Hindawi
Advances in High Energy Physics
Volume 2018, Article ID 9105825, 7 pages
https://doi.org/10.1155/2018/9105825

Research Article

Semiexact Solutions of the Razavy Potential

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Received 9 May 2018; Accepted 10 June 2018; Published 28 August 2018

Academic Editor: Saber Zarrinkamar

In this work, we study the quantum system with the symmetric Razavy potential and show how to find its exact solutions. We find that the solutions are given by the confluent Heun functions. The eigenvalues have to be calculated numerically. The properties of the wave functions depending on m are illustrated graphically for a given potential parameter ξ. We find that the even and odd wave functions with definite parity are changed to odd and even wave functions when the potential parameter m increases. This arises from the fact that the parity, which is a defined symmetry for very small m, is completely violated for large m. We also notice that the energy levels ϵ_i decrease with the increasing potential parameter m.

1. Introduction

It is well-known that the exact solutions of quantum systems play an important role since the early foundation of the quantum mechanics. Generally speaking, two typical examples are studied for the hydrogen atom and harmonic oscillator in classical quantum mechanics textbooks [1, 2]. Up till now, there are a few main methods to solve the quantum soluble systems. The first is called the functional analysis method. That is to say, one solves the second-order differential equation and obtains their solutions [3], which are expressed by some well-known special functions. The second is called the algebraic method, which is realized by studying the Hamiltonian of quantum system. This method is also related to supersymmetric quantum mechanics (SUSYQM) [4], further closely with the factorization method [5]. The third is called the exact quantization rule method [6], from which we proposed proper quantization rule [7], which shows more beauty and symmetry than exact quantization rule. It should be recognized that almost all soluble potentials mentioned above belong to single well potentials. The double-well potentials have not been studied well due to their complications [8–17], in which many authors have been searching the solutions of the double-well potentials for a long history. This is because the double-well potentials could be used in the quantum theory of molecules to describe the motion of the particle in the presence of two centers of force, the heterostructures, Bose-Einstein condensates, superconducting circuits, etc.

Almost forty years ago, Razavy proposed a bistable potential [18]:

$$V(x) = \frac{\hbar^2\beta^2}{2\mu}\left[\frac{1}{8}\xi^2\cosh(4\beta x) - (m+1)\,\xi\cosh(2\beta x) - \frac{1}{8}\xi^2\right], \tag{1}$$

which depends on three potential parameters β, ξ, and a positive integer m. In Figure 1 we plot it as the function of the variables x with various m, in which we take $\beta = 1$ and $\xi = 3$. Choose atomic units $\hbar = \mu = 1$ and also take $\mathscr{V}(x) = 2V(x)$. Using series expansion around the origin, we have

$$\mathscr{V}(x) = (-m\xi - \xi) + x^2\left(-2m\xi + \xi^2 - 2\xi\right)$$

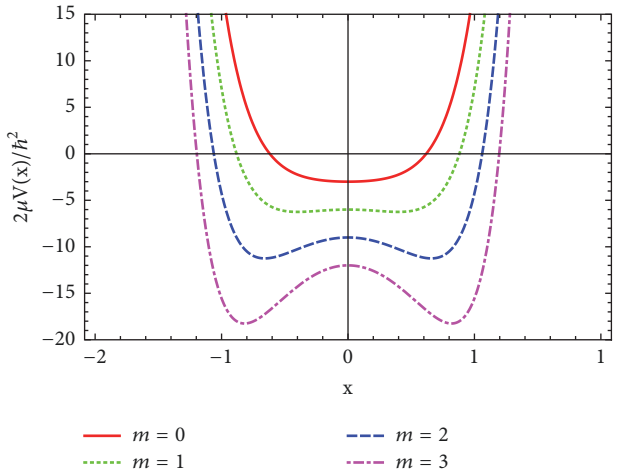


FIGURE 1: (Color online) A plot of potential as function of the variables x and m.

$$+\frac{2}{3}x^4\left(-m\xi+2\xi^2-\xi\right)+\frac{4}{45}x^6\left(-m\xi+8\xi^2-\xi\right)+O\left(x^7\right), \tag{2}$$

which shows that $\mathscr{V}(x)$ is symmetric to variable x. We find that the minimum value of the potential $\mathscr{V}_{\min}(x) = -(m+1)^2 - \xi^2/4$ at two minimum values $x = \pm(1/2)\cosh^{-1}[2(m+1)/\xi]$. For a given value $\xi = 3$, we find that the potential has a flat bottom for $n = 0$, but for $n > 1$ it takes the form of a double-well. Razavy presented the so-called exact solutions by using the "polynomial method" [18]. After studying it carefully, we find that the solutions cannot be given exactly due to the complicated three-term recurrence relation. The method presented there [18] is more like the Bethe Ansatz method as summarized in our recent book [19]. That is, the solutions cannot be expressed as one of special functions because of three-term recurrence relations. In order to obtain some so-called exact solutions, the author has to take some constraints on the coefficients in the recurrence relations as shown in [18]. Inspired by recent study of the hyperbolic type potential well [20–28], in which we have found that their solutions can be exactly expressed by the confluent Heun functions [23], in this work we attempt to study the solutions of the Razavy potential. We shall find that the solutions can be written as the confluent Heun functions but their energy levels have to be calculated numerically since the energy term is involved within the parameter η of the confluent Heun functions $H_c(\alpha,\beta,\gamma,\delta,\eta,z)$. This constraints us to use the traditional Bethe Ansatz method to get the energy levels. Even though the Heun functions have been studied well, its main topics are focused in the mathematical area. Only recent connections with the physical problems have been discovered; in particular the quantum systems for those hyperbolic type potential have been studied [20–28]. The terminology "semiexact" solutions used in [21] arise from the fact that the wave functions can be obtained analytically, but the eigenvalues cannot be written out explicitly.

This paper is organized as follows. In Section 2, we present the solutions of the Schrödinger equation with the Razavy potential. It should be recognized that the Razavy potential is single or double-well depends on the potential parameter m. In Section 3 some fundamental properties of the solutions are studied. The energy levels for different m are calculated numerically. Some concluding remarks are given in Section 4.

2. Semiexact Solutions

Let us consider the one-dimensional Schrödinger equation:

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x)=E\psi(x). \tag{3}$$

Substituting potential (1) into (3), we have

$$\frac{d^2}{dx^2}\psi(x)+\left\{\epsilon-\left[\frac{1}{8}\xi^2\cosh(4x)-(m+1)\xi\cosh(2x)-\frac{1}{8}\xi^2\right]\right\}\cdot\psi(x)=0, \tag{4}$$

$$\epsilon = 2E.$$

Take the wave functions of the form

$$\psi(x)=e^{\xi\cosh^2(x)/2}y(x). \tag{5}$$

Substituting this into (4) allows us to obtain

$$y''(x)+\xi\sinh(2x)\,y'(x)+\left[(m+2)\xi\cosh(2x)+\epsilon\right]y(x)=0. \tag{6}$$

Take a new variable $z = \cosh^2(x)$. The above equation becomes

$$4(z-1)zy''(z)+\left[4z(\xi(z-1)+1)-2\right]y'(z)+((m+2)\xi(2z-1)+\epsilon)\,y(z)=0 \tag{7}$$

which can be rearranged as

$$y''(z)+\left[\xi+\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z-1}\right)\right]y'(z)+\frac{(m+2)\xi(2z-1)+\epsilon}{4(z-1)z}y(z)=0. \tag{8}$$

When comparing this with the confluent Heun differential equation in the simplest uniform form [13]

$$\frac{d^2H(z)}{dz^2}+\left(\alpha+\frac{1+\beta}{z}+\frac{1+\gamma}{z-1}\right)\frac{dH(z)}{dz}+\left(\frac{\mu}{z}+\frac{\nu}{z-1}\right)H(z)=0, \tag{9}$$

we find the solution to (8) is given by the acceptable confluent Heun function $H_c(\alpha, \beta, \gamma, \delta, \eta; z)$ with

$$\begin{aligned}
\alpha &= \xi, \\
\beta &= -\frac{1}{2}, \\
\gamma &= -\frac{1}{2}, \\
\mu &= \frac{\xi(m+2)-\varepsilon}{4}, \\
\nu &= \frac{\xi(m+2)+\varepsilon}{4},
\end{aligned} \tag{10}$$

from which we are able to calculate the parameters δ and η involved in $H_c(\alpha, \beta, \gamma, \delta, \eta; z)$ as

$$\begin{aligned}
\delta &= \mu + \nu - \frac{1}{2}\alpha(\beta+\gamma+2) = \frac{1}{2}(m+1)\xi, \\
\eta &= \frac{1}{2}\alpha(\beta+1) - \mu - \frac{1}{2}(\beta+\gamma+\beta\gamma) \\
&= \frac{1}{8}\left[-2(m+1)\xi + 2\epsilon + 3\right].
\end{aligned} \tag{11}$$

It is found that the parameter η related to energy levels is involved in the confluent Heun function. The wave function given by this function seems to be analytical, but the key issue is how to first get the energy levels. Otherwise, the solution becomes unsolvable. Generally, the confluent Heun function can be expressed as a series of expansions:

$$H_C(\alpha, \beta, \gamma, \delta, \eta, z) = \sum_{n=0}^{\infty} \nu_n(\alpha, \beta, \gamma, \delta, \eta, \xi) z^n, \quad |z| < 1. \tag{12}$$

The coefficients ν_n are given by a three-term recurrence relation:

$$\begin{aligned}
A_n \nu_n - B_n \nu_{n-1} - C_n \nu_{n-2} &= 0, \\
\nu_{-1} &= 0, \\
\nu_0 &= 1,
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
A_n &= 1 + \frac{\beta}{n}, \\
B_n &= 1 + \frac{1}{n}(\beta+\gamma-\alpha-1) \\
&\quad + \frac{1}{n^2}\left\{\eta - \frac{1}{2}(\beta+\gamma-\alpha) - \frac{\alpha\beta}{2} + \frac{\beta\gamma}{2}\right\}, \\
C_n &= \frac{\alpha}{n^2}\left(\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + n - 1\right).
\end{aligned} \tag{14}$$

To make the confluent Heun functions reduce to polynomials, two termination conditions have to be satisfied [13, 14]:

$$\begin{aligned}
\mu + \nu + N\alpha &= 0, \\
\Delta_{N+1}(\mu) &= 0,
\end{aligned} \tag{15}$$

where

$$\begin{vmatrix}
\mu - p_1 & (1+\beta) & 0 & \cdots & 0 & 0 & 0 \\
N\alpha & \mu - p_2 + \alpha & 2(2+\beta) & \cdots & 0 & 0 & 0 \\
0 & (N-1)\alpha & \mu - p_3 + 2\alpha & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mu - p_{N-1} + (N-2)\alpha & (N-1)(N-1+\beta) & 0 \\
0 & 0 & 0 & \cdots & 2\alpha & \mu - p_N + (N-1)\alpha & N(N+\beta) \\
0 & 0 & 0 & \cdots & 0 & \alpha & \mu - p_{N+1} + N\alpha
\end{vmatrix} = 0 \tag{16}$$

with

$$p_N = (N-1)(N+\beta+\gamma). \tag{17}$$

For present problem, it is not difficult to see that the first condition is violated. That is, $\mu + \nu + \alpha = 0$ when $N = 1$. From this we have $m = -4$. This is contrary to the fact that m is a positive integer. Therefore, we cannot use this method to obtain the eigenvalues. On the other hand, we know that $z \in [1, \infty)$. Thus, the series expansion method is invalid. This is unlike previous study [22, 24], in which the quasiexact wave functions and eigenvalues can be obtained by studying those two constraints. The present case is similar to our previous study [20, 21], in which some constraint is violated. We have to choose other approach to study the eigenvalues as used in [20, 21].

3. Fundamental Properties

In this section we are going to study some basic properties of the wave functions as shown in Figures 2–4. We first consider the positive integer m. Since the energy spectrum

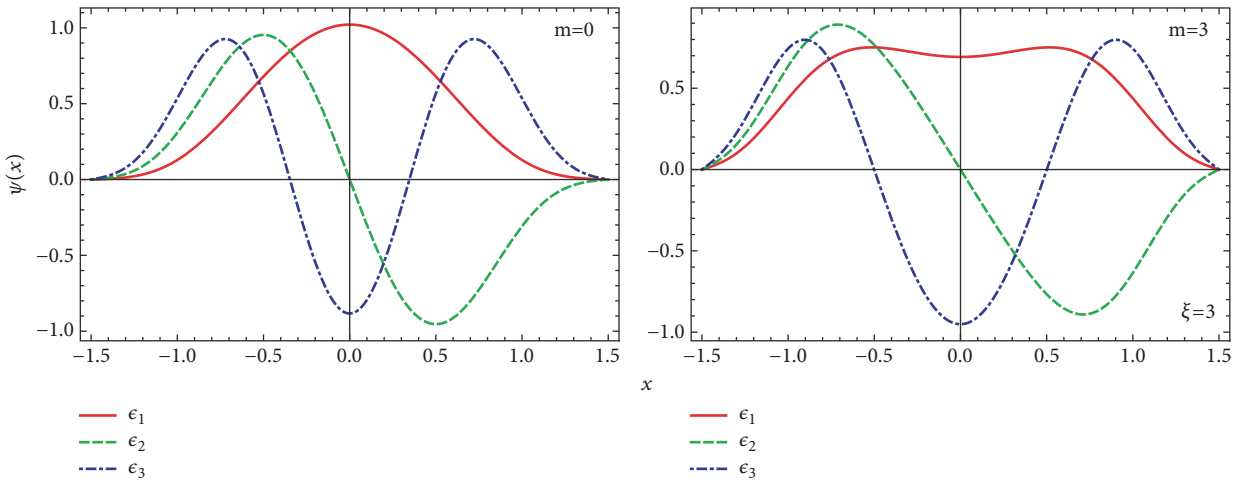


Figure 2: (Color online) The characteristics of the potential $V(z)$ as a function of the position z. We take $m = 0, 1$ and $\xi = 3$.

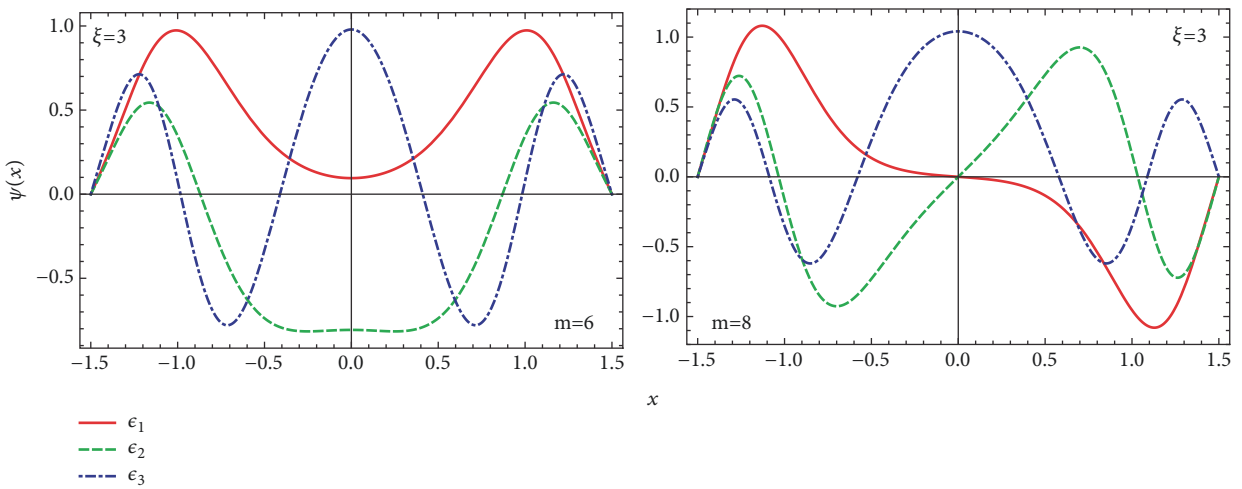
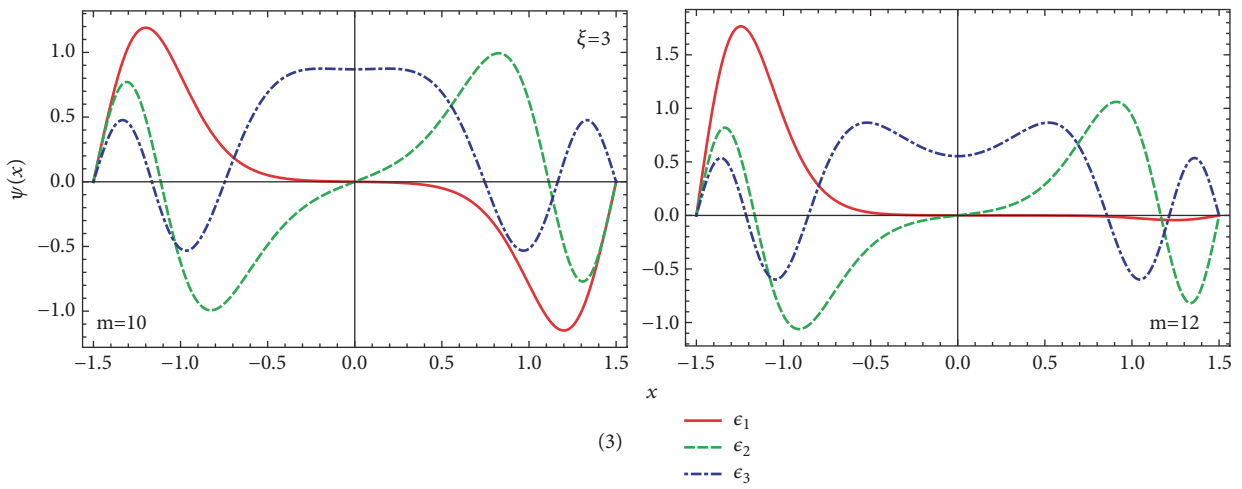


Figure 3: (Color online) The characteristics of the potential $V(z)$ as a function of the position z. We take $m = 6, 8$ and $\xi = 3$.



(3)

Figure 4: (Color online) The same as the above case but $m = 10, 12$.

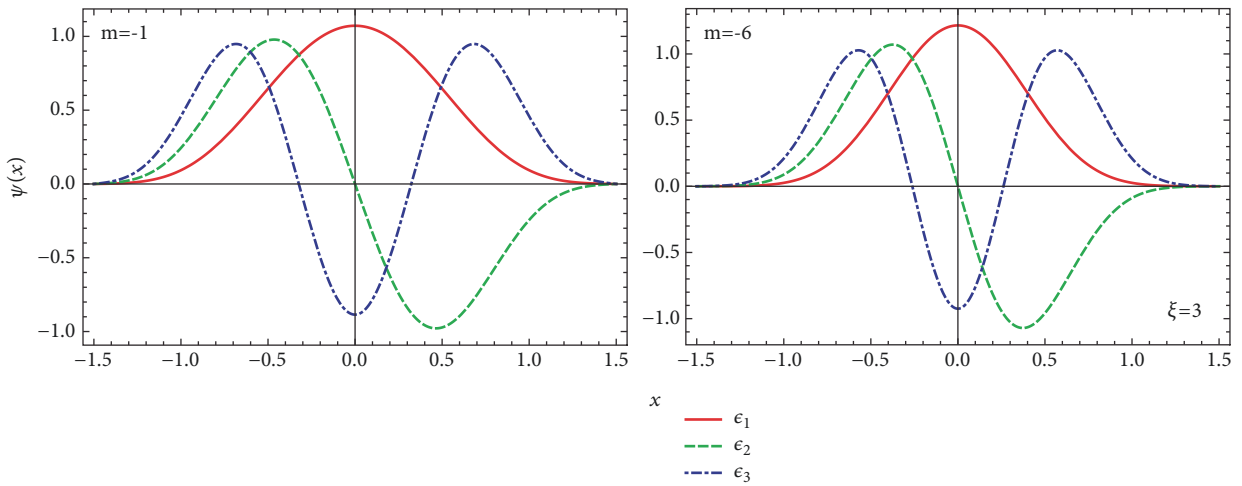


FIGURE 5: (Color online) The characteristics of the potential $V(z)$ as a function of the position z. We take $m = -1, -6$ and $\xi = 3$.

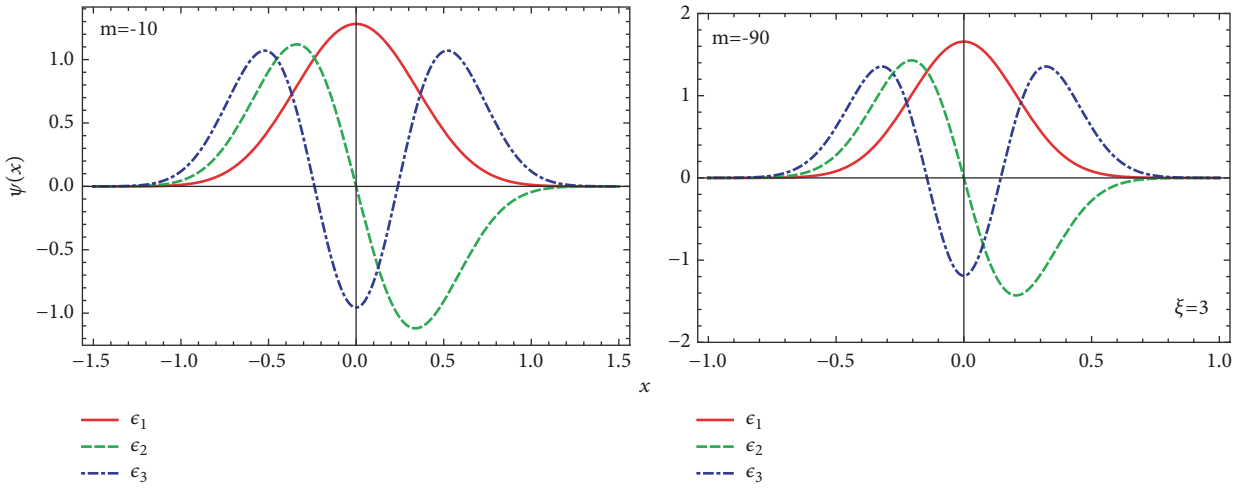


FIGURE 6: (Color online) The same as the above case but $m = -10, -90$.

cannot be given explicitly we have to solve the second-order differential equation (4) numerically. We denote the energy levels as ϵ_i ($i \in [1, 6]$) in Table 1. We find that the energy levels ϵ_i decrease with the increasing m. Originally, we wanted to calculate the energy levels numerically by using powerful MAPLE, which includes some special functions such as the confluent Heun function that cannot be found in MATHEMATICA. As we know, the wave function is given by $\psi(z) = \exp(z\xi/2)H_c(\alpha, \beta, \gamma, \delta, \eta, z)$. Generally speaking, the wave function requires $\psi(z) \longrightarrow 0$ when $z \longrightarrow \infty$; i.e., $x \longrightarrow \infty$. Unfortunately, the present study is unlike our previous study [20, 21], in which $z \longrightarrow 1$ when x goes to infinity. The energy spectra can be calculated by series expansions through taking $z \longrightarrow 1$. On the other hand, the wave functions have a definite parity; e.g., for $m = 0$ some wave functions are symmetric. It is found that such properties are violated when the potential parameter m becomes larger as shown in Figure 4. That is, the wave functions for $m = 12$ are nonsymmetric. In addition, on the contrary to the case discussed by Razavy [18], in which he supposed the m is taken as positive integers, we are going to show what happens to the negative m case. We display the graphics in Figures 5 and 6 for this case. We find that the wave functions are shrunk towards the origin. This makes the amplitude of the wave function increase.

4. Conclusions

In this work we have studied the quantum system with the Razavy potential, which is symmetric with respect to the variable x and showed how its exact solutions are found by transforming the original differential equation into a confluent type Heun differential equation. It is found that the solutions can be expressed by the confluent Heun functions $Hc(\alpha, \beta, \gamma, \delta, \eta)$, in which the energy levels are involved inside the parameter η. This makes us calculate the eigenvalues numerically. The properties of the wave functions depending on m are illustrated graphically for a given potential parameter ξ. We have found that the even and odd wave functions with definite parity are changed to odd and even

Table 1: Energy levels of the Schrödinger equation with potential (1).

ν	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
$m = -6$	21.6608	35.7557	51.3448	68.3341	86.6500	106.233
$m = -5$	18.1891	31.3844	46.1503	62.3746	79.9715	98.8740
$m = -4$	14.6806	26.9167	40.8214	56.2549	73.1150	91.3249
$m = -3$	11.1259	22.3314	35.3346	49.9525	66.0599	83.5680
$m = -2$	7.51110	17.5996	29.6610	43.4412	58.7838	75.5860
$m = -1$	3.81463	12.6800	23.7644	36.6914	51.2639	67.3635
$m = 0$	0.00007	7.51170	17.6027	29.6729	43.4799	58.8919
$m = 1$	-3.99968	2.00200	11.1343	22.3606	35.4208	50.1750
$m = 2$	-8.32288	-3.99300	4.34771	14.7494	27.0959	41.2385
$m = 3$	-13.2815	-10.6927	-2.64788	6.87526	18.5501	32.1389
$m = 4$	-19.5196	-9.46859	-1.17161	9.87916	22.9677	38.0537
$m = 5$	-27.7547	-15.7094	-9.29612	1.24110	13.8439	28.5940
$m = 6$	-38.0314	-21.6913	-17.5131	-7.12621	4.89289	19.3065
$m = 7$	-49.9928	-28.2027	-25.9897	-14.8827	-3.78434	10.2625
$m = 8$	-63.3335	-35.8866	-21.7455	-12.1464	1.51447	17.5661
$m = 9$	-77.8339	-44.5255	-27.8571	-20.2355	-6.89162	8.76577
$m = 10$	-93.3024	-54.9017	-33.6970	-28.1690	-14.8944	0.229704
$m = 11$	-109.592	-65.743	-39.7373	-36.1005	-22.4007	-8.04337
$m = 12$	-126.580	-77.2416	-46.3335	-29.3139	-16.0647	1.06475

wave functions when the potential parameter m increases. This arises from the fact that the parity, which is a defined symmetry for very small m, is completely violated for large m. We have also noticed that the energy levels ϵ_i decrease with the increasing potential parameter m.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by Project 20180677-SIP-IPN, COFAA-IPN, Mexico, and partially by the CONACYT project under Grant no. 288856-CB-2016 and partially by NSFC with Grant no. 11465006.

References

[1] L. I. Schiff, *Quantum Mechanics*, McGraw-Hill Book Co, New York, NY, USA, 3rd edition, 1955.

[2] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Non-Relativistic Theory)*, Pergamon, New York, NY, USA, 3rd edition, 1977.

[3] D. ter Haar, *Problems in Quantum Mechanics*, Pion Ltd, Londonm, UK, 3rd edition, 1975.

[4] F. Cooper, A. Khare, and U. Sukhatme, "Supersymmetry and quantum mechanics," *Physics Reports*, vol. 251, no. 5-6, pp. 267–385, 1995.

[5] S. H. Dong, *Factorization Method in Quantum Mechanics*, Springer, Kluwer Academic Publisher, 2007.

[6] Z. Q. Ma and B. W. Xu, "Quantum correction in exact quantization rules," *Europhys. Lett*, vol. 69, no. 5, 685 pages, 2005.

[7] W. C. Qiang and S. H. Dong, "Proper quantization rule," *EPL*, vol. 89, no. 1, Article ID 10003, 2010.

[8] N. Rosen and P. M. Morse, "On the vibrations of polyatomic molecules," *Physical Review A: Atomic, Molecular and Optical Physics*, vol. 42, no. 2, pp. 210–217, 1932.

[9] M. F. Manning, "Energy levels of a symmetrical double minima problem with applications to the NH_3 and ND_3 molecules," *The Journal of Chemical Physics*, vol. 3, no. 3, pp. 136–138, 1935.

[10] M. Baradaran and H. Panahi, "Lie Symmetry and the Bethe Ansatz Solution of a New Quasi-Exactly Solvable Double-Well Potential," *Advances in High Energy Physics*, vol. 2017, Article ID 2181532, 8 pages, 2017.

[11] H. Konwent, "One-dimensional Schrödinger equation with a new type double-well potential," *Physics Letters A*, vol. 118, no. 9, pp. 467–470, 1986.

[12] H. Konwent, P. Machnikowski, and A. Radosz, "A certain double-well potential related to SU(2) symmetry," *Journal of Physics A: Mathematical and General*, vol. 28, no. 13, pp. 3757–3762, 1995.

[13] Q.-T. Xie, "New quasi-exactly solvable double-well potentials," *Journal of Physics A: Mathematical and General*, vol. 45, no. 17, Article ID 175302, 2012.

[14] B. Chen, Y. Wu, and Q. Xie, "Heun functions and quasi-exactly solvable double-well potentials," *Journal of Physics A: Mathematical and Theoretical*, vol. 46, no. 3, Article ID 035301, 2013.

[15] R. R. Hartmann, "Bound states in a hyperbolic asymmetric double-well," *Journal of Mathematical Physics*, vol. 55, no. 1, 012105, 6 pages, 2014.

[16] A. E. Sitniksky, "Exactly solvable Schrödinger equation with double-well potential for hydrogen bond," *Chemical Physics Letters*, vol. 676, pp. 169–173, 2017.

[17] D.-N. Le, N. D. Hoang, and V.-H. Le, "Exact analytical solutions of the Schrödinger equation for a two dimensional purely sextic double-well potential," *Journal of Mathematical Physics*, vol. 59, no. 3, 032101, 15 pages, 2018.

[18] M. Razavy and Am. J. Phys, "An exactly soluble Schrödinger equation with a bistable potential," *American Journal of Physics*, vol. 48, no. 4, 285 pages, 1980.

[19] S.-H. Dong, *Wave Equation in Higher Dimensions*, Springer, Berlin, Germany, 2011.

[20] S. Dong, Q. Fang, B. J. Falaye, G. Sun, C. Yáñez-Márquez, and S. Dong, "Exact solutions to solitonic profile mass Schrödinger problem with a modified Pöschl–Teller potential," *Modern Physics Letters A*, vol. 31, no. 04, p. 1650017, 2016.

[21] S. Dong, G. H. Sun, B. J. Falaye, and S. H. Dong, "Semi-exact solutions to position-dependent mass Schrödinger problem with a class of hyperbolic potential $V_0\tanh(ax)$," *The European Physical Journal Plus*, vol. 131, no. 176, 2016.

[22] G. H. Sun, S. H. Dong, K. D. Launey et al., "Shannon information entropy for a hyperbolic double-well potential," *International Journal of Quantum Chemistry*, vol. 115, no. 14, pp. 891–899, 2015.

[23] N. Ainsworth-Vaughn, *Heun's Differential Equations*, A. Ronveaux, Ed., Oxford University Press, Oxford, UK, 1995.

[24] C. A. Downing, "On a solution of the Schrödinger equation with a hyperbolic double-well potential," *Journal of Mathematical Physics*, vol. 54, no. 7, 072101, 8 pages, 2013.

[25] P. P. Fiziev, "Novel relations and new properties of confluent Heun's functions and their derivatives of arbitrary order," *Journal of Physics A: Mathematical and General*, vol. 43, no. 3, article 035203, 2010.

[26] R. Hartmann and M. E. Portnoi, "Quasi-exact solution to the Dirac equation for the hyperbolic-secant potential," *Physical Review A*, vol. 89, Article ID 012101, 2014.

[27] D. Agboola, "On the solvability of the generalized hyperbolic double-well models," *Journal of Mathematical Physics*, vol. 55, no. 5, 052102, 8 pages, 2014.

[28] F.-K. Wen, Z.-Y. Yang, C. Liu, W.-L. Yang, and Y.-Z. Zhang, "Exact polynomial solutions of Schrödinger equation with various hyperbolic potentials," *Communications in Theoretical Physics*, vol. 61, no. 2, pp. 153–159, 2014.