^{1/3} - \left(\frac{\ell^4}{3}\right)^{1/3}\frac{1}{\left(9M + \sqrt{3}\sqrt{\ell^2 + 27M^2}\right)^{1/3}}. \quad (43)$$

The thermodynamic volume is computed to be

$$\mathcal{V}_h = \frac{4}{3}\pi r_h^3 = \frac{4}{3} \cdot \pi\left[\left(\frac{\ell}{3}\right)^{2/3}\left(9M + \sqrt{3}\sqrt{\ell^2 + 27M^2}\right)^{1/3} - \left(\frac{\ell^4}{3}\right)^{1/3}\frac{1}{\left(9M + \sqrt{3}\sqrt{\ell^2 + 27M^2}\right)^{1/3}}\right]^3. \quad (44)$$

Since it is an isolated case and the thermodynamic volume is mass-dependent, therefore it is not universal nor does it quantized. We do not consider the other AdS spacetime because it has already been discussed in [14].

2.5. RN-AdS BH. For this BH, the metric function is given by

$$\mathcal{Z}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}. \quad (45)$$

The horizon radii could be found from the following equation:

$$r^4 + \ell^2 r^2 - 2M\ell^2 r + \ell^2 Q^2 = 0. \quad (46)$$

Among the four roots, two roots are real and two roots are imaginary. Thus the OH and IH radii become

$$r_{h,c} = \frac{1}{2}\sqrt{\frac{1}{3}\left(\frac{x}{2}\right)^{1/3} + \left(\frac{2}{x}\right)^{1/3}\frac{\ell^2\left(\ell^2 + 12Q^2\right)}{3} - \frac{2\ell^2}{3}} \pm \frac{1}{2} \cdot \sqrt{\frac{4M\ell^2}{\sqrt{(1/3)(x/2)^{1/3} + (2/x)^{1/3}\ell^2\left(\ell^2 + 12Q^2\right)/3 - 2\ell^2/3}} - \frac{1}{3}\left(\frac{x}{2}\right)^{1/3} - \left(\frac{2}{x}\right)^{1/3}\frac{\ell^2\left(\ell^2 + 12Q^2\right)}{3} - \frac{4\ell^2}{3}}, \quad (47)$$

where

$$x = 2\ell^6 + 108M^2\ell^4 - 72\ell^4Q^2 + \sqrt{\left(2\ell^6 + 108M^2\ell^4 - 72\ell^4Q^2\right)^2 - 4\ell^6\left(\ell^2 + 12Q^2\right)^3} \quad (48)$$

The thermodynamic volume product of RN-AdS BH for OH and IH is computed to be

$$\mathcal{V}_h\mathcal{V}_c = \frac{\pi^2}{36}\left[\frac{2}{3}\left(\frac{x}{2}\right)^{1/3} + \left(\frac{2}{x}\right)^{1/3}\frac{2\ell^2\left(\ell^2 + 12Q^2\right)}{3} + \frac{2\ell^2}{3} - \frac{4M\ell^2}{\sqrt{(1/3)(x/2)^{1/3} + (2/x)^{1/3}\ell^2\left(\ell^2 + 12Q^2\right)/3 - 2\ell^2/3}}\right]^3. \quad (49)$$

It is clearly evident from the above expression that the product is strictly mass-dependent. Thus the product is not universal. But below we would like to determine that *somewhat complicated function of inner and outer horizon* *volume* is indeed mass-independent. To proceed it we would like to use Vieta's theorem. Therefore from (46), we get

$$\sum_{i=1}^{4} r_i = 0. \quad (50)$$

$$\sum_{1\leq i<j\leq 4} r_i r_j = \ell^2. \quad (51)$$

$$\sum_{1\leq i<j<k\leq 4} r_i r_j r_k = 2M\ell^2. \quad (52)$$

$$\prod_{i=1}^{4} r_i = \ell^2 Q^2. \quad (53)$$

Hence the mass-independent *volume sum* and *volume product* relations are

$$\sum_{i=1}^{4} \mathcal{V}_i^{1/3} = 0. \quad (54)$$

$$\sum_{1\leq i<j\leq 4} \left(\mathcal{V}_i \mathcal{V}_j\right)^{1/3} = \left(\frac{3}{32\pi}\right)^{1/3} \frac{8\pi\ell^2}{3}. \quad (55)$$

$$\prod_{i=1}^{4} \left(\mathcal{V}_i\right)^{1/3} = \left(\frac{\pi}{6}\right)^{1/3} \frac{8\pi\ell^2 Q^2}{3}. \quad (56)$$

The mass-independent volume functional relations in terms of two horizons are

$$f\left(\mathcal{V}_h, \mathcal{V}_c\right) = \ell^2, \quad (57)$$

where

$$\begin{aligned} f\left(\mathcal{V}_h, \mathcal{V}_c\right) &= \left(\frac{3}{32\pi}\right)^{1/3} \frac{\left(8\pi\ell^2 Q^2/3\right)}{\left(\mathcal{V}_h \mathcal{V}_c\right)^{1/3}} \\ &- \left(\frac{3}{4\pi}\right)^{2/3} \left[\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3} + \left(\mathcal{V}_h \mathcal{V}_c\right)^{1/3}\right]. \end{aligned} \quad (58)$$

These are explicitly mass-independent volume functional relations in the extended phase space.

2.6. Hořava Lifshitz-AdS BH. The metric function for Hořava Lifshitz BH in AdS space [28, 29] is given by

$$\mathcal{Z}(r) = 1 + \left(1 - \frac{2\Lambda}{3\omega}\right)\omega r^2 - \omega r^2 \sqrt{1 - \frac{4\Lambda}{3\omega} + \frac{4M}{\omega r^3}}. \quad (59)$$

The horizon radii could be calculated from the following equation:

$$4r^4 + 2\left(\omega\ell^2 + 2\right)\ell^2 r^2 - 4M\omega\ell^4 r + \ell^4 = 0. \quad (60)$$

Similarly, among the four roots, two roots are real and two roots are imaginary. Thus the OH and IH radii become

$$\begin{aligned} r_h &= \frac{(a+b)}{2}, \\ r_c &= \frac{(a-b)}{2}, \end{aligned} \quad (61)$$

where

$$a = \sqrt{\frac{2^{1/3}\left(\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4\right) - \left(\ell^2/3\right)\left(\omega\ell^2 + 2\right)}{3y^{1/3}} + \frac{1}{12}\left(\frac{y}{2}\right)^{1/3}} \quad (62)$$

$$b = \sqrt{\frac{2M\omega\ell^4}{a} - \frac{1}{12}\left(\frac{y}{2}\right)^{1/3} - \frac{\left[2^{1/3}\left(\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4\right) + (2/3)\,\ell^2\left(\omega\ell^2 + 2\right)\right]}{3y^{1/3}}} \quad (63)$$

and

$$\begin{aligned} y &= 1728M^2\omega^2\ell^8 - 576\ell^6\left(\omega\ell^2 + 2\right) + 16\ell^6\left(\omega\ell^2 + 2\right)^3 \\ &+ \sqrt{\left[1728M^2\omega^2\ell^8 - 576\ell^6\left(\omega\ell^2 + 2\right) + 16\ell^6\left(\omega\ell^2 + 2\right)^3\right]^2 - 256\ell^{12}\left(\omega^2\ell^4 + 4\omega\ell^2 + 16\right)^3}. \end{aligned} \quad (64)$$

The thermodynamic volume for this BH is quite different from RN-AdS spacetime and it has been calculated in [29]:

$$\mathcal{V}_h = \frac{4}{3}\pi r_h^3 \left[\frac{4}{\ell^2} + \frac{2}{\omega r_h^2}\right] \quad (65)$$

and

$$\mathscr{V}_c = \frac{4}{3}\pi r_c^3 \left[\frac{4}{\ell^2} + \frac{2}{\omega r_c^2}\right]. \quad (66)$$

The volume product is calculated to be

$$\begin{aligned}\mathscr{V}_h\mathscr{V}_c &= \frac{16\pi^2}{9}\left[\frac{2^{4/3}\left(\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4\right) + \left(\ell^2/3\right)\left(\omega\ell^2 + 2\right)}{3y^{1/3}} + \frac{1}{6}\left(\frac{y}{2}\right)^{1/3} - \frac{2M\omega\ell^4}{a}\right] \\ &\times \left[\frac{1}{4\ell^4}\left\{\frac{2^{4/3}\left(\omega^2\ell^8 + 4\omega\ell^6 + 16\ell^4\right) + \left(\ell^2/3\right)\left(\omega\ell^2 + 2\right)}{3y^{1/3}} + \frac{1}{6}\left(\frac{y}{2}\right)^{1/3} - \frac{2M\omega\ell^4}{a}\right\}^2 + \frac{1}{\omega^2} + \frac{1}{\omega\ell^2}\left\{\frac{2M\omega\ell^4}{a} - \frac{\ell^2\left(\omega\ell^2 + 2\right)}{3y^{1/3}}\right\}\right]. \quad (67)\end{aligned}$$

From the above expression, we can conclude that the volume product for Hořava Lifshitz BH in AdS space is strictly mass-dependent. Thus this product is not a universal quantity. Below we will derive *more complicated function of inner and outer horizon volume* that is indeed mass-independent. To compute it, we should apply Vieta's theorem. Thus from (60), we find

$$\sum_{i=1}^{4} r_i = 0. \quad (68)$$

$$\sum_{1\le i<j\le 4} r_i r_j = \frac{\omega\ell^4}{2}\left(1 + \frac{2}{\omega\ell^2}\right). \quad (69)$$

$$\sum_{1\le i<j<k\le 4} r_i r_j r_k = M\omega\ell^4. \quad (70)$$

$$\sum_{1\le i<j<k<l\le 4} r_i r_j r_k r_l = \frac{\ell^4}{4}. \quad (71)$$

Eliminating the mass parameter in terms of two horizons, one could obtain the following mass-independent volume functional relation:

$$g\left(\mathscr{V}_h, \mathscr{V}_c\right) = \frac{\omega\ell^4}{2}\left(1 + \frac{2}{\omega\ell^2}\right), \quad (72)$$

where

$$g\left(\mathscr{V}_h, \mathscr{V}_c\right) = \frac{\left(\ell^4/4\right)}{r_h r_c} - \left(r_h^2 + r_c^2 + r_h r_c\right), \quad (73)$$

where the parameters r_h and r_c could be obtained by solving (65) and (66) in terms of thermodynamic volume as

$$r_h = \frac{1}{2}\left[\left(\frac{u_h}{9}\right)^{1/3}\frac{1}{\omega} - \frac{2\ell^2}{\left(3u_h\right)^{1/3}}\right], \quad (74)$$

$$r_c = \frac{1}{2}\left[\left(\frac{u_c}{9}\right)^{1/3}\frac{1}{\omega} - \frac{2\ell^2}{\left(3u_c\right)^{1/3}}\right], \quad (75)$$

and

$$u_h = \sqrt{3}\sqrt{8\omega^3\ell^6 + 27\ell^4\omega^6 + 27\ell^4\omega^6\left(\frac{3V_h}{4\pi}\right)^3} - 9\ell^2\omega^3\left(\frac{3V_h}{4\pi}\right), \quad (76)$$

$$u_c = \sqrt{3}\sqrt{8\omega^3\ell^6 + 27\ell^4\omega^6 + 27\ell^4\omega^6\left(\frac{3V_c}{4\pi}\right)^3} - 9\ell^2\omega^3\left(\frac{3V_c}{4\pi}\right). \quad (77)$$

Now (72) is completely mass-independent volume functional relation.

2.7. Thermodynamic Volume Products for Phantom BHs. In this section, we would like to discuss the thermodynamic volume products for phantom AdS BH [30]. The phantom fields are exotic fields in BH physics. They could be generated via negative energy density. They could explain the acceleration of our universe. Thus one could expect that these exotic fields might have an important role in BH thermodynamics. We want to study here what is the key role of these phantom fields in thermodynamic volume functional relation? This is the main motivation behind this work. For phantom BH, the metric function is given by

$$\mathscr{Z}(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \eta\frac{Q^2}{r^2}, \quad (78)$$

where the parameter η determines the nature of electromagnetic (EM) field. For $\eta = 1$, one obtains the classical EM theory but when $\eta = -1$, one obtains the Maxwell field which is *phantom*.

Therefore for phantom BH, the horizon radii could be found from the following equation:

$$r^4 + \ell^2 r^2 - 2M\ell^2 r - \ell^2 Q^2 = 0. \quad (79)$$

The above equation has four roots; among them the two roots are real and other two roots are imaginary. Thus the OH and IH radii are

$$r_{h,c} = \frac{1}{2}\sqrt{\frac{1}{3}\left(\frac{z}{2}\right)^{1/3} + \left(\frac{2}{z}\right)^{1/3}\frac{\ell^2\left(\ell^2 - 12Q^2\right)}{3} - \frac{2\ell^2}{3}} \pm \frac{1}{2} \cdot \sqrt{\frac{4M\ell^2}{\sqrt{(1/3)(z/2)^{1/3} + (2/z)^{1/3}\ell^2\left(\ell^2 - 12Q^2\right)/3 - 2\ell^2/3}} - \frac{1}{3}\left(\frac{z}{2}\right)^{1/3} - \left(\frac{2}{z}\right)^{1/3}\frac{\ell^2\left(\ell^2 - 12Q^2\right)}{3} - \frac{4\ell^2}{3}}, \tag{80}$$

where

$$z = 2\ell^6 + 108M^2\ell^4 + 72\ell^4Q^2 + \sqrt{\left(2\ell^6 + 108M^2\ell^4 + 72\ell^4Q^2\right)^2 - 4\ell^6\left(\ell^2 - 12Q^2\right)^3} \tag{81}$$

Now we compute the thermodynamic volume product which turns out to be

$$\mathcal{V}_h\mathcal{V}_c = \frac{\pi^2}{36}\left[\frac{2}{3}\left(\frac{z}{2}\right)^{1/3} + \left(\frac{2}{z}\right)^{1/3}\frac{2\ell^2\left(\ell^2 - 12Q^2\right)}{3} + \frac{2\ell^2}{3} - \frac{4M\ell^2}{\sqrt{(1/3)(z/2)^{1/3} + (2/z)^{1/3}\ell^2\left(\ell^2 - 12Q^2\right)/3 - 2\ell^2/3}}\right]^3. \tag{82}$$

The above product indicates that it is strictly mass-dependent. Therefore the product is not universal. Below we would like to prove that *more complicated function of inner and outer horizon volume* is indeed mass-independent.

To do this we would like to use Vieta's theorem. Thus from (79), we find

$$\sum_{i=1}^{4} r_i = 0. \tag{83}$$

$$\sum_{1\le i<j\le 4} r_i r_j = \ell^2. \tag{84}$$

$$\sum_{1\le i<j<k\le 4} r_i r_j r_k = 2M\ell^2. \tag{85}$$

$$\prod_{i=1}^{4} r_i = -\ell^2 Q^2. \tag{86}$$

Thus the mass-independent *volume sum* and *volume product* relations should read

$$\sum_{i=1}^{4} \mathcal{V}_i^{1/3} = 0. \tag{87}$$

$$\sum_{1\le i<j\le 4} \left(\mathcal{V}_i\mathcal{V}_j\right)^{1/3} = \left(\frac{3}{32\pi}\right)^{1/3}\frac{8\pi\ell^2}{3}. \tag{88}$$

$$\prod_{i=1}^{4} \left(\mathcal{V}_j\right)^{1/3} = \left(\frac{\pi}{6}\right)^{1/3}\frac{8\pi\ell^2 Q^2}{3}. \tag{89}$$

Therefore the mass-independent volume functional relations in terms of two horizons are

$$f\left(\mathcal{V}_h, \mathcal{V}_c\right) = -\ell^2, \tag{90}$$

where

$$f\left(\mathcal{V}_h, \mathcal{V}_c\right) = \left(\frac{3}{32\pi}\right)^{1/3}\frac{\left(8\pi\ell^2Q^2/3\right)}{\left(\mathcal{V}_h\mathcal{V}_c\right)^{1/3}} + \left(\frac{3}{4\pi}\right)^{2/3}\left[\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3} + \left(\mathcal{V}_h\mathcal{V}_c\right)^{1/3}\right]. \tag{91}$$

These are explicitly mass-independent volume functional relations in the extended phase space.

2.8. Thermodynamic Volume Products for AdS BH in $f(R)$ Gravity. In this section, we are interested in deriving the thermodynamic volume products for a static, spherically symmetric AdS BH in $f(R)$ gravity. To some extent, it is called modified gravity. It is a very crucial tool for explaining the current and future status of the accelerating universe. Thus it is very important to investigate the thermodynamic volume products for this gravity. The metric [14, 31] function for this kind of gravity can be written as

$$\mathcal{Z}(r) = 1 - \frac{2m}{r} + \frac{q^2}{\alpha r^2} - \frac{R_0}{12}r^2, \tag{92}$$

where $\alpha = 1 + f'(R_0)$. The parameters m and q are related to the ADM mass, M, and electric charge, Q, by the following expression:

$$m = \frac{M}{\alpha}, \qquad q = \sqrt{\alpha}Q. \tag{93}$$

In this gravity, the thermodynamic pressure could be written as $P = -(\Lambda/8\pi)\alpha = 3/8\pi\ell^2$ and the scalar curvature constant as $R_0 = -12/\ell^2 = 4\Lambda$. Thus the horizon equation for $f(R)$ gravity becomes

$$r^4 + \ell^2r^2 - 2m\ell^2r + \frac{\ell^2q^2}{\alpha} = 0. \tag{94}$$

The EH radius and CH radius are

$$r_h = \frac{(a+b)}{2}, \qquad r_c = \frac{(a-b)}{2}, \tag{95}$$

where

$$a = \sqrt{\frac{1}{3\alpha}\left(\frac{\mu}{2}\right)^{1/3} + \left(\frac{2}{\mu}\right)^{1/3}\frac{\ell^2\left(\alpha\ell^2+12q^2\right)}{3} - \frac{2\ell^2}{3}} \tag{96}$$

and

$$b = \sqrt{\frac{4m\ell^2}{a} - \frac{1}{3\alpha}\left(\frac{\mu}{2}\right)^{1/3} - \left(\frac{2}{\mu}\right)^{1/3}\frac{\ell^2\left(\alpha\ell^2+12q^2\right)}{3} - \frac{4\ell^2}{3}}, \tag{97}$$

where

$$\mu = 2\alpha^3\ell^6 + 108\alpha^3 m^2\ell^4 - 72\alpha^2\ell^4 q^2 + \sqrt{\left(2\alpha^3\ell^6 + 108\alpha^3 m^2\ell^4 - 72\alpha^2\ell^4 q^2\right)^2 - 4\ell^6\left(\alpha^2\ell^2 + 12\alpha q^2\right)^3} \tag{98}$$

The volume products for $f(R)$ gravity are derived to be

$$\mathcal{V}_h\mathcal{V}_c = \frac{\pi^2}{36}\left[\frac{2}{3\alpha}\left(\frac{\mu}{2}\right)^{1/3} + \left(\frac{2}{\mu}\right)^{1/3}\frac{2\ell^2\left(\alpha\ell^2+12q^2\right)}{3} + \frac{2\ell^2}{3} - \frac{4m\ell^2}{a}\right]^3. \tag{99}$$

It indicates that the volume product is not mass-independent. Now we shall give an alternative approach where we would see that more complicated function of volume functional relation is quite mass-independent. To derive it, we should use Vieta's theorem; then one could find

$$\sum_{i=1}^{4} r_i = 0. \tag{100}$$

$$\sum_{1\le i<j\le 4} r_i r_j = \ell^2. \tag{101}$$

$$\sum_{1\le i<j<k\le 4} r_i r_j r_k = 2m\ell^2. \tag{102}$$

$$\prod_{i=1}^{4} r_i = \frac{q^2\ell^2}{\alpha}. \tag{103}$$

Eliminating third and fourth roots, the mass-independent volume functional relation is derived as

$$f\left(\mathcal{V}_h, \mathcal{V}_c\right) = \ell^2, \tag{104}$$

where

$$f\left(\mathcal{V}_h, \mathcal{V}_c\right) = \left(\frac{3}{32\pi}\right)^{1/3}\frac{\left(8\pi\ell^2 q^2/3\right)}{\alpha\left(\mathcal{V}_h\mathcal{V}_c\right)^{1/3}} - \left(\frac{3}{4\pi}\right)^{2/3}\left[\mathcal{V}_h^{\,2/3} + \mathcal{V}_c^{\,2/3} + \left(\mathcal{V}_h\mathcal{V}_c\right)^{1/3}\right]. \tag{105}$$

This equation is explicitly mass-independent.

2.9. Thermodynamic Volume Products for Regular BH. In this section, we compute the thermodynamic volume products for a regular BH derived by Ayón-Beato and García (ABG) [32, 33]. It is a spherically symmetric solution of Einstein's general relativity and it is a curvature singularity free solution. The metric function form of ABG BH is given by

$$\mathcal{Z}(r) = 1 - \frac{2mr^2}{\left(r^2+q^2\right)^{3/2}} + \frac{q^2 r^2}{\left(r^2+q^2\right)^2}. \tag{106}$$

m is the mass of the BH and q is the monopole charge. The horizon radii could be found from the following equation:

$$r^8 + \left(6q^2 - 4m^2\right)r^6 + \left(11q^4 - 4m^2q^2\right)r^4 + 6q^6r^2 + q^8 = 0. \tag{107}$$

This is a polynomial equation of order 8^{th}. This could be reduced to fourth-order polynomial equation by putting $r^2 = z$; then one obtains [33]

$$z^4 + \left(6q^2 - 4m^2\right)z^3 + \left(11q^4 - 4m^2q^2\right)z^2 + 6q^6 z + q^8 = 0. \tag{108}$$

The EH and CH are located at

$$r_h = \sqrt{\frac{\left(2m^2-3q^2\right)}{2} + \frac{a}{2} + \frac{b}{2}} \tag{109}$$

$$r_c = \sqrt{\frac{\left(2m^2-3q^2\right)}{2} - \frac{a}{2} - \frac{b}{2}} \tag{110}$$

and the other horizons (we have considered only here EH and CH; the other horizons are discarded) are located at

$$r_{hc} = \sqrt{\frac{\left(2m^2-3q^2\right)}{2} + \frac{a}{2} - \frac{b}{2}} \tag{111}$$

$$r_{ch} = \sqrt{\frac{\left(2m^2-3q^2\right)}{2} - \frac{a}{2} + \frac{b}{2}} \tag{112}$$

where

$$a = \sqrt{4m^2q^2 - 11q^4 + \left(2m^2 - 3q^2\right)^2 + \frac{\left(11q^4 - 4m^2q^2\right)}{3} + \left(\frac{2}{\delta}\right)^{1/3} \frac{\left(16m^4q^4 - 16m^2q^6 + 25q^8\right)}{3} + \frac{1}{3}\left(\frac{\delta}{2}\right)^{1/3}} \quad (113)$$

and

$$b = \sqrt{c} \quad (114)$$

where

$$c = 4m^2q^2 - 11q^4 + 2\left(2m^2 - 3q^2\right)^2 + \frac{\left(4m^2q^2 - 11q^4\right)}{3} - \left(\frac{2}{\delta}\right)^{1/3} \frac{\left(16m^4q^2 - 16m^2q^6 + 25q^8\right)}{3} - \frac{1}{3}\left(\frac{\delta}{2}\right)^{1/3} + \frac{\left\{48q^6 - 8\left(2m^2 - 3q^2\right)^3 + 8\left(2m^2 - 3q^2\right)\left(11q^4 - 4m^2q^2\right)\right\}}{4a} \quad (115)$$

and

$$\delta = 624m^4q^8 - 128m^6q^6 - 240m^2q^{10} + 250q^{12} + \sqrt{324864m^8q^{16} - 110592m^{10}q^{14} - 193536m^6q^{18} + 172800m^4q^{20}}. \quad (116)$$

The volume product of $\mathcal{H}^{\pm}$ is evaluated to be

$$\mathcal{V}_h\mathcal{V}_c = \frac{\pi^2}{36}\left[\left(2m^2 - 3q^2\right)^2 - (a+b)^2\right]. \quad (117)$$

As usual, the volume product is not mass-independent. Now we would see below that *somewhat more complicated function of inner and outer horizon volume* is indeed mass-independent. To derive it, we have to apply Vieta's theorem in (108); thus one obtains

$$\sum_{i=1}^{4} z_i = 4m^2 - 6q^2. \quad (118)$$

$$\sum_{1\le i<j\le 4} z_iz_j = 11q^4 - 4m^2q^2. \quad (119)$$

$$\sum_{1\le i<j<k\le 4} z_iz_jz_k = -6q^2. \quad (120)$$

$$\prod_{i=1}^{4} z_i = q^8. \quad (121)$$

Eliminating the mass parameter, one obtains the mass-independent equation in terms of two horizons

$$z_1z_2\left(z_1+z_2\right) + 6q^2z_1z_2 - q^8\frac{\left(z_1+z_2\right)}{z_1z_2} - \frac{1}{z_1+z_2+q^2}\left[\left(z_1+z_2\right)^2 + 6q^2\left(z_1+z_2\right) - z_1z_2 - \frac{q^8}{z_1z_2} + 11q^4\right] = 6q^6. \quad (122)$$

It should be noted that the symbols (h, c) and $(1, 2)$ both have the same meaning. Now in terms of volume of $\mathcal{H}^{\pm}$ the mass-independent volume functional relation becomes

$$f\left(\mathcal{V}_h, \mathcal{V}_c\right) = 6q^6, \quad (123)$$

where

$$\begin{aligned} f\left(\mathcal{V}_h, \mathcal{V}_c\right) = &\left(\frac{3}{4\pi}\right)^2 \left(\mathcal{V}_h\mathcal{V}_c\right)^{2/3}\left\{\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3}\right\} + 6q^2\left(\frac{3}{4\pi}\right)^{4/3}\left(\mathcal{V}_h\mathcal{V}_c\right)^{2/3} - q^8\left(\frac{4\pi}{3}\right)^{2/3} \\ &\cdot \frac{\left(\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3}\right)}{\left(\mathcal{V}_h\mathcal{V}_c\right)^{2/3}} - \left(\frac{3}{4\pi}\right)^{2/3} \cdot \frac{\left(\mathcal{V}_h\mathcal{V}_c\right)^{2/3}}{\left\{\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3} + \left(\frac{4\pi}{3}\right)^{2/3} q^2\right\}} \\ &\times \left[\left(\frac{3}{4\pi}\right)^{4/3}\left(\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3}\right)^2 + 6q^2\left(\frac{3}{4\pi}\right)^{2/3}\left(\mathcal{V}_h^{2/3} + \mathcal{V}_c^{2/3}\right) - \left(\frac{3}{4\pi}\right)^{4/3}\left(\mathcal{V}_h\mathcal{V}_c\right)^{2/3} - \left(\frac{4\pi}{3}\right)^{2/3}\frac{q^8}{\left(\mathcal{V}_h\mathcal{V}_c\right)^{2/3}} + 11q^4\right] \end{aligned} \quad (124)$$

Now we are moving to axisymmetric spacetime, to see what happens there?

3. Axisymmetric Spacetime

In this section, we have considered only the various axisymmetric BHs. It is easy to compute volume products for spherically symmetric cases because of $\mathcal{V}_h \propto \mathcal{A}_h r_h$ for OH and $\mathcal{V}_c \propto \mathcal{A}_c r_c$ for IH. For axisymmetric spacetime, this proportionality is quite different because here the spin parameter is present. Now see what happens in this case by starting with Kerr BH.

3.1. Kerr BH.

The Kerr BH is a solution of Einstein equation. The OH radius and IH radius for this BH should read

$$r_h = M + \sqrt{M^2 - a^2} \tag{125}$$

$$r_c = M - \sqrt{M^2 - a^2}, \tag{126}$$

where $a = J/M$. J is angular momentum of the BH. When $M^2 > a^2$, it describes a BH; when $M^2 < a^2$, it describes a naked singularity. The thermodynamic volume for OH [34] and IH [14] becomes

$$\mathcal{V}_h = \frac{\mathcal{A}_h r_h}{3}\left[1 + \frac{a^2}{2r_h^2}\right] \tag{127}$$

$$\mathcal{V}_c = \frac{\mathcal{A}_c r_c}{3}\left[1 + \frac{a^2}{2r_c^2}\right]. \tag{128}$$

The thermodynamic volume product of Kerr BH for OH and IH is calculated to be

$$\mathcal{V}_h\mathcal{V}_c = \frac{128}{9}\pi^2 J^2 M^2\left(1 + \frac{a^2}{8M^2}\right). \tag{129}$$

The volume sum for OH and IH is

$$\mathcal{V}_h + \mathcal{V}_c = \frac{32}{3}\pi M^3\left(1 - \frac{a^2}{4M^2}\right). \tag{130}$$

Similarly, the volume minus for OH and IH is

$$\mathcal{V}_h - \mathcal{V}_c = \frac{32}{3}\pi M^2\sqrt{M^2 - a^2}. \tag{131}$$

The volume division is

$$\frac{\mathcal{V}_h}{\mathcal{V}_c} = \left(\frac{4M^2 - a^2 + 4M\sqrt{M^2 - a^2}}{4M^2 - a^2 - 4M\sqrt{M^2 - a^2}}\right). \tag{132}$$

It indicates that the volume product, volume sum, volume minus, and volume division for Kerr BH are mass-dependent. Therefore the product, the sum, the minus, and the division all are *not* universal.

3.2. Kerr-AdS BH.

The horizon function for Kerr-AdS BH [35] is given by

$$\Delta_r = \left(r^2 + a^2\right)\left(1 + \frac{r^2}{\ell^2}\right) - 2Mr = 0, \tag{133}$$

which gives the quartic order of horizon equation

$$\frac{r^4}{\ell^2} + \left(1 + \frac{a^2}{\ell^2}\right)r^2 - 2mr + a^2 = 0. \tag{134}$$

The quantities m and a are related to the parameters mass M and angular momentum J as follows:

$$m = M\Xi^2, \qquad a = \frac{J}{m}\Xi^2 \tag{135}$$

where $\Xi = 1 - a^2/\ell^2$. To obtain the roots of (134), we apply Vieta's theorem, and we find

$$\sum_{i=1}^{4} r_i = 0. \tag{136}$$

$$\sum_{1\le i<j\le 4} r_i r_j = \ell^2\left(1 + \frac{a^2}{\ell^2}\right). \tag{137}$$

$$\sum_{1\le i<j<k\le 4} r_i r_j r_k = 2m\ell^2. \tag{138}$$

$$\prod_{i=1}^{4} r_i = a^2\ell^2. \tag{139}$$

There are at least two real zeros of (134) which is OH radius and IH radius. After some algebraic computation, we have

$$r_h + r_c = \frac{2m\ell^2}{a^2 + \ell^2 + r_h^2 + r_c^2}, \tag{140}$$

$$r_h r_c = \frac{a^2\ell^2 - \left(r_h r_c\right)^2}{a^2 + \ell^2 + r_h^2 + r_c^2}. \tag{141}$$

The area of the BH for OH is

$$\mathcal{A}_h = \frac{4\pi\left(r_h^2 + a^2\right)}{\Xi}, \tag{142}$$

and for IH is

$$\mathcal{A}_c = \frac{4\pi\left(r_c^2 + a^2\right)}{\Xi}. \tag{143}$$

The thermodynamic volume for OH [17, 22] becomes

$$\mathcal{V}_h = \frac{2\pi\left[\left(r_h^2 + a^2\right)\left(2r_h^2\ell^2 + a^2\ell^2 - r_h^2 a^2\right)\right]}{3r_h\ell^2\Xi^2}. \tag{144}$$

And we derive that the thermodynamic volume for IH becomes

$$\mathcal{V}_c = \frac{2\pi\left[\left(r_c^2 + a^2\right)\left(2r_c^2\ell^2 + a^2\ell^2 - r_c^2 a^2\right)\right]}{3r_c\ell^2\Xi^2}. \tag{145}$$

The thermodynamic volume product for Kerr-AdS BH is calculated in

$$\mathcal{V}_h\mathcal{V}_c = \frac{4\pi^2\left\{r_h^2 r_c^2 + a^2\left(r_h^2 + r_c^2\right) + a^4\right\}}{9\Xi^4 r_h r_c} \times\left[3r_h^2 r_c^2 + 2a^2\left(r_h^2 + r_c^2\right) + \Xi^2 r_h^2 r_c^2\right]. \tag{146}$$

Using (140), (141), (142), and (143), we observe that there is no way to eliminate the mass parameter from (146); therefore the volume product for Kerr-AdS BH is not mass-independent; thus it is not universal and not quantized.

3.3. Kerr-Newman BH. It is an axisymmetric solution of Einstein-Maxwell equations. The OH radius and IH radius for this BH become

$$r_h = M + \sqrt{M^2 - a^2 - Q^2} \quad (147)$$

$$r_c = M - \sqrt{M^2 - a^2 - Q^2}. \quad (148)$$

The thermodynamic volume for OH [17] is

$$\mathscr{V}_h = \frac{2\pi\left[\left(r_h^2 + a^2\right)\left(2r_h^2 + a^2\right) + a^2Q^2\right]}{3r_h}. \quad (149)$$

And we derive that the thermodynamic volume for IH is

$$\mathscr{V}_c = \frac{2\pi\left[\left(r_c^2 + a^2\right)\left(2r_c^2 + a^2\right) + a^2Q^2\right]}{3r_c}. \quad (150)$$

The thermodynamic volume product for KN BH is computed to be

$$\mathscr{V}_h\mathscr{V}_c = \frac{16\pi^2}{9} \times \frac{\left[J^2\left(8J^2 + a^4 - a^2Q^2 - 2Q^4 + 8M^2Q^2\right) + Q^4\left(a^2 + Q^2\right)^2\right]}{a^2 + Q^2}. \quad (151)$$

It also indicates that the thermodynamic volume for KN BH is mass-dependent. Thus the volume product is not universal for any axisymmetric spacetime. In the appropriate limit, i.e., when $a = J = 0$, one obtains the thermodynamic volume product for Reissner Nordström BH and when $Q = 0$, one obtains the volume product for Kerr BH.

3.4. Kerr-Newman-AdS BH. The horizon function for Kerr-Newman-AdS BH [36] reads

$$\Delta_r = \left(r^2 + a^2\right)\left(1 + \frac{r^2}{\ell^2}\right) - 2Mr + q^2 = 0, \quad (152)$$

which has the quartic order of horizon equation

$$\frac{r^4}{\ell^2} + \left(1 + \frac{a^2}{\ell^2}\right)r^2 - 2mr + a^2 + q^2 = 0. \quad (153)$$

The quantity q is related to the charge parameter Q as

$$q = Q\Xi \quad (154)$$

To determine the roots of (153) again we apply Vieta's rule; then one obtains

$$\sum_{i=1}^{4} r_i = 0. \quad (155)$$

$$\sum_{1\le i<j\le 4} r_i r_j = \ell^2\left(1 + \frac{a^2}{\ell^2}\right). \quad (156)$$

$$\sum_{1\le i<j<k\le 4} r_i r_j r_k = 2m\ell^2. \quad (157)$$

$$\prod_{i=1}^{4} r_i = \left(a^2 + q^2\right)\ell^2. \quad (158)$$

Similarly, there are at least two real zeros of (153) which is OH radius and IH radius. After some algebraic derivation, one gets

$$r_h + r_c = \frac{2m\ell^2}{a^2 + \ell^2 + r_h^2 + r_c^2}, \quad (159)$$

$$r_h r_c = \frac{\left(a^2 + q^2\right)\ell^2 - \left(r_h r_c\right)^2}{a^2 + \ell^2 + r_h^2 + r_c^2}. \quad (160)$$

The area of this BH for OH is

$$\mathscr{A}_h = \frac{4\pi\left(r_h^2 + a^2\right)}{\Xi} \quad (161)$$

and for IH is

$$\mathscr{A}_c = \frac{4\pi\left(r_c^2 + a^2\right)}{\Xi}. \quad (162)$$

The thermodynamic volume for OH [17, 22] becomes

$$\mathscr{V}_h = \frac{2\pi\left[\left(r_h^2 + a^2\right)\left(2r_h^2\ell^2 + a^2\ell^2 - r_h^2a^2\right) + \ell^2q^2a^2\right]}{3r_h\ell^2\Xi^2}. \quad (163)$$

And we derive that the thermodynamic volume for IH becomes

$$\mathscr{V}_c = \frac{2\pi\left[\left(r_c^2 + a^2\right)\left(2r_c^2\ell^2 + a^2\ell^2 - r_c^2a^2\right) + \ell^2q^2a^2\right]}{3r_c\ell^2\Xi^2}. \quad (164)$$

The thermodynamic volume product for Kerr-Newman-AdS BH is computed in

$$\mathscr{V}_h\mathscr{V}_c = \frac{4\pi^2}{9\Xi^4 r_h r_c} \times \left[\left\{r_h^2 r_c^2 + a^2\left(r_h^2 + r_c^2\right) + a^4\right\}^2 + \left\{r_h^2 r_c^2 + a^2\left(r_h^2 + r_c^2\right) + a^4\right\}\left\{2r_h^2 r_c^2 + a^2\left(r_h^2 + r_c^2\right)\right\}\right]$$
$$+ \frac{4\pi^2\left[a^2q^2\left\{r_h^4 + r_c^4 + 2a^2\left(r_h^2 + r_c^2\right) + 2a^4\right\} + \Xi^2 r_h^2 r_c^2\left\{r_h^2 r_c^2 + a^2\left(r_h^2 + r_c^2\right) + a^4\right\}\right]}{9\Xi^4 r_h r_c}$$
$$+ \frac{4\pi^2\left[a^2q^2\Xi\left\{r_h^4 + r_c^4 + a^2\left(r_h^2 + r_c^2\right)\right\} + a^4q^4\right]}{9\Xi^4 r_h r_c}. \quad (165)$$

Again using (159), (160), (161), and (162), we speculate that there has been no chance to eliminate the mass parameter from (165); thus the volume product for Kerr-Newman-AdS BH is not mass-independent; therefore it is not universal and not quantized.

3.5. Spinning BTZ BH. The metric for rotating BTZ BH [37] in 2 + 1 dimension is given by

$$ds^2 = -\left(\frac{r^2}{\ell^2} + \frac{J^2}{4r^2} - M\right)dt^2 + \frac{dr^2}{\left(r^2/\ell^2 + J^2/4r^2 - M\right)} + r^2\left(-\frac{J}{2r^2}dt + d\phi\right)^2. \quad (166)$$

M and J represent the ADM mass and the angular momentum of the BH. $-\Lambda = 1/\ell^2 = 8\pi PG_3$ denotes the cosmological constant. Here we have set $8G_3 = 1 = c = \hbar = k$. When $J = 0$, one obtains the static BTZ BH.

The BH OH radius and IH radius are [37, 38]

$$r_h = \sqrt{\frac{M\ell^2}{2}\left(1 + \sqrt{1 - \frac{J^2}{M^2\ell^2}}\right)}. \quad (167)$$

$$r_c = \sqrt{\frac{M\ell^2}{2}\left(1 - \sqrt{1 - \frac{J^2}{M^2\ell^2}}\right)}. \quad (168)$$

The thermodynamic volume for 3D spinning BTZ BH for OH and IH is

$$\mathscr{V}_h = \left(\frac{\partial M}{\partial P}\right)_J = \pi r_h^2 \quad (169)$$

$$\mathscr{V}_c = \left(\frac{\partial M}{\partial P}\right)_J = \pi r_c^2 \quad (170)$$

The thermodynamic volume product is computed to be

$$\mathscr{V}_h\mathscr{V}_c = \frac{\pi^2 J^2 \ell^2}{4}. \quad (171)$$

Interestingly, the thermodynamic volume product for rotating BTZ BH is *mass-independent, i.e., universal,* and it is also quantized. This is the only example for *rotating* cases; the volume product is universal. This is an interesting result of this work.

4. Discussion

In this work, we have demonstrated the thermodynamic products, in particular thermodynamic volume products, of spherically symmetric spacetime and axisymmetric spacetime by incorporating the extended phase space formalism. In this formalism, the cosmological constant should be considered as a thermodynamic pressure and its conjugate parameter as thermodynamic volume. In addition to that, the mass parameter should be treated as enthalpy of the system rather than internal energy. Then in this phase space the first law of BH thermodynamics should be satisfied for both the OH and IH.

We explicitly computed the thermodynamic volume products both for OH and IH of several classes of spherically symmetric and axisymmetric BHs including the AdS spacetime. In this case, the simple volume product of $\mathscr{H}^\pm$ is not mass-independent. Rather slightly more complicated volume functional relations are indeed mass-independent. We have proved that, for simple Reissner Nordström BH of Einstein gravity and Kehagias-Sfetsos BH of Hořava Lifshitz gravity, the thermodynamic volume product of $\mathscr{H}^\pm$ is indeed *universal*. Such products are *mass-independent* for spherically symmetric cases because of $\mathscr{V}_h \propto \mathscr{A}_h r_h$ for OH and $\mathscr{V}_c \propto \mathscr{A}_c r_c$ for IH.

Axisymmetric spacetime does not satisfy this proportionality due to presence of the spin parameter; thus such spacetime shows *no* mass-independent features except the rotating BTZ BH; the only axisymmetric spacetime in 3D showed *universal* features; thus it has been quantized in this sense. We also computed thermodynamic volume sum but they are always mass-dependent so they are not universal as well as they are not quantized.

Like area (or entropy) products, the simple thermodynamic volume product of $\mathscr{H}^\pm$ is not mass-independent; rather more complicated function of volume functional relation is indeed mass-independent. This is often true for spherically symmetric BHs including AdS spacetime. This scenario for axisymmetric spacetime (except 3D BTZ BH) is quite different. In this case, the area functional relation becomes mass-independent whereas the volume functional relation is not mass-independent. For volume products, this is the main difference between spherically symmetric spacetime and axisymmetric spacetime. To sum up, the volume functional relation that we have studied in this work in spherically symmetric cases (but not for axisymmetric cases) further provides some universal properties of the BH which gives some insight into microscopic origin of BH entropy of both outer and inner horizons.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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