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Research Article

Black Hole Thermodynamics and Generalized Uncertainty Principle with Higher Order Terms in Momentum Uncertainty

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We study the modification of thermodynamic properties of Schwarzschild and Reissner-Nordström black hole in the framework of generalized uncertainty principle with correction terms up to fourth order in momentum uncertainty. The mass-temperature relation and the heat capacity for these black holes have been investigated. These have been used to obtain the critical and remnant masses. The entropy expression using this generalized uncertainty principle reveals the area law up to leading order logarithmic corrections and subleading corrections of the form $1/A^n$. The mass output and radiation rate using Stefan-Boltzmann law have been computed which show deviations from the standard case and the case with the simplest form for the generalized uncertainty principle.

1. Introduction

The consistent unification of quantum mechanics (QM) with general relativity (GR) is one of the major tasks in theoretical physics. GR deals with the definition of world-lines of particles, which is in contradiction with QM since it does not allow the notion of trajectory due to the presence of an uncertainty in the determination of the momentum and position of a quantum particle. It has been the aim to unify these two theories into one theory known as quantum gravity. It is quite interesting that all approaches towards quantum gravity such as black hole physics [1–3], string theory [4, 5], or even *Gedanken* experiment [6] predict the existence of a minimum measurable length. The occurrence of such a minimal length also arises in various theories of quantum gravity phenomenology, namely, the generalized uncertainty principle (GUP) [5, 7, 8], modified dispersion relation (MDR) [9–12], and deformed special relativity (DSR) [13], to name a few. It is now widely accepted that Heisenberg uncertainty principle would involve corrections from gravity at energies close to the Planck scale. Thus, emergence of a minimal length seems to be inevitable when gravitational effects are taken into account. There has been a lot of work incorporating the existence of a minimal length scale in condensed matter and atomic physics experiments such as Lamb Shift, Landau levels and the scanning Tunneling Microscope [14–19], loop quantum gravity [20–22], noncommutative geometry [23], computing Planck scale corrections to the phenomena of superconductivity and quantum Hall effect [24], and understanding its consequences in cosmology [25, 26].

The incorporation of the GUP to study black hole thermodynamics has been another interesting area of active research [27–38]. It has been observed that the GUP reveals a self-complete characteristic of gravity which basically amounts to hiding any curvature singularity behind an event horizon as a consequence of matter compression at the Planck scale [39–41]. Further, the effects of the GUP have also been considered in the tunneling formalism for Hawking radiation to evaluate the quantum-corrected Hawking temperature and entropy of a Schwarzschild black hole [42–46]. In our earlier findings [32–36], we have studied the modification of thermodynamic properties, namely, the temperature, heat capacity, and entropy of black holes due to the simplest form of the GUP. Interestingly the correction to the Schwarzschild black hole temperature due to quadratic and linear-quadratic GUP has also been compared with the corrections from the quantum Raychaudhuri equation [47]. Very recently the Lorentz-invariance-violating class of dispersion relations has been applied to study the thermodynamics of black holes [48]. It would therefore be interesting to compare these results with those coming from the GUP.

The above studies motivate us to investigate the modification of thermodynamic properties for Schwarzschild and Reissner-Nordström (RN) black holes using the form of the GUP proposed in [30]. This GUP involves higher order terms in the momentum uncertainty. We compute the remnant and critical masses analytically for these black holes below where the temperature becomes ill-defined. We then use the Stefan-Boltzmann law to estimate the mass and the energy output as a function of time. We finally compute the entropy and obtain the well-known area theorem containing corrections from the GUP with higher order terms in momentum uncertainty.

The paper is organized as follows. In Section 2, we study the thermodynamics of Schwarzschild black hole taking into account the effect of the GUP, with higher order terms in momentum uncertainty. In Section 2.1, we also obtain the mass and radiation rate characteristics for the Schwarzschild black hole as a function of time by using the Stefan-Boltzmann law. In Section 3, we study the thermodynamics of Reissner-Nordström black holes taking into account the effect of the GUP. Finally, we conclude in Section 4.

2. Thermodynamics of Schwarzschild Black Hole

In this paper, we work with the following form of the GUP [49]:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \sum_{i=0}^{\infty} a_i \left(\frac{l_p \Delta p}{\hbar} \right)^{2i}; \qquad [a_0 = 1;\ a_i > 0,\ i = 1, 2, \ldots] \tag{1}$$

where l_p is the Planck length ($\sim 10^{-35} m$). Keeping terms up to fourth order in momentum uncertainty, we have

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left\{ a_0 + a_1 \left(\frac{l_p \Delta p}{\hbar} \right)^2 + a_2 \left(\frac{l_p \Delta p}{\hbar} \right)^4 \right\}. \tag{2}$$

We now consider a Schwarzschild black hole of mass M. In the vicinity of the event horizon of the black hole, let a pair (particle-antiparticle) production occurs. For simplicity we consider the particle to be massless. The particle with negative energy falls inside the horizon and that with positive energy escapes outside the horizon and gets observed by some observer at infinity. The momentum of the emitted particle (p), which also characterizes the temperature (T), is of the order of its uncertainty in momentum Δp. Consequently

$$T = \frac{(\Delta p) c}{k_B} \tag{3}$$

where c is the speed of light and k_B is the Boltzmann constant.

The Hawking temperature of the black hole will be equal to the temperature of the particle when thermodynamic equilibrium is reached. The uncertainty in the position of a particle near the event horizon of the Schwarzschild black hole will be of the order of the Schwarzschild radius of the black hole

$$\Delta x = \epsilon r_s; \quad r_s = \frac{2GM}{c^2} \tag{4}$$

where ϵ is a calibration factor, r_s is the Schwarzschild radius, and G is Newton's universal gravitational constant.

To relate the Hawking temperature of the black hole with the mass of the black hole, we consider the saturated form of the GUP (2)

$$\Delta x \Delta p = \frac{\hbar}{2} \left\{ a_0 + a_1 \left(\frac{l_p \Delta p}{\hbar} \right)^2 + a_2 \left(\frac{l_p \Delta p}{\hbar} \right)^4 \right\}. \tag{5}$$

Substituting (3) and (4) in (5) gives

$$M = \frac{{M_p}^2 c^2}{4\epsilon k_B T} \left\{ 1 + a_1 \frac{k_B^2 T^2}{M_p^2 c^4} + a_2 \frac{k_B^4 T^4}{M_p^4 c^8} \right\} \tag{6}$$

where the relations $c\hbar / l_p = M_p c^2$ and $M_p = c^2 l_p / G$ (M_p being the Planck mass) have been used. In the absence of corrections due to GUP, (6) reduces to

$$M = \frac{M_p^2 c^2}{4\epsilon k_B T}. \tag{7}$$

The value of ϵ now gets fixed to 2π by comparing this expression with the semiclassical Hawking temperature $T = M_p^2 c^2 / 8\pi M k_B$ [50, 51]. The mass-temperature relation (6) finally takes the form

$$M = \frac{{M_p}^2 c^2}{8\pi} \left\{ \frac{1}{k_B T} + a_1 \frac{k_B T}{M_p^2 c^4} + a_2 \frac{k_B^3 T^3}{M_p^4 c^8} \right\}. \tag{8}$$

The heat capacity of the black hole therefore reads

$$C = c^2 \frac{dM}{dT} \tag{9}$$

$$\begin{aligned} &= \frac{{M_p}^2 c^4}{8\pi} \left\{ -\frac{1}{k_B T^2} + a_1 \frac{k_B}{M_p^2 c^4} + 3a_2 \frac{k_B^3 T^2}{M_p^4 c^8} \right\} \\ &= \frac{k_B}{8\pi} \left\{ -\left(\frac{M_p c^2}{k_B T} \right)^2 + a_1 + 3a_2 \left(\frac{k_B T}{M_p c^2} \right)^2 \right\}. \end{aligned} \tag{10}$$

The above expression takes the form

$$C = \frac{k_B}{8\pi} \left\{ -\frac{1}{T'^2} + a_1 + 3a_2 T'^2 \right\} \tag{11}$$

after the following notations are introduced

$$M' = \frac{8\pi M}{M_p};$$
$$T' = \frac{k_B T}{M_p c^2}. \tag{12}$$

The mass of the black hole decreases due to radiation from the black hole. This leads to an increase in the temperature of the black hole. It can be observed from (7) and (10) that there exists a finite temperature at which the heat capacity vanishes. To find out this temperature, we set C = 0. This gives

$$3a_2 T'^4 + a_1 T'^2 - 1 = 0. \tag{13}$$

Solving this, we get

$$T'^2 = \frac{1}{6a_2}\left\{-a_1 + \sqrt{a_1^2 + 12a_2}\right\} \tag{14}$$

where the positive sign before the square root has been taken so that the above result reduces to corresponding result when $a_2 = 0$ [33].

Finally we get the expression for T' to be

$$T' = \frac{1}{\sqrt{6a_2}}\left\{-a_1 + \sqrt{a_1^2 + 12a_2}\right\}^{1/2}. \tag{15}$$

Now in terms of T', M' the mass-temperature relation (8) can be represented as

$$a_2 T'^4 + a_1 T'^2 - M'T' + 1 = 0. \tag{16}$$

The remnant mass can now be obtained by substituting (15) in (16). This yields

$$M'_{rem} = \frac{1}{T'}\left\{a_2 T'^4 + a_1 T'^2 + 1\right\} \Longrightarrow$$
$$M_{rem} = \frac{M_p}{8\pi}\frac{\sqrt{6a_2}}{9a_2\left\{-a_1 + \sqrt{a_1^2 + 12a_2}\right\}^{1/2}}\left[-a_1^2 + a_1\sqrt{a_1^2 + 12a_2} + 12a_2\right]. \tag{17}$$

Reassuringly the above result reduces to the result in $a_2 \to 0$ limit [33]

$$M_{rem} = \frac{M_p\sqrt{a_1}}{4\pi}. \tag{18}$$

Now for $a_1 \to 0$, $a_2 \neq 0$, the remnant mass is given by

$$M_{rem} = \frac{M_p}{6\pi}\left(3a_2\right)^{1/4}. \tag{19}$$

Also for $a_1 \to 0$, the mass-temperature relation (16) reads

$$T'^4 - \frac{M'}{a_2}T' + \frac{1}{a_2} = 0. \tag{20}$$

The solution of this biquadratic equation in T' yields

$$T' = \left[\frac{M'^2}{16a_2^2} + \sqrt{\frac{M'^4}{256a_2^4} - \frac{1}{27a_2^3}}\right]^{1/3} + \left[\frac{M'^2}{16a_2^2} - \sqrt{\frac{M'^4}{256a_2^4} - \frac{1}{27a_2^3}}\right]^{1/3}. \tag{21}$$

The above relation readily implies the existence of a critical mass below which the temperature will be a complex quantity

$$M_{cr} = \frac{M_p}{6\pi}\left(3a_2\right)^{1/4}. \tag{22}$$

This demonstrates that the remnant and critical masses are equal.

At this point, we would like to make a comment. It can be observed from the above analysis that analytical expressions for the remnant and critical masses can be obtained even if one retains terms of order of $(\Delta p)^8$ in the momentum uncertainty. This is because it leads to an equation of the form $aT'^8 + bT'^6 + cT'^4 + dT'^2 + e = 0$ when the condition $C = 0$ is imposed. This equation can be solved analytically to obtain the remnant mass. If we keep terms beyond this order in momentum uncertainty, analytical expressions for the remnant and critical masses can not be obtained.

The black hole entropy from the first law of black hole thermodynamics is given by

$$S = \int c^2\frac{dM}{T} = \int C\frac{dT}{T} = \int \frac{k_B}{8\pi}\left[-\left(\frac{M_p c^2}{k_B T}\right)^2 + a_1 + 3a_2\left(\frac{k_B T}{M_p c^2}\right)^2\right]\frac{dT}{T}$$
$$= \frac{k_B}{8\pi}\left[\frac{1}{2}\left(\frac{M_p c^2}{k_B T}\right)^2 + a_1 \ln\left(\frac{k_B T}{M_p c^2}\right) + \frac{3a_2}{2}\left(\frac{M_p c^2}{k_B T}\right)\right] = \frac{k_B}{8\pi}\left[\frac{1}{2T'^2} + a_1 \log T' + \frac{3}{2} \cdot a_2 T'^2\right]. \tag{23}$$

To obtain the entropy S in terms of the mass M of the black hole, we need to consider (16) to obtain an expression for the temperature T in terms of mass M.

Equation (16) yields up to $\mathcal{O}(a_1^2, a_2^2, a_1 a_2)$

$$T' = \frac{1}{M'} + \frac{a_1}{M'^3} + \frac{a_2}{M'^5} + \frac{2a_1^2}{M'^5} + \frac{6a_1 a_2}{M'^7} + \frac{4a_2^2}{M'^9}. \tag{24}$$

Now the entropy expression in terms of the mass can be written as

$$\begin{aligned}\frac{S}{k_B} &= \frac{1}{8\pi}\left[\frac{1}{2}M'^2 - a_1 - a_1\log M' + \frac{a_1{}^2}{2M'^2} + \frac{a_2}{2M'^2}\right.\\ &\quad\left. + \frac{a_1a_2}{M'^4} + \frac{a_2{}^2}{2M'^6}\right] + \mathcal{O}\left(a_1{}^2a_2, a_1a_2{}^2, a_1{}^3, a_2{}^3\right)\\ &= \frac{S_{BH}}{k_B} - \frac{a_1}{16\pi}\log\left(\frac{S_{BH}}{k_B}\right) - \frac{a_1}{16\pi}\log(16\pi) - \frac{a_1}{8\pi}\\ &\quad + \frac{a_1{}^2 + a_2}{(16\pi)^2}\left(\frac{S_{BH}}{k_B}\right)^{-1} + \frac{2a_1a_2}{(16\pi)^3}\left(\frac{S_{BH}}{k_B}\right)^{-2}\\ &\quad + \frac{a_2{}^2}{(16\pi)^4}\left(\frac{S_{BH}}{k_B}\right)^{-3} + \mathcal{O}\left(a_1{}^2a_2, a_1a_2{}^2, a_1{}^3, a_2{}^3\right)\end{aligned} \tag{25}$$

where $S_{BH}/k_B = 4\pi M^2/M_p^2$ is the semiclassical Bekenstein-Hawking entropy for the Schwarzschild black hole. In terms of the black hole horizon area $A = 4\pi r_s^2 = 16\pi(G^2M^2/c^4) = 4l_p^2(S_{BH}/k_B)$, the above entropy expression can be written as

$$\begin{aligned}\frac{S}{k_B} &= \frac{A}{4l_p^2} - \frac{a_1}{16\pi}\log\left(\frac{A}{4l_p^2}\right) - \frac{a_1}{16\pi}\log(16\pi) - \frac{a_1}{8\pi}\\ &\quad + \frac{a_1{}^2 + a_2}{(16\pi)^2}\left(\frac{A}{4l_p^2}\right)^{-1} + \frac{2a_1a_2}{(16\pi)^3}\left(\frac{A}{4l_p^2}\right)^{-2}\\ &\quad + \frac{a_2{}^2}{(16\pi)^4}\left(\frac{A}{4l_p^2}\right)^{-3}\\ &\quad + \mathcal{O}\left(a_1{}^2a_2, a_1a_2{}^2, a_1{}^3, a_2{}^3\right).\end{aligned} \tag{26}$$

This completes our discussion of the effect of the GUP on the thermodynamic properties of the Schwarzschild black hole. In Figure 1, we present the plot of the entropy of the black hole versus the horizon area for the GUP case and compare it with the standard case.

2.1. Energy Output as a Function of Time. Due to radiation of the black hole, the mass of the black hole reduces, while its temperature keeps on increasing. If one assumes that the energy loss is dominated by photons, then one can apply the Stefan-Boltzmann law to estimate the energy radiated as a function of time

$$\frac{dM}{dt} = -\sigma A T_H{}^4 \tag{27}$$

where σ is the Stefan-Boltzmann constant. In terms of Schwarzschild black hole mass M with the horizon area $A = 4\pi r_s^2 = 16\pi G^2M^2/c^4$, the above equation implies

$$\frac{d}{dt}\left(\frac{8\pi M}{M_p}\right) = -\sigma\frac{2G^2c^4M_p{}^5}{k_B{}^4}\left(\frac{8\pi M}{M_p}\right)^2 T'^4 \tag{28}$$

where we have used $T' = k_BT_H/M_pc^2$.

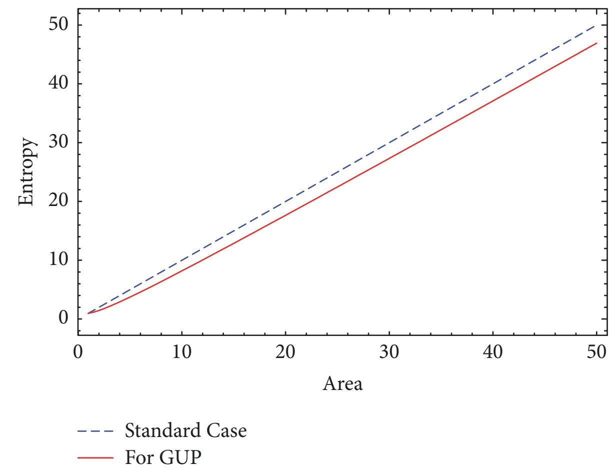

FIGURE 1: The entropy versus area plot. Here the solid line (lower curve) represents the GUP case (considering only the first two terms in the right hand side of (26)), and the dashed line (upper curve) represents the standard case.

We now write the above equation taking into account the effect of the GUP. Thus, considering the mass-temperature relation (24), the radiation rate takes the following form:

$$\frac{dx}{dt} = -\frac{1}{t_{ch}x^2}\left[\frac{1}{x} + \frac{a_1}{x^3} + \frac{a_2}{x^5} + \frac{2a_1{}^2}{x^5} + \frac{6a_1a_2}{x^7} + \frac{4a_2{}^2}{x^9}\right]^4 \tag{29}$$

where we have set $x = 8\pi M/M_p$ and the characteristic time t_{ch} is being defined as $t_{ch} = k_B{}^4/2\sigma M_p{}^5c^4G^2$. If x_i refers to the initial mass at time $t = 0$, the solution of the above equation yields the mass-time relation. Up to $\mathcal{O}(a_1, a_2)$, we have

$$\begin{aligned}x &= \left[-\frac{3t}{t_{ch}} + x_i^3 - 12a_1x_i + \frac{12a_2}{x_i}\right.\\ &\quad\left. + 12a_1\left(x_i{}^3 - \frac{3t}{t_{ch}}\right)^{1/3} - 12a_2\left(x_i^3 - \frac{3t}{t_{ch}}\right)^{-1/3}\right]^{1/3}\end{aligned} \tag{30}$$

where

$$\frac{t}{t_{ch}} = \frac{x_i{}^3}{3} - 4a_1x_i + \frac{4a_2}{x_i}. \tag{31}$$

In Figures 2 and 3, we have plotted the mass of the black hole as a function of time and the radiation rate as a function of time.

3. Thermodynamics of Reissner-Nordström Black Hole

In this section, we consider the Reissner-Nordström (RN) black hole of mass M and charge Q. In this case, near

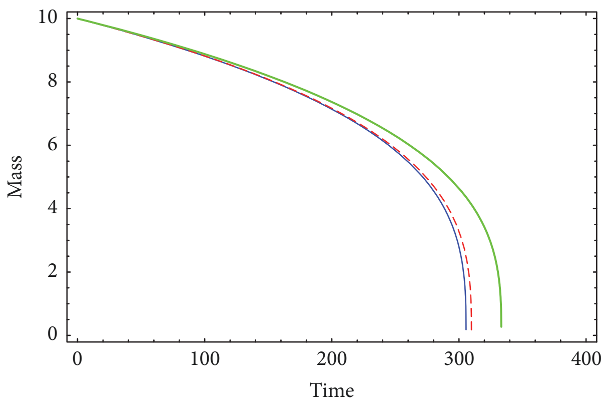

FIGURE 2: The mass of the black hole versus time. The mass is in units of Planck mass and the time is in units of characteristic time. Here thin line (lower curve) represents GUP case (considering both a_2 and a_1), the dashed line (middle curve) represents GUP case (considering only a_1), and thick line (upper curve) represents the standard case.

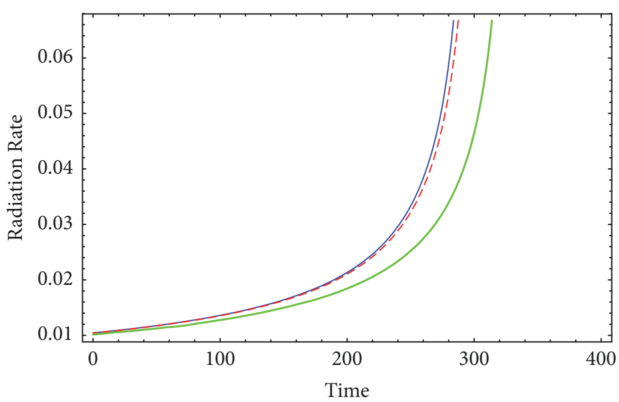

FIGURE 3: The radiation rate of the black hole versus time. The rate is in units of Planck mass per characteristic time and the time is in units of characteristic time. Here thin line (upper curve) represents GUP case (considering both a_2 and a_1), the dashed line (middle curve) represents GUP case (considering only a_1), and thick line (lower curve) represents the standard case.

the horizon of the black hole, the position uncertainty of a particle will be of the order of the RN radius of the black hole

$$\begin{aligned} \Delta x &= \epsilon r_h \\ r_h &= \frac{G r_0}{c^2} \\ r_0 &= M + \sqrt{M^2 - Q^2} \end{aligned} \tag{32}$$

where r_h is the radius of the horizon of the RN black hole. Substituting the value of Δp and Δx from (3) and (32), the GUP (5) can be rewritten as

$$r_0 = \frac{\hbar c^3}{2\epsilon G k_B T}\left[1 + a_1\left(\frac{l_p \Delta p}{\hbar}\right)^2 + a_2\left(\frac{l_p \Delta p}{\hbar}\right)^4\right]. \tag{33}$$

Once again, in the absence of correction due to GUP, (33) reduces to

$$r_0 = \frac{M_p^2 c^2}{2\epsilon k_B T}. \tag{34}$$

Comparing the above relation with the semiclassical Hawking temperature $T = M_p^2 c^2 (M r_0 - Q^2)/2\pi k_B r_0^3$ yields the value of ϵ to be

$$\epsilon = \frac{\pi r_0^2}{(M r_0 - Q^2)}. \tag{35}$$

This finally fixes the form of the mass-charge-temperature relation (33) to be

$$\begin{aligned} &\frac{r_0^2}{(r_0 - M)} \\ &\quad = \frac{M_p}{2\pi}\left[\frac{M_p c^2}{k_B T} + a_1 \frac{k_B T}{M_p c^2} + a_2\left(\frac{k_B T}{M_p c^2}\right)^3\right] \end{aligned} \tag{36}$$

where the identity

$$\frac{r_0}{(M r_0 - Q^2)} = \frac{1}{(r_0 - M)} \tag{37}$$

has been used.

The heat capacity of the black hole can now be calculated using relation (9) and equation (36):

$$\begin{aligned} C = \frac{k_B (r_0 - M)^3}{2\pi {r_0}^2 (2r_0 - 3M)} &\left[-\left(\frac{M_p c^2}{k_B T}\right)^2 + a_1\right. \\ &\left. + 3a_2\left(\frac{k_B T}{M_p c^2}\right)^2\right]. \end{aligned} \tag{38}$$

To express the heat capacity in terms of the mass, once again we make use of relation (12) to recast (36) in the form

$$a_2 {T'}^4 + a_1 {T'}^2 - \frac{g(r_0)}{M_p} T' + 1 = 0 \tag{39}$$

where

$$g(r_0) = \frac{2\pi r_0^2}{(r_0 - M)}. \tag{40}$$

Now to find out the temperature where the radiation process stops, we set C = 0. Equation (38) therefore yields

$$3a_2 {T'}^4 + a_1 {T'}^2 - 1 = 0 \tag{41}$$

from which solution of T' reads

$$T' = \frac{1}{\sqrt{6a_2}}\left\{-a_1 + \sqrt{{a_1}^2 + 12a_2}\right\}^{1/2} \tag{42}$$

where the positive sign before the square root has been taken to reproduce the result corresponding to the limit $a_2 \to 0$ [33].

The remnant mass can now be computed by substituting (42) in (39). This would then give

$$\frac{g(r_0)}{M_p} = \frac{\sqrt{6a_2}}{9a_2\left\{-a_1+\sqrt{a_1^{\,2}+12a_2}\right\}^{1/2}}\left[-a_1^{\,2} + a_1\sqrt{a_1^{\,2}+12a_2}+12a_2\right]. \tag{43}$$

Thus, we finally obtain the following cubic equation for the remnant mass:

$$\left(\frac{2M_pZ}{\pi}\right)M_{rem}^3 - \left(\frac{M_pZ}{2\pi}\right)^2 M_{rem}^2 - \left(\frac{2M_pZ}{\pi}\right)Q^2M_{rem} + Q^4 + \left(\frac{M_pZ}{2\pi}\right)^2 Q^2 = 0 \tag{44}$$

where

$$Z = \frac{\sqrt{6a_2}}{9a_2\left\{-a_1+\sqrt{a_1^{\,2}+12a_2}\right\}^{1/2}}\left[-a_1^{\,2} + a_1\sqrt{a_1^{\,2}+12a_2}+12a_2\right]. \tag{45}$$

Solution of (44) gives us the exact expression of remnant mass for the RN black hole. This yields

$$M_{rem} = \frac{M_pZ}{24\pi}\left[1+\frac{\left(M_p^2Z^2/4\pi^2+48Q^2\right)}{(B/2)^{1/3}} + \frac{4\pi^2}{M_p^2Z^2}\left(\frac{B}{2}\right)^{1/3}\right] \tag{46}$$

where

$$B = \frac{M_p^6Z^6}{32\pi^6} - 18\frac{M_p^4Z^4}{\pi^4}Q^2 - 108\frac{M_p^2Z^2}{\pi^2}Q^4 + \sqrt{-\frac{27}{16}\frac{M_p^{10}Z^{10}}{\pi^{10}}Q^2 - \frac{350}{27}\frac{M_p^8Z^8}{\pi^8}Q^4 - 4968\frac{M_p^6Z^6}{\pi^6}Q^6 + 11664\frac{M_p^4Z^4}{\pi^4}Q^8}. \tag{47}$$

The above expression for the remnant mass reduces to the remnant mass for the Schwarzschild black hole (17) in the $Q \to 0$ limit.

Finally, we proceed to compute the entropy of the RN black hole. To do that, we first obtain an expression of $T'(T)$ from (39) in terms of the mass and the charge of the RN black hole. This gives up to $\mathcal{O}(a_1^{\,2}, a_1a_2, a_2^{\,2})$

$$T' = \frac{M_p}{g(r_0)}\left[1 + a_1\left(\frac{M_p}{g(r_0)}\right)^2 + a_2\left(\frac{M_p}{g(r_0)}\right)^4 + 2a_1^{\,2}\left(\frac{M_p}{g(r_0)}\right)^4 + 6a_1a_2\left(\frac{M_p}{g(r_0)}\right)^6 + 4a_2^{\,2}\left(\frac{M_p}{g(r_0)}\right)^8\right]. \tag{48}$$

From this one now can calculate the entropy for the RN black hole using (38) and (48):

$$\begin{aligned}
\frac{S}{k_B} &= \frac{\pi r_h^2}{l_p^2} - \frac{a_1}{16\pi}\ln\left(\frac{\pi r_h^2}{l_p^2}\right) - \left(\frac{a_1Q^2}{2^3M_p^2} - \frac{a_2}{2^8\pi^2}\right)\left(\frac{\pi r_h^2}{l_p^2}\right)^{-1} - \left(\frac{a_2}{2^7\pi M_p^2} - \frac{a_1\pi Q^4}{2^5M_p^5}\right)\left(\frac{\pi r_h^2}{l_p^2}\right)^{-2} \\
&\quad - \left(\frac{a_2Q^4}{2^7M_p^4}\right)\left(\frac{\pi r_h^2}{l_p^2}\right)^{-3} - \left(\frac{a_2\pi Q^6}{2^8M_p^6}\right)\left(\frac{\pi r_h^2}{l_p^2}\right)^{-4} - \frac{1}{5}\left(\frac{a_2\pi^2Q^8}{2^8M_p^8}\right)\left(\frac{\pi r_h^2}{l_p^2}\right)^{-5} + \mathcal{O}\left(a_1^{\,2}, a_1a_2, a_2^{\,2}\right) \\
&= \frac{S_{BH}}{k_B} - \frac{a_1}{16\pi}\ln\left(\frac{S_{BH}}{k_B}\right) - \left(\frac{a_1Q^2}{2^3M_p^2} - \frac{a_2}{2^8\pi^2}\right)\left(\frac{S_{BH}}{k_B}\right)^{-1} - \left(\frac{a_2}{2^7\pi M_p^2} - \frac{a_1\pi Q^4}{2^5M_p^5}\right)\left(\frac{S_{BH}}{k_B}\right)^{-2} \\
&\quad - \left(\frac{a_2Q^4}{2^7M_p^4}\right)\left(\frac{S_{BH}}{k_B}\right)^{-3} - \left(\frac{a_2\pi Q^6}{2^8M_p^6}\right)\left(\frac{S_{BH}}{k_B}\right)^{-4} - \frac{1}{5}\left(\frac{a_2\pi^2Q^8}{2^8M_p^8}\right)\left(\frac{S_{BH}}{k_B}\right)^{-5} + \mathcal{O}\left(a_1^{\,2}, a_1a_2, a_2^{\,2}\right)
\end{aligned} \tag{49}$$

where $S_{BH}/k_B = \pi r_h^2/l_p^2$ is the semiclassical Bekenstein-Hawking entropy for the RN black hole. In terms of the area of the horizon $A = 4\pi r_h^2 = 4l_p^2(S_{BH}/k_B)$, the above equation can be written as

$$\begin{aligned}\frac{S}{k_B} &= \frac{A}{4l_p^2} - \frac{a_1}{16\pi}\ln\left(\frac{A}{4l_p^2}\right) \\ &\quad - \left(\frac{a_1Q^2}{2^3M_p^2} - \frac{a_2}{2^8\pi^2}\right)\left(\frac{A}{4l_p^2}\right)^{-1} \\ &\quad - \left(\frac{a_2}{2^7\pi M_p^2} - \frac{a_1\pi Q^4}{2^5M_p^5}\right)\left(\frac{A}{4l_p^2}\right)^{-2} \\ &\quad - \left(\frac{a_2Q^4}{2^7M_p^4}\right)\left(\frac{A}{4l_p^2}\right)^{-3} - \left(\frac{a_2\pi Q^6}{2^8M_p^6}\right)\left(\frac{A}{4l_p^2}\right)^{-4} \\ &\quad - \frac{1}{5}\left(\frac{a_2\pi^2Q^8}{2^8M_p^8}\right)\left(\frac{A}{4l_p^2}\right)^{-5} + \mathcal{O}\left({a_1}^2, a_1a_2, {a_2}^2\right)\end{aligned} \tag{50}$$

which is the area theorem for the RN black hole with corrections from the GUP containing higher order terms in the momentum uncertainty.

We would like to conclude this section by mentioning that, in [52], it has been pointed out that there is a part of the information (leaking out of the black hole due to Hawking radiation) related to nonthermal GUP correlations. This insight may be important to provide a solution for the well-known information loss paradox and is worth investigating in future.

4. Conclusions

In this paper, we have investigated the modifications of the various thermodynamic properties of Schwarzschild and Reissner-Nordström black holes using higher order momentum uncertainty terms in the GUP. We obtain the GUP modified mass-temperature relation. This then leads to the existence of a remnant mass thereby preventing the complete evaporation of the black hole. The expression for the remnant and critical masses have been obtained analytically. In this regard, we observe that analytical expressions for these masses can be obtained even if we keep terms of the order of $(\Delta p)^8$ in the momentum uncertainty. Beyond this, it is no longer possible to obtain analytical expression for the critical and remnant masses. We also compute the mass and energy outputs as functions of time using the Stefan-Boltzman law. We observe that these expressions get modified from the standard case as well as the case where the simplest form of the GUP is used. The expression for the entropy exhibits the well-known area theorem in terms of the horizon area in both cases up to leading order corrections from the GUP.

Data Availability

No data has been used.

Conflicts of Interest

There are no conflicts of interest with any funding agency regarding the publication of this manuscript.

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References

[1] M. Maggiore, "A generalized uncertainty principle in quantum gravity," *Physics Letters B*, vol. 304, no. 1-2, pp. 65–69, 1993.

[2] D. J. Gross and P. F. Mende, "The high-energy behavior of string scattering amplitudes," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 197, no. 1-2, pp. 129–134, 1987.

[3] D. J. Gross and P. F. Mende, "String theory beyond the Planck scale," *Nuclear Physics. B. Theoretical, Phenomenological, and Experimental High Energy Physics. Quantum Field Theory and Statistical Systems*, vol. 303, no. 3, pp. 407–454, 1988.

[4] D. Amati, M. Ciafaloni, and G. Veneziano, "Can spacetime be probed below the string size?" *Physics Letters B*, vol. 216, no. 1-2, pp. 41–47, 1989.

[5] K. Konishi, G. Paffuti, and P. Provero, "Minimum physical length and the generalized uncertainty principle in string theory," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 234, no. 3, pp. 276–284, 1990.

[6] F. Scardigli, "Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment," *Physics Letters B*, vol. 452, no. 1-2, pp. 39–44, 1999.

[7] M. Maggiore, "The algebraic structure of the generalized uncertainty principle," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 319, no. 1-3, pp. 83–86, 1993.

[8] M. Maggiore, "Quantum groups, gravity, and the generalized uncertainty principle," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 49, no. 10, pp. 5182–5187, 1994.

[9] G. Amelino-Camelia, M. Arzano, and A. Procaccini, "Severe constraints on the loop-quantum-gravity energy-momentum dispersion relation from the black-hole area-entropy law," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 70, no. 10, Article ID 107501, 2004.

[10] G. Amelino-Camelia, A. Procaccini, and M. Arzano, "A glimpse at the flat-spacetime limit of quantum gravity using the Bekenstein argument in reverse," *International Journal of Modern Physics D*, vol. 13, no. 10, pp. 2337–2343, 2004.

[11] G. Amelino-Camelia, M. Arzano, Y. Ling, and G. Mandanici, "Black-hole thermodynamics with modified dispersion relations and generalized uncertainty principles," *Classical and Quantum Gravity*, vol. 23, no. 7, pp. 2585–2606, 2006.

[12] S. D. Sadatian and H. Dareyni, "Correction of Cardy-Verlinde formula for fermions and bosons with modified dispersion relation," *Annals of Physics*, vol. 380, pp. 71–77, 2017.

[13] J. Magueijo and L. Smolin, "String theories with deformed energy-momentum relations, and a possible nontachyonic bosonic string," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 71, no. 2, Article ID 026010, 6 pages, 2005.

[14] S. Das and E. C. Vagenas, "Universality of quantum gravity corrections," *Physical Review Letters*, vol. 101, Article ID 221301, 4 pages, 2008.

[15] S. Das and E. C. Vagenas, "Phenomenological implications of the generalized uncertainty principle," *Canadian Journal of Physic*, vol. 87, no. 3, pp. 233–240, 2009.

[16] S. Das and E. C. Vagenas, "Reply to 'Comment on 'Universality of Quantum Gravity Corrections," *Physical Review Letters*, vol. 101, 2010.

[17] A. F. Ali, S. Das, and E. C. Vagenas, "Discreteness of space from the generalized uncertainty principle," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 678, no. 5, pp. 497–499, 2009.

[18] M. Wang, "Erratum: "Solitary wave solutions for variant Boussinesq equations" [Phys. Lett. A **199** (1995), no. 3-4, 169–172]," *Physics Letters A*, vol. 212, no. 6, 353 pages, 1996.

[19] A. F. Ali, S. Das, and E. C. Vagenas, "Proposal for testing quantum gravity in the lab," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 84, no. 4, Article ID 044013, 2011.

[20] R. Gambini and J. Pullin, "Nonstandard optics from quantum space-time," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 59, no. 12, 124021, 4 pages, 1999.

[21] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, "Quantum gravity corrections to neutrino propagation," *Physical Review Letters*, vol. 84, no. 11, pp. 2318–2321, 2000.

[22] G. Amelino-Camelia, L. Smolin, and A. Starodubtsev, "Quantum symmetry, the cosmological constant and Planck-scale phenomenology," *Classical and Quantum Gravity*, vol. 21, no. 13, pp. 3095–3110, 2004.

[23] F. Girelli, E. R. Livine, and D. Oriti, "Deformed special relativity as an effective flat limit of quantum gravity," *Nuclear Physics. B. Theoretical, Phenomenological, and Experimental High Energy Physics. Quantum Field Theory and Statistical Systems*, vol. 708, no. 1-3, pp. 411–433, 2005.

[24] S. Das and R. B. Mann, "Planck scale effects on some low energy quantum phenomena," *Physics Letters B*, vol. 704, no. 5, pp. 596–599, 2011.

[25] S. Basilakos, S. Das, and E. C. Vagenas, "Quantum gravity corrections and entropy at the planck time," *Journal of Cosmology and Astroparticle Physics*, vol. 2010, no. 9, article no. 027, 2010.

[26] W. Chemissany, S. Das, A. F. Ali, and E. C. Vagenas, "Effect of the generalized uncertainty principle on post-inflation preheating," *Journal of Cosmology and Astroparticle Physics*, vol. 2011, no. 12, p. 17, 2011.

[27] F. Scardigli, "Some heuristic semi-classical derivations of the Planck length, the Hawking effect and the Unruh effect," *Il Nuovo Cimento B*, vol. 110, no. 9, pp. 1029–1034, 1995.

[28] K. Nozari and S. H. Mehdipour, "Gravitational uncertainty and black hole remnants," *Modern Physics Letters A*, vol. 20, no. 38, pp. 2937–2948, 2005.

[29] K. Nozari and S. H. Mehdipour, "Black holes remnants in extra dimensions and dark matter," *International Journal of Modern Physics A*, vol. 21, no. 23-24, pp. 4979–4992, 2006.

[30] R. Banerjee and S. Ghosh, "Generalised uncertainty principle, remnant mass and singularity problem in black hole thermodynamics," *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, vol. 688, no. 2-3, pp. 224–229, 2010.

[31] K. Nozari and S. Saghafi, "Natural cutoffs and quantum tunneling from black hole horizon," *Journal of High Energy Physics*, vol. 2012, no. 11, article 005, 2012.

[32] S. Gangopadhyay, A. Dutta, and A. Saha, "Generalized uncertainty principle and black hole thermodynamics," *General Relativity and Gravitation*, vol. 46, no. 2, Art. 1661, 10 pages, 2014.

[33] A. Dutta and S. Gangopadhyay, "Remnant mass and entropy of black holes and modified uncertainty principle," *General Relativity and Gravitation*, vol. 46, no. 6, Art. 1747, 10 pages, 2014.

[34] S. Gangopadhyay, A. Dutta, and M. Faizal, "Constraints on the generalized uncertainty principle from black-hole thermodynamics," *EPL (Europhysics Letters)*, vol. 112, no. 2, Article ID 20006, 2015.

[35] M. A. Anacleto, F. A. Brito, G. C. Luna, E. Passos, and J. Spinelly, "Quantum-corrected finite entropy of noncommutative acoustic black holes," *Annals of Physics*, vol. 362, pp. 436–448, 2015.

[36] A. Dutta and S. Gangopadhyay, "Thermodynamics of Black Holes and the Symmetric Generalized Uncertainty Principle," *International Journal of Theoretical Physics*, vol. 55, no. 6, pp. 2746–2754, 2016.

[37] S. Zhou and G.-R. Chen, "Corrected black hole thermodynamics in Damour-Ruffini's method with generalized uncertainty principle," *International Journal of Modern Physics D: Gravitation, Astrophysics, Cosmology*, vol. 26, no. 7, 1750062, 8 pages, 2017.

[38] Z. W. Feng, H. L. Li, X. T. Zu, and S. Z. Yang, "Quantum corrections to the thermodynamics of Schwarzschild-Tangherlini black hole and the generalized uncertainty principle," *The European Physical Journal C*, vol. 76, article 212, 2016.

[39] G. Dvali, S. Folkerts, and C. Germani, "Physics of trans-Planckian gravity," *Physical Review D: Particles, Fields, Gravitation and Cosmology*, vol. 84, no. 2, Article ID 024039, 2011.

[40] G. Dvali and C. Gomez, "Ultra-high energy probes of classicalization," *Journal of Cosmology and Astroparticle Physics*, vol. 2012, no. 7, article 015, 2012.

[41] M. Isi, J. Mureika, and P. Nicolini, "Self-completeness and the generalized uncertainty principle," *Journal of High Energy Physics*, no. 11, 139, front matter+17 pages, 2013.

[42] K. Nozari and S. H. Mehdipour, "Hawking radiation as quantum tunneling from a noncommutative Schwarzschild black hole," *Classical and Quantum Gravity*, vol. 25, no. 17, Article ID 175015, 2008.

[43] K. Nozari and S. H. Mehdipour, "Quantum gravity and recovery of information in black hole evaporation," *EPL (Europhysics Letters)*, vol. 84, no. 2, Article ID 20008, 2008.

[44] M. Dehghani, "Hawking tunneling radiation of the spherically symmetric black holes at the Planck scale," *Astrophysics and Space Science*, vol. 357, no. 2, article no. 169, 2015.

[45] S. Gangopadhyay, "Minimal Length Effects in Black Hole Thermodynamics from Tunneling Formalism," *International Journal of Theoretical Physics*, vol. 55, no. 1, pp. 617–624, 2016.

[46] S.-Z. Yang, Z.-W. Feng, and H.-L. Li, "The tunneling radiation from non-stationary spherical symmetry black holes and the Hamilton-Jacobi equation," *International Journal of Theoretical Physics*, vol. 56, no. 2, pp. 546–553, 2017.

[47] E. C. Vagenas, L. Alasfar, S. M. Alsaleh, and A. F. Ali, "The GUP and quantum Raychaudhuri equation," *Nuclear Physics B*, vol. 931, pp. 72–78, 2018.

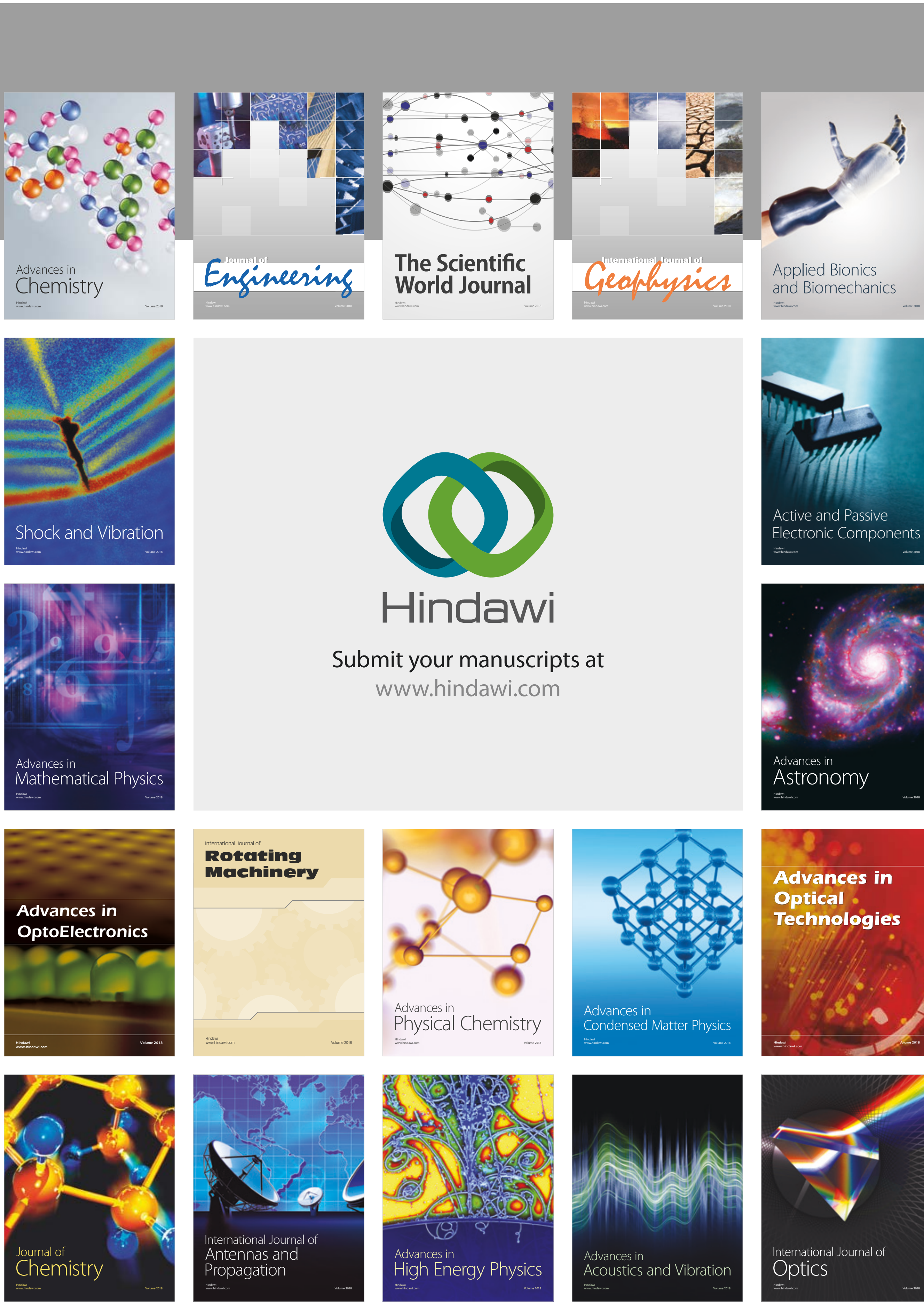
[48] G.-P. Li, J. Pu, Q.-Q. Jiang, and X.-T. Zu, "An application of Lorentz-invariance violation in black hole thermodynamics," *The European Physical Journal C*, vol. 77, no. 10, article no. 666, 2017.

[49] K. Nozari and A. S. Sefiedgar, "Comparison of approaches to quantum correction of black hole thermodynamics," *Physics Letters B*, vol. 635, no. 2-3, pp. 156–160, 2006.

[50] S. W. Hawking, "Black hole explosions?" *Nature*, vol. 248, no. 5443, pp. 30-31, 1974.

[51] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975.

[52] K. Nozari and S. H. Mehdipour, "Parikh-Wilczek tunneling from noncommutative higher dimensional black holes," *Journal of High Energy Physics. A SISSA Journal*, no. 3, 061, 24 pages, 2009.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