Hindawi
Advances in High Energy Physics
Volume 2020, Article ID 4269246, 9 pages
https://doi.org/10.1155/2020/4269246

Research Article

Zeeman Effect in Phase Space

R. A. S. Paiva,[1] R. G. G. Amorim,[1,2] S. C. Ulhoa,[1] A. E. Santana,[1] and F. C. Khanna[3]

[1]*International Center of Physics, Instituto de Física, Universidade de Brasília, 70910-900 Brasília, DF, Brazil*
[2]*Faculdade Gama, Universidade de Brasília, 72444-240 Brasília, DF, Brazil*
[3]*Department of Physics and Astronomy, University of Victoria, BC, Canada V8P 5C2*

Correspondence should be addressed to R. G. G. Amorim; ronniamorim@gmail.com

Received 9 September 2019; Revised 21 October 2019; Accepted 21 November 2019; Published 8 January 2020

Academic Editor: Smarajit Triambak

The two-dimensional hydrogen atom in an external magnetic field is considered in the context of phase space. Using the solution of the Schrödinger equation in phase space, the Wigner function related to the Zeeman effect is calculated. For this purpose, the Bohlin mapping is used to transform the Coulomb potential into a harmonic oscillator problem. Then, it is possible to solve the Schrödinger equation easier by using the perturbation theory. The negativity parameter for this system is realised.

1. Introduction

Since the early years of the development of quantum mechanics, its formulation in phase space has been troublesome. The seminal paper by Wigner in 1932 addressed such a problem in an attempt to deal with the superfluidity of helium [1]. The Wigner function, $f_W(q,p)$, was introduced as a Fourier transform of the density matrix, $\rho(q,q')$. Then, the phase space manifold (Γ), which has a symplectic structure, is described by the coordinates (q,p) [1–4]. The Wigner function is identified as a quasiprobability density since $f_W(q,p)$ is real but it is not positively defined. However, the integrals $\rho(q)=\int fw(q,p)dp$ and $\rho(p)=\int fw(q,p)dp$ are distribution functions.

In the Wigner formalism, each operator, A, defined in the Hilbert space, $\mathscr{H}$, is associated with a function, $a_W(q,p)$, in Γ. This procedure is precisely specified by a mapping $\Omega_W : A \longrightarrow a_W(q,p)$, such that the associative algebra of operators defined in $\mathscr{H}$ leads to be an algebra in Γ, given by $\Omega_W : AB \longrightarrow a_W \star b_W$, where the star product, $\star$, is defined by

$$a_W \star b_W = a_W(q,p)\exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right]b_W(q,p). \quad (1)$$

As a consequence, a noncommutative structure in Γ is obtained that has been explored in different ways [2–25]. Recently [26–30], unitary representations of symmetry Lie groups have been obtained on a symplectic manifold, exploring the noncommutative nature of the star product and using the mapping Ω_W [26–28]. As a result, using a specific representation of a Galilei group, the Schrödinger equation in phase space is obtained. On the other hand, the scalar representation of the Lorentz group for spin 0 and spin 1/2 leads to the Klein-Gordon and Dirac equations in phase space. In relativist and nonrelativistic approaches, the wave functions are closely associated with the Wigner function [26, 27]. This provides a fundamental ingredient for the physical interpretation of the formalism showing its advantage in relation to other attempts.

In recent years, the two-dimensional physical systems have been investigated due to both experimental and theoretical interests. For example, it is possible to cite the fractional Hall effect in a tilted magnetic field [31, 32], superconductivity in two-dimensional organic conductors induced by magnetic field [33], investigations of graphene [34, 35], etc. Particularly, a two-dimensional model of hydrogen atom was considered in several contexts; for instance, it is possible to describe highly three-dimensional anisotropic crystals [36], semiconductor heterostructures [37–39], and astrophysical applications [40–42]. In addition, the hydrogen atom in a uniform magnetic field can present a nonclassical behavior with the increase of strength of magnetic field [43–46]. In this paper, the objective is to investigate the two-dimensional hydrogen atom in a constant magnetic field in phase space picture, i.e., the Schrödinger equation in phase space is used to study the Zeeman effect. The Zeeman effect is applied in a variety of systems, including intense laser lights, comic rays, and the study of intergalactic and interstellar medium [47, 48]. The importance to study the Wigner function for such an effect is in order to obtain information about the chaotic nature of such systems.

In Section 2, a review about the formalism of quantum mechanics in phase space and its connection with Wigner function is presented. In Section 3, the Hamiltonian of hydrogen atom in an external magnetic field and Bohlin mapping are discussed. The solution for the Schrödinger equation for the Zeeman effect in phase space is solved by perturbative method in Section 4. In Section 5, a summary and concluding remarks are presented.

2. Symplectic Quantum Mechanics

In this section, the representation of the Galilei group in $\mathscr{H}(\Gamma)$ is presented. This procedure leads us to the Schrödinger equation in phase space. Then, a connection between this representation and the Wigner formalism is established.

Using the star operator, $\widehat{A} = a\star$, the position and momentum operators, respectively, are defined by

$$\widehat{Q} = q\star = q + \frac{i\hbar}{2}\partial_p, \quad (2)$$

$$\widehat{P} = p\star = p - \frac{i\hbar}{2}\partial_q. \quad (3)$$

The operators given in equations (2) and (3) satisfy the Heisenberg commutation relation,

$$\left[\widehat{Q}, \widehat{P}\right] = i\hbar. \quad (4)$$

In addition, the following operators are introduced:

$$\begin{aligned} \widehat{K} &= m\widehat{Q}_i - t\widehat{P}_i, \\ \widehat{L}_i &= \epsilon_{ijk}\widehat{Q}_j\widehat{P}_k, \\ \widehat{H} &= \frac{\widehat{P}^2}{2m} = \frac{1}{2m}\left(\widehat{P}_1^2 + \widehat{P}_2^2 + \widehat{P}_3^2\right). \end{aligned} \quad (5)$$

From this set of unitary operators, after simple calculations, the following set of commutation relations are obtained:

$$\begin{aligned} \left[\widehat{L}_i, \widehat{L}_j\right] &= i\hbar\epsilon_{ijk}\widehat{L}_k, \\ \left[\widehat{L}_i, \widehat{K}_j\right] &= i\hbar\epsilon_{ijk}\widehat{K}_k, \\ \left[\widehat{L}_i, \widehat{P}_j\right] &= i\hbar\epsilon_{ijk}\widehat{P}_k, \\ \left[\widehat{K}_i, \widehat{P}_j\right] &= i\hbar m\delta_{ij}1, \\ \left[\widehat{K}_i, \widehat{H}\right] &= i\hbar\widehat{P}_i, \end{aligned} \quad (6)$$

with all other commutation relations being null. This is the Galilei-Lie algebra with a central extension characterized by m. The operators defining the Galilei symmetry $\widehat{P}$, $\widehat{K}$, $\widehat{L}$, and $\widehat{H}$ are then generators of translations, boost, rotations, and time translations, respectively.

Defining the operators

$$\begin{aligned} \bar{Q} &= q1, \\ \bar{P} &= p1, \end{aligned} \quad (7)$$

for boost, operators $\bar{Q}$ and $\bar{P}$ transform as

$$\exp\left(-iv\frac{\widehat{K}}{\hbar}\right)2\bar{Q}\exp\left(-iv\frac{\widehat{K}}{\hbar}\right) = 2\bar{Q} + vt1, \quad (8)$$

$$\exp\left(-iv\frac{\widehat{K}}{\hbar}\right)2\bar{P}\exp\left(iv\frac{\widehat{K}}{\hbar}\right) = 2\bar{P} + mv1. \quad (9)$$

This shows that $\bar{Q}$ and $\bar{P}$ transform as position and momentum variables, respectively. These operators satisfy $[\bar{Q}, \bar{P}] = 0$. Then, $\bar{Q}$ and $\bar{P}$ cannot be interpreted as observables. Nevertheless, they can be used to construct a Hilbert space framework in phase space. Then, we define the functions $\phi(q, p)$ in $\mathscr{H}(\Gamma)$ that satisfy the condition

$$\int dqdp\phi^*(q, p)\phi(q, p) < \infty. \quad (10)$$

The wave function $\psi(q, p, t) = \langle q, p \mid \psi(t)\rangle$ associated with the state of the system is defined, but does not have the content of the usual quantum mechanics state.

The time evolution equation for $\psi(q, p, t)$ is derived by using the generator of time translations, i.e.,

$$\psi(t) = e^{-i\widehat{H}t/\hbar}\psi(0). \quad (11)$$

Then, this leads to

$$i\hbar\partial_t\psi(q, p\,; t) = \widehat{H}(q, p)\psi(q, p\,; t), \quad (12)$$

or

$$i\hbar\partial_t\psi(q, p\,; t) = H(q, p)\star\psi(q, p\,; t), \quad (13)$$

which is the Schrödinger equation in phase space [26].

The association of $\psi(q,p,t)$ with the Wigner function is given by [26]

$$fw(q,p)=\psi(q,p,t)\star\psi^{\dagger}(q,p,t). \tag{14}$$

This function satisfies the Liouville-von Neumann equation [26].

3. Two-Dimensional Hydrogen Atom in an External Magnetic Field and Bohlin Mapping

The Hamiltonian for the two-dimensional hydrogen atom in a constant and uniform magnetic field $\mathbf{B}=B\hat{z}$ is given as [44, 49, 50]

$$H=\frac{(\mathbf{P}\text{-}e\mathbf{A})^2}{2m}-\frac{k}{(x^2+y^2)^{1/2}}, \tag{15}$$

where m and e represent the electron mass and charge, respectively, A is the magnetic potential vector and k is a constant. In a two-dimensional case, equation (15) is written as

$$H=\frac{\mathbf{P^2}}{2m}-\frac{k}{(x^2+y^2)^{1/2}}+\frac{m\omega}{2}\left(x^2+y^2\right)+\omega L_z, \tag{16}$$

where $\omega=eB/2mc$ is the frequency and L_z is the angular momentum in the z-direction. Here, the constant ωL_z will be neglected once the energy is defined up to a constant.

In order to solve the Schrödinger equation for this Hamiltonian, the Bohlin mapping is used.

3.1. Bohlin Mapping. Bohlin mapping is defined by [51–53]

$$x+iy=\left(q_1^2-q_2^2\right)+i(2q_1q_2), \tag{17}$$

or

$$x=q_1^2-q_2^2, \tag{18}$$

$$y=2q_1q_2. \tag{19}$$

Defining

$$P_x+iP_y=\frac{p_1+ip_2}{2(q_1+iq_2)} \tag{20}$$

leads to

$$P_x=\frac{p_1q_1+p_2q_2}{2\left(q_1^2+q_2^2\right)}, \tag{21}$$

$$P_y=\frac{p_2q_1-p_1q_2}{2\left(q_1^2+q_2^2\right)}. \tag{22}$$

Substituting equations (18), (19), (21), and (22) in equation (16) leads to the Hamiltonian

$$H=\frac{1}{2}\frac{p_1^2+p_2^2}{\left(q_1^2+q_2^2\right)}-\frac{k}{\left(q_1^2+q_2^2\right)}+\frac{B^2}{8}\left(q_1^2+q_2^2\right)^3. \tag{23}$$

Using $\hbar=\omega=e=m=1$ and taking the hypersurface given by $H=E$ leads to

$$\frac{1}{2}\left(p_1^2+p_2^2\right)+\frac{B^2}{8}\left(q_1^2+q_2^2\right)^3-E\left(q_1^2+q_2^2\right)-k=0, \tag{24}$$

which is the Hamiltonian to be used in the next section. It should be noted that the Bohlin transformation is a canonical transformation [54].

4. Zeeman Effect in Phase Space

Using equation (24), the equation is written as

$$\left[\frac{1}{2}\left(p_1^2+p_2^2\right)+\frac{B^2}{8}\left(q_1^2+q_2^2\right)^3-E\left(q_1^2+q_2^2\right)-k\right]\star\psi(q_1,p_1,q_2,p_2)=0. \tag{25}$$

It should be noted that the above equation is obtained from the classical Hamiltonian by means of the star product. Thus, the Bohlin mapping that leads to equation (24) is a classical transformation. This equation is analyzed by the perturbation theory. The equation in phase space is defined as

$$\left(\widehat{H}_0+\widehat{H}_1\right)\psi(q_1,p_1,q_2,p_2)=k\psi(q_1,p_1,q_2,p_2), \tag{26}$$

where $\widehat{H}_0=(1/2)(p_1^2+p_2^2)\star-E(q_1^2+q_2^2)\star$ and $\widehat{H}_1=(B^2/8)(q_1^2+q_2^2)\star$.

The equation for $\widehat{H}_0$ has the form

$$\widehat{H}_0\psi^{(0)}(q_1,p_1,q_2,p_2)=k^{(0)}\psi^{(0)}(q_1,p_1,q_2,p_2), \tag{27}$$

where $\psi^{(0)}(q_1,p_1,q_2,p_2)$ and $k^{(0)}$ represent, respectively, the eigenfunction and eigenvalue of the unperturbed Hamiltonian.

Defining the operators

$$\begin{aligned}\widehat{a}&=\left(\sqrt{\frac{W}{2}}q_1\star+i\sqrt{\frac{1}{2W}}p_1\star\right),\\ \widehat{a}^{\dagger}&=\left(\sqrt{\frac{W}{2}}q_1\star-i\sqrt{\frac{1}{2W}}p_1\star\right),\\ \widehat{b}&=\left(\sqrt{\frac{W}{2}}q_2\star+i\sqrt{\frac{1}{2W}}p_2\star\right),\\ \widehat{b}^{\dagger}&=\left(\sqrt{\frac{W}{2}}q_2\star-i\sqrt{\frac{1}{2W}}p_2\star\right),\end{aligned} \tag{28}$$

where $W^2/2 = -E$, the star operators $q_1\star$ and $p_1\star$ are given by

$$q_i\star = q_i + \frac{i}{2}\frac{\partial}{\partial p_i}, \quad p_i\star = p_i - \frac{i}{2}\frac{\partial}{\partial q_i}, \tag{29}$$

and perturbed Hamiltonian is

$$\widehat{H} = W\left(\widehat{a}\widehat{a}^\dagger + \widehat{b}\widehat{b}^\dagger + 1\right) + \frac{B^2}{8}\left[\left(\widehat{a}+\widehat{a}^\dagger\right)^2 + \left(\widehat{b}+\widehat{b}^\dagger\right)^2\right]^3. \tag{30}$$

The unperturbed Hamiltonian is defined as

$$\widehat{H}_0 = W\left(\widehat{a}\widehat{a^\dagger} + \widehat{b}\widehat{b^\dagger} + 1\right). \tag{31}$$

Then, the perturbed part is

$$\widehat{H}_1 = \frac{B^2}{8}\left[\left(\widehat{a}+\widehat{a}^\dagger\right)^2 + \left(\widehat{b}+\widehat{b}^\dagger\right)^2\right]^3. \tag{32}$$

The equation that is to be analyzed is given as

$$H\star\psi(q_1, p_1, q_2, p_2) = k\psi(q_1, p_1, q_2, p_2). \tag{33}$$

The unperturbed equation is

$$H_0\star\psi^{(0)}_{n_1,n_2}(q_1, p_1, q_2, p_2) = k^{(0)}_{n_1,n_1}\psi^{(0)}_{n_1,n_2}(q_1, p_1, q_2, p_2), \tag{34}$$

The unperturbed part, $\widehat{H}_0$, has solutions given by

$$\psi^{(0)}_{n_1,n_2}(q_1, p_1, q_2, p_2) = \phi_{n_1}(q_1, p_1)\Gamma_{n_2}(q_2, p_2), \tag{35}$$

where $\phi_{n_1}(q_1, p_1)$ and $\Gamma_{n_2}(q_2, p_2)$ are solutions. The eigenvalue equations are given by

$$\widehat{a}\phi_{n_1} = \sqrt{n_1}\phi_{n_1} - 1, \tag{36}$$

$$\widehat{a}^\dagger\phi_{n_1} = \sqrt{n_1+1}\phi_{n_1} + 1, \tag{37}$$

$$\widehat{b}\Gamma_{n_2} = \sqrt{n_2}\Gamma_{n_2-1}, \tag{38}$$

$$\widehat{b^\dagger}\Gamma_{n_2} = \sqrt{n_2+1}\Gamma_{n_2+1}. \tag{39}$$

Using the relations $\widehat{a}\phi_0 = 0$ and $\widehat{b}\Gamma_0 = 0$, the ground state solution is

$$\psi^{(0)}_{0,0}(q_1, p_1, q_2, p_2) = \mathcal{N}e^{-\left(Wq_1^2+p_1^2\right)}L_{n_1}\left(Wq_1^2+p_1^2\right)e^{-\left(Wq_2^2+p_2^2\right)}L_{n_2} \cdot \left(Wq_2^2+p_2^2\right), \tag{40}$$

where L_{n_1} and L_{n_2} are Laguerre polynomials; and $\mathcal{N}$ is a normalization constant. The eigenvalue solutions, given in equation (34), are

$$k^{(0)}_{n_1,n_2} = (n_1 + n_2 + 1)W. \tag{41}$$

The excited states are obtained from equation (40) using operators given in equations (36), (37), and (38).

Then, the solution for the first-order perturbed Hamiltonian is given by

$$\psi^{(1)}_{n_1,n_2}(q_1, p_1, q_2, p_2) = \psi^{(0)}_{n_1,n_2}(q_1, p_1, q_2, p_2) + \sum_{m_1\neq n_1; m_2\neq n_2} \cdot \left(\frac{\int \psi^{*(0)}_{m_1,m_2}(q_1, p_1, q_2, p_2)\widehat{H}_1\psi^{(0)}_{n_1,n_2}(q_1, p_1, q_2, p_2)dq_1dp_1dq_2dp_2}{k^{(0)}_{n_1,n_2} - k^{(0)}_{m_1,m_2}}\right) \times \psi^{(0)}_{m_1,m_2}(q_1, p_1, q_2, p_2). \tag{42}$$

It is to be noted that the following integral needs to be solved

$$I = \int \psi^{*(0)}_{m_1,m_2}(q_1, p_1, q_2, p_2)\frac{B^2}{8}\left[\left(\widehat{a}+\widehat{a}^\dagger\right)^2 + \left(\widehat{b}+\widehat{b}^\dagger\right)^2\right]^3 \cdot \psi^{(0)}_{n_1,n_2}(q_1, p_1, q_2, p_2)dq_1dp_1dq_2dp_2, \tag{43}$$

before a solution for equation (42). Using the orthogonality relations

$$\int \phi^*_n(q_1, p_1)\phi_m(q_1, p_1)dq_1dp_1 = \delta_{n,m}, \quad \int \Gamma^*_n(q_2, p_2)\Gamma_m(q_2, p_2)dq_2dp_2 = \delta_{n,m}, \tag{44}$$

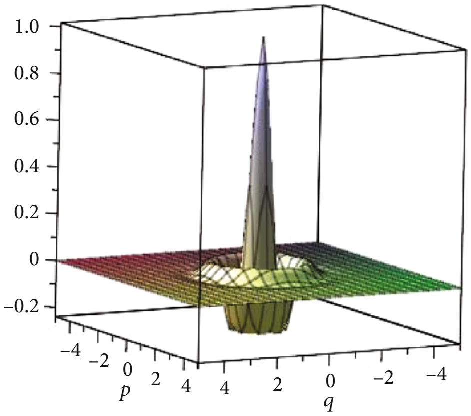

Figure 1: Wigner function zero order, Zeeman effect, $E = 1$ and $B = 1$.

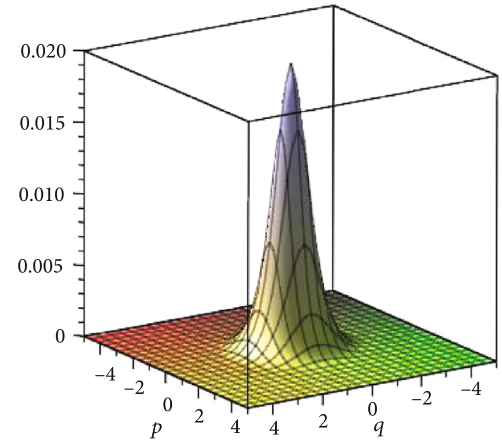

Figure 2: Wigner function zero order, Zeeman effect, $E = 1$ and $B = 0.1$.

the ground state is

$$\begin{aligned}
&\psi_{0,0}^{(1)}(q_1, p_1, q_2, p_2) \\
&\quad = \psi_{0,0}^{(0)}(q_1, p_1, q_2, p_2) + \frac{B^2}{8W} \\
&\qquad \cdot \left[\left(-21\sqrt{2} - 18 - 25\sqrt{10}\right) \psi_{2,0}^{(0)} + \left(-3\frac{\sqrt{2}}{2} - 3\right) \psi_{2,2}^{(0)} \right. \\
&\qquad \left. + \left(-30\sqrt{21} - 3\sqrt{6}\right) \psi_{4,0}^{(0)} + 4\sqrt{3}\psi_{4,2}^{(0)} - 8\sqrt{1155}\psi_{6,0}^{(0)} \right].
\end{aligned} \tag{45}$$

And for excited states, the wave functions are

$$\begin{aligned}
\psi_{1,0}^{(1)} &= \psi_{1,0}^{(0)} + \frac{B^2}{8W} \left[89,30\psi_{1,2}^{(0)} + 13,47\psi_{1,4}^{(0)} + 6,32\psi_{1,6}^{(0)} \right. \\
&\quad - 89,43\psi_{3,0}^{(0)} - 19,33\psi_{3,2}^{(0)} - 23,51\psi_{5,0}^{(0)} - 10,31\psi_{5,4}^{(0)} \\
&\quad \left. - 11,83\psi_{7,0}^{(0)} \right], \\
\psi_{0,1}^{(1)} &= \psi_{0,1}^{(0)} + \frac{B^2}{8W} \left[-89,30\psi_{2,1}^{(0)} - 13,47\psi_{4,1}^{(0)} - 6,32\psi_{6,1}^{(0)} \right. \\
&\quad + 89,43\psi_{0,3}^{(0)} + 19,33\psi_{2,3}^{(0)} + 23,51\psi_{0,5}^{(0)} + 10,31\psi_{4,5}^{(0)} \\
&\quad \left. + 11,83\psi_{0,7}^{(0)} \right].
\end{aligned} \tag{46}$$

The Wigner function for the hydrogen atom in a constant magnetic field is given by

$$fw(q_1, p_1, q_2, p_2) = \psi_{n_1,n_2}^{*(1)}(q_1, p_1, q_2, p_2) \star \psi_{n_1,n_2}^{(1)}(q_1, p_1, q_2, p_2). \tag{47}$$

It should be noted that all plots consider $q_2 = p_2 = 1$ in order to show a 3D figure, thus $q_1 = q$ and $p_1 = p$. In Figures 1 and 2, the behavior of the Wigner function is presented for order zero with magnetic field assuming values $B = 1$ and $B = 0.1$, respectively, with $E = 1$. In Figures 3 and 4, the behavior of the Wigner function is shown for the first order with magnetic field assuming values $B = 1$ and $B = 0.1$, respectively, with $E = 10$.

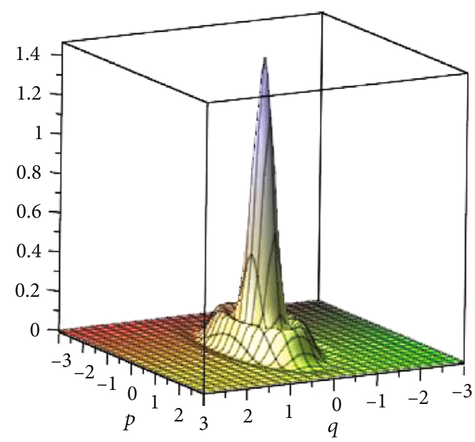

Figure 3: Wigner function first order, Zeeman effect, $E = 10$ and $B = 1$.

Comparing the graphics given in Figures 1–4, the negative part of the Wigner function increases with larger values of energy and magnetic field.

The Wigner function to the first order for magnetic field values $B = 1$ and $B = 0.1$ is shown in Figures 5 and 6 for $E = 1$. The behavior of the Wigner function to the first order with magnetic field value $B = 0.5$ is shown in Figures 7 and 8, for $E = 1$ and $E = 10$, respectively. The correction of the first order of eigenvalue of equation (33) is given by

$$\delta k_{n_1,n_2}^{(1)} = \int \psi_{n_1,n_2}^{(0)} \widehat{H}_1 \psi_{n_1,n_2}^{*(0)} dq_1 dp_1 dq_2 dp_2. \tag{48}$$

Performing calculations for $\widehat{H}_1$ leads to

$$k_{n_1,n_2}^{(1)} = (n_1 + n_2 + 1)W + \frac{B^2}{8}\delta, \tag{49}$$

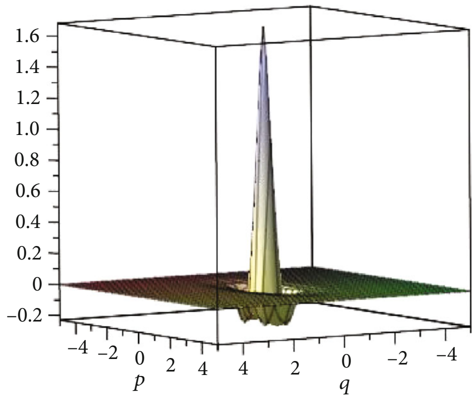

Figure 4: Wigner function first order, Zeeman effect, $E = 10$ and $B = 0.1$.

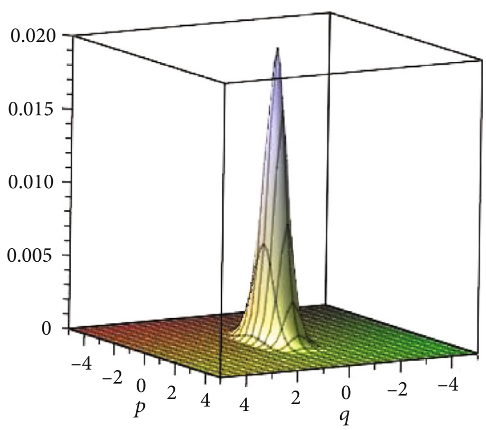

Figure 5: Wigner function first order, Zeeman effect, $E = 1$ and $B = 1$.

where

$$\begin{aligned}\delta = {} & (n_1+1)(n_1+2)(n_1+3) + (n_1+1)(n_1+2)^2 \\ & + (n_1-1)n_1(n_1+1)^2 + (n_1+1)n_1(n_1+1) \\ & + \sqrt{n_1^3(n_1+1)^3} + (n_1+1)n_1^2 + (n_1+1)n_1(n_1-1) \\ & + n_1(n_1-1)^2 + n_1(n_1-1)(n_1-2) + 3(n_1+1)n_1(n_2+1) \\ & + 3(n_1+1)n_1n_2 + 3n_1^2(n_2+1)3n_1^2n_2 \\ & + 3(n_1-1)n_1(n_2+1) + 3(n_1-1)n_1n_2 \\ & + 3(n_2+1)n_2(n_1+1) + 3(n_2+1)n_2n_1 \\ & + 3n_2^2(n_1+1)3n_2^2n_1 + 3(n_2-1)n_2(n_1+1) \\ & + 3(n_2-1)n_2n_1 + (n_2+1)(n_2+2)(n_2+3) \\ & + (n_2+1)(n_2+2)^2 + (n_2-1)n_2(n_2+1)^2 \\ & + \sqrt{n_2^3(n_2+1)^3} + \sqrt{(n_2+1)^2n_2^4} + (n_2+1)n_2(n_2-1) \\ & + n_2(n_2-1)^2 + n_2(n_2-1)(n_2-2).\end{aligned} \quad (50)$$

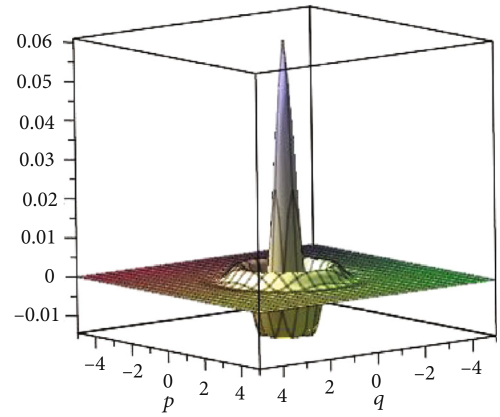

Figure 6: Wigner function first order, Zeeman effect, $E = 1$ and $B = 0.1$.

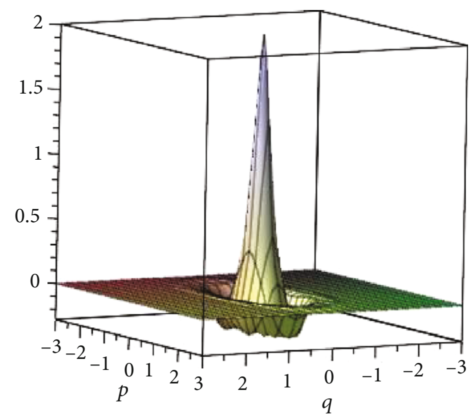

Figure 7: Wigner function first order, Zeeman effect, $E = 1$ and $B = 0.5$.

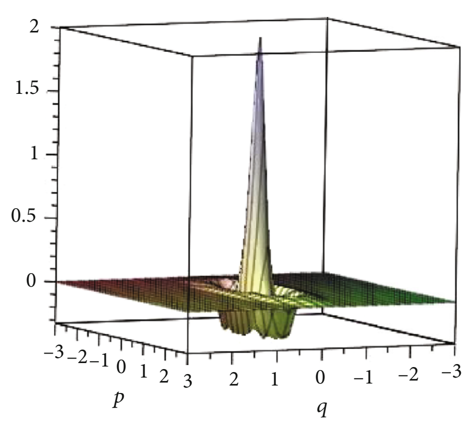

Figure 8: Wigner function first order, Zeeman effect, $E = 10$ and $B = 0.5$.

Then, the eigenvalue is

$$W = \frac{k^{(1)}_{n_1,n_2} - (B^2/8)\delta}{n_1 + n_2 + 1}. \quad (51)$$

Table 1: Negativity parameter, $B = 1$.

n_1, n_2	$\eta(\psi)$
0, 0	0.14345
0, 1	0.32645
1, 0	0.32645
1, 1	0.45786
2, 0	0.45786
0, 2	0.45786

Table 2: Negativity parameter, $B = 0.1$.

n_1, n_2	$\eta(\psi)$
0, 0	0.0034
1, 0	0.0562
0, 1	0.0562
1, 1	0.0635
2, 0	0.0635
0, 2	0.0635

With the relation $W^2/2 = -E$, the eigenvalues associated to the Zeeman effect in phase space are given by

$$E_{n_1,n_2} = -\frac{1}{2}\left[\frac{k^{(1)}_{n_1,n_2} - (B^2/8)\delta}{n_1 + n_2 + 1}\right]^2,$$
$$E_N = -\frac{1}{2}\left[\frac{k^{(1)}_{n_1,n_2} - (B^2/8)\delta}{N}\right]^2, \quad (52)$$

where $N = n_1 + n_2 + 1$. Note that if $B \longrightarrow 0$, the known results are obtained [55].

Using the Wigner function, the negative parameter for the system is calculated. The results are presented in Tables 1 and 2. It is to be noted that when the magnetic field increases the negativity parameter also increases. In addition, for a given value of the magnetic field, the negativity parameter increases when the sum $n_1 + n_2$ increases. This result is presented in the graphics above.

5. Concluding Remarks

The Zeeman effect in phase space for the Schrödinger equation, which endows the Galilean symmetry, is analyzed. The Wigner function is calculated numerically and presented in the panels for several parameters. Such a function has a clear interpretation in the classical limit and can be projected in the momenta or coordinate space for experimental purpose. The modulus of the Wigner function is also finite that allows a calculation its negativity. The results are presented in Tables 1 and 2. It indicates a direct relation between the magnetic field and the discrete parameter $N = n_1 + n_2$. The increase of the magnetic field is related to the departure from the classical behavior since the negativity parameter increases accordingly.

Data Availability

In this paper, we did not use any experimental data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was partially supported by CNPq of Brazil.

References

[1] E. Wigner, "On the quantum correction for thermodynamic equilibrium," *Physical Review*, vol. 19, p. 749, 1932.

[2] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, "Distribution functions in physics: fundamentals," *Physics Reports*, vol. 106, no. 3, pp. 121–167, 1984.

[3] Y. S. Kim and M. E. Noz, *Phase Space Picture and Quantum Mechanics-Group Theoretical Approach*, World Scientific, London, 1991.

[4] T. Curtright, D. Fairlie, and C. Zachos, "Features of time-independent Wigner functions," *Physical Review D*, vol. 58, no. 2, p. 25002, 1998.

[5] C. K. Zachos, "Deformation quantization: quantum mechanics lives and works in phase-space," *International Journal of Modern Physics A*, vol. 17, no. 3, pp. 297–316, 2002.

[6] F. C. Khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, and A. E. Santana, *Thermal Quantum Field Theory: Algebraic Aspects and Applications*, World Scientific, Singapore, 2009.

[7] J. D. Vianna, M. C. B. Fernandes, and A. E. Santana, "Galilean-covariant Clifford algebras in the phase-space representation," *Foundations of Physics*, vol. 35, no. 1, pp. 109–129, 2005.

[8] H. Weyl, "Quantenmechanik und Gruppentheorie," *Zeitschrift für Physik*, vol. 46, no. 1-2, pp. 1–46, 1927.

[9] J. E. Moyal, "Quantum mechanics as a statistical theory," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 45, no. 1, pp. 99–124, 1949.

[10] S. A. Smolyansky, A. V. Prozorkevich, G. Maino, and S. G. Mashnic, "A covariant generalization of the real-time Green's functions method in the theory of kinetic equations," *Annals of Physics*, vol. 277, no. 2, pp. 193–218, 1999.

[11] T. Curtright and C. Zachos, "Wigner trajectory characteristics in phase space and field theory," *Journal of Physics A: Mathematical and General*, vol. 32, no. 5, pp. 771–779, 1999.

[12] I. Galaviz, H. García-Compeán, M. Przanowski, and F. J. Turrubiates, "Weyl-Wigner-Moyal for Fermi classical systems," https://arxiv.org/abs/hep-th/0612245v1.

[13] J. Dito, "Star-products and nonstandard quantization for Klein–Gordon equation," *Journal of Mathematical Physics*, vol. 33, no. 2, pp. 791–801, 1992.

[14] G. Torres-Vega, "Quantum mechanics in phase space: new approaches to the correspondence principle," *The Journal of Chemical Physics*, vol. 93, no. 12, pp. 8862–8874, 1990.

[15] M. A. de Gosson, "Symplectically covariant Schrödinger equation in phase space," *Journal of Physics A: Mathematical and General*, vol. 38, no. 42, article 9263, 2005.

[16] M. A. de Gosson, "Semi-classical propagation of wavepackets for the phase space Schrödinger equation: interpretation in

terms of the Feichtinger algebra," *Journal of Physics A: Mathematical and Theoretical*, vol. 41, no. 9, article 095202, 2008.

[17] D. Galetti and A. F. R. de Toledo Piza, "Symmetries and time evolution in discrete phase spaces: a soluble model calculation," *Physica A: Statistical Mechanics and its Applications*, vol. 214, no. 2, pp. 207–228, 1995.

[18] L. P. Horwitz, S. Shashoua, and W. C. Schive, "A manifestly covariant relativistic Boltzmann equation for the evolution of a system of events," *Physica A: Statistical Mechanics and its Applications*, vol. 161, no. 2, pp. 300–338, 1989.

[19] P. R. Holland, "Relativistic algebraic spinors and quantum motions in phase space," *Foundations of Physics*, vol. 16, no. 8, pp. 701–719, 1986.

[20] M. C. B. Fernandes and J. D. M. Vianna, "On the Duffin-Kemmer-Petiau algebra and the generalized phase space," *Brazilian Journal of Physics*, vol. 28, no. 4, 1999.

[21] M. C. B. Fernandes, A. E. Santana, and J. D. M. Vianna, "Galilean Duffin Kemmer Petiau algebra and symplectic structure," *Journal of Physics A: Mathematical and General*, vol. 36, no. 13, pp. 3841–3854, 2003.

[22] A. E. Santana, A. Matos Neto, J. D. M. Vianna, and F. C. Khanna, "Symmetry groups, density-matrix equations and covariant Wigner functions," *Physica A: Statistical Mechanics and its Applications*, vol. 280, no. 3-4, pp. 405–436, 2001.

[23] D. Bohm and B. J. Hiley, "On a quantum algebraic approach to a generalized phase space," *Foundations of Physics*, vol. 11, no. 3-4, pp. 179–203, 1981.

[24] M. C. B. Andrade, A. E. Santana, and J. D. M. Vianna, "Poincaré-Lie algebra and relativistic phase space," *Journal of Physics A: Mathematical and General*, vol. 33, no. 22, pp. 4015–4024, 2000.

[25] M. A. Alonso, G. S. Pogosyan, and K. B. Wolf, "Wigner functions for curved spaces. I. On hyperboloids," *Journal of Mathematical Physics*, vol. 43, no. 12, pp. 5857–5871, 2002.

[26] M. D. Oliveira, M. C. B. Fernandes, F. C. Khanna, A. E. Santana, and J. D. M. Vianna, "Symplectic quantum mechanics," *Annals of Physics*, vol. 312, no. 2, pp. 492–510, 2004.

[27] R. G. G. Amorim, M. C. B. Fernandes, F. C. Khanna, A. E. Santana, and J. D. M. Vianna, "Non-commutative geometry and symplectic field theory," *Physics Letters A*, vol. 361, no. 6, pp. 464–471, 2007.

[28] R. G. G. Amorim, F. C. Khanna, A. E. Santana, and J. D. M. Vianna, "Perturbative symplectic field theory and Wigner function," *Physica A: Statistical Mechanics and its Applications*, vol. 388, no. 18, pp. 3771–3778, 2009.

[29] M. C. B. Fernandes, F. C. Khanna, M. G. R. Martins, A. E. Santana, and J. D. M. Vianna, "Non-linear Liouville and Shrödinger equations in phase space," *Physica A: Statistical Mechanics and its Applications*, vol. 389, no. 17, pp. 3409–3419, 2010.

[30] L. M. Abreu, A. E. Santana, and A. Ribeiro Filho, "The Cangemi–Jackiw manifold in high dimensions and symplectic structure," *Annals of Physics*, vol. 297, no. 2, pp. 396–408, 2002.

[31] L. W. Engel, S. W. Hwang, T. Sajoto, D. C. Tsui, and M. Shayegan, "Fractional quantum Hall effect at ν=2/3 and 3/5 in tilted magnetic fields," *Physical Review B*, vol. 45, no. 7, pp. 3418–3425, 1992.

[32] J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, K. W. West, and S. He, "New fractional quantum Hall state in double-layer two-dimensional electron systems," *Physical Review Letters*, vol. 68, p. 1383, 1992.

[33] S. Uji, H. Shinagawa, T. Terashima et al., "Magnetic-field-induced superconductivity in a two-dimensional organic conductor," *Nature*, vol. 410, no. 6831, pp. 908–910, 2001.

[34] J. O. Sofo, A. S. Chaudhari, and G. D. Barber, "Graphane: a two-dimensional hydrocarbon," *Physical Review B*, vol. 75, no. 15, p. 153401, 2007.

[35] D. C. Elias, R. R. Nair, T. M. G. Mohiuddin et al., "Control of graphene's properties by reversible hydrogenation: evidence for graphane," *Science*, vol. 323, no. 5914, pp. 610–613, 2009.

[36] W. Kohn and J. M. Luttinger, "Theory of donor states in silicon," *Physical Review*, vol. 98, no. 4, pp. 915–922, 1955.

[37] D. G. W. Parfitt and M. E. Portnoi, "The two-dimensional hydrogen atom revisited," *Journal of Mathematical Physics*, vol. 43, no. 10, p. 4681, 2002.

[38] A. Cisneros, "Symmetry of the two-dimensional hydrogen atom," *Journal of Mathematical Physics*, vol. 10, no. 2, pp. 277–286, 1969.

[39] M. Robnik, "Hydrogen atom in a strong magnetic field: on the existence of the third integral of motion," *Journal of Physics A: Mathematical and General*, vol. 14, no. 12, pp. 3195–3216, 1981.

[40] M. G. Dimova, M. S. Kaschiev, and S. I. Vinitsky, "The Kantorovich method for high-accuracy calculations of a hydrogen atom in a strong magnetic field: low-lying excited states," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 38, no. 14, pp. 2337–2352, 2005.

[41] O. Chuluunbaatar, A. A. Gusev, V. L. Derbov et al., "Calculation of a hydrogen atom photoionization in a strong magnetic field by using the angular oblate spheroidal functions," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 38, pp. 11485–11524, 2007.

[42] M. Robnik and V. G. Romanovski, "Two-dimensional hydrogen atom in a strong magnetic field," *Journal of Physics A: Mathematical and General*, vol. 36, no. 29, pp. 7923–7951, 2003.

[43] M. C. Gutzwiller and J. Math, "Periodic orbits and classical quantization conditions," *Journal of Mathematical Physics*, vol. 12, no. 3, pp. 343–358, 1971.

[44] H. Friedrich and H. Wintgen, "The hydrogen atom in a uniform magnetic field — an example of chaos," *Physics Reports*, vol. 183, no. 2, pp. 37–79, 1989.

[45] A. Harada and H. Hasegawa, "Correspondence between classical and quantum chaos for hydrogen in a uniform magnetic field," *Journal of Physics A: Mathematical and General*, vol. 16, no. 8, pp. L259–L263, 1983.

[46] M. C. Gutzwiller, "Chaos in classical and quantum mechanics," in *Interdisciplinary Applied Mathematics*, vol. 1, Springer, New York, NY, USA, 1990.

[47] E. Stambulchik and Y. Maron, "Zeeman effect induced by intense laser light," *Physical Review Letters*, vol. 113, no. 8, article 083002, 2014.

[48] T. Jaffe, "Practical modeling of large-scale galactic magnetic fields: status and prospects," *Galaxies*, vol. 7, no. 2, p. 52, 2019.

[49] L. E. Reichl, *The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations*, SpringVerlag, Berlin, 1992.

[50] F. Haake, *Quantum Signatures of Chaos*, Spring, Berlin, 1992.

[51] K. Bohlin, "Note sur le problème des deux corps et sur une intégration nouvelle dans le problème des trois corps," *Bulletin Astronomique*, vol. 28, pp. 113–119, 1911.

[52] T. Levi-Civita, *Opere Mat.*, vol. 2, pp. 411–417, 1906.

[53] T. Levi-Civita, “Sur la régularisation du problème des trois corps,” *Acta Mathematica*, vol. 42, pp. 99–144, 1920.

[54] S. Chanda, “Jacobi–Maupertuis metric and Kepler equation,” *International Journal of Geometric Methods in Modern Physics*, vol. 14, no. 7, article 1730002, 2017.

[55] P. Campos, M. G. R. Martins, and J. D. M. Vianna, “Quantum mechanics on phase space and the coulomb potential,” *Physics Letters A*, vol. 381, no. 13, pp. 1129–1133, 2017.