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Research Article

On Regular Black Holes at Finite Temperature

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The Thermo Field Dynamics (TFD) formalism is used to investigate the regular black holes at finite temperature. Using the Teleparalelism Equivalent to General Relativity (TEGR), the gravitational Stefan-Boltzmann law and the gravitational Casimir effect at zero and finite temperature are calculated. In addition, the first law of thermodynamics is considered. Then, the gravitational entropy and the temperature of the event horizon of a class of regular black holes are determined.

1. Introduction

The existence of singularities in the theory of general relativity has been problematic since its inception, especially those linked to black holes. The so-called fundamental singularities do not allow the application of the laws of physics, which are a type of difficult giving rise to some proposals to address the problem. One of them is the well-known cosmic censorship that was proposed by Penrose in the last century [1]. Another more particular proposal came with the solution of regular black holes whose first solution was obtained by Bardeen [2]. It is interesting to note that solutions describing regular black holes have an event horizon and therefore share many features with singular black holes. So, if black holes are real objects, there is a good chance that the regular ones are the objects that will be experimentally perceived. In this way, the theoretical study of such solutions becomes very relevant. Particularly the analysis of the thermodynamics of regular black holes can reveal substantial experimental implications. For this, it is necessary to define how the temperature is introduced and how from it the gravitational entropy is obtained. In addition, other effects can be predicted by means of this termalization, such as the gravitational Stefan-Boltzmann law and Casimir effect. In this sense, we will work with Thermo Field Dynamics (TFD) because it has proved to be a very powerful tool to deal with the termalization of a given field [3, 4]. This approach requires doubling the Fock space which allows both time and temperature to be system variables. This is an advantage over Matsubara's approach [5]. Thermo Field Dynamics also requires the characterization of the field under analysis by means of the corresponding Green function. Regarding the gravitational field, we will use an alternative description that is known as Teleparalelism Equivalent to General Relativity (TEGR). When the gravitational field is described in this alternative way, some unique predictions are revealed.

Teleparalelism Equivalent to General Relativity is described in terms of tetrads which has many advantages over the metric formulation of gravitation as general relativity, since equivalence occurs only in relation to field equations. Among them, the most notorious is the existence of a gravitational energy that is obtained naturally through the Hamiltonian formulation of the TEGR [6]. Moreover, the propagator of graviton obtained in general relativity does not coincide with that predicted by the TEGR [7]. This opens up scope for the exploration of regular black holes via

Thermo Field Dynamics. In this paper, we will use TFD to calculate the Stefan-Boltzmann law and the Casimir effect through the TEGR.

This article is organized as follows. In Section 2, Thermo Field Dynamics is introduced. In Section 3, the ideas of the Teleparallelism Equivalent to General Relativity are briefly recalled. In Section 4, the gravitational Stefan-Boltzmann and entropy are calculated for regular black holes. In addition, we also calculate for the same gravitational system, the Casimir effect, which is given at zero and finite temperature. Finally in the last section, we present our conclusions. In this article, we use the natural unities system, $G = c = 1$. We denote the Lorentz symmetry by Latin indices, $a = (0), (1), (2), (3)$, while diffeomorphisms are denoted by greek indices, $\mu = 0, 1, 2, 3$.

2. Thermo Field Dynamics (TFD)

TFD is a thermal quantum field theory [3, 8–15]. This formalism is used when it is desirable to have explicit time dependence in addition to the temperature. TFD is based on two basic ingredients: (i) a doubling of the Hilbert space, S, of the original field system, giving rise to $S_T = S \otimes \tilde{S}$, where $\tilde{S}$ is the tilde (dual) space. This doubling is defined by the tilde conjugation rules. (ii) The Bogoliubov transformation which introduces thermal effects through a rotation between tilde ($\tilde{S}$) and nontilde (S) operators. These ingredients allow to interpret the statistical average of an arbitrary operator A, as the expectation value in a thermal vacuum. The thermal vacuum, $|0(\beta)\rangle$, describes the thermal equilibrium of the system, where $\beta = 1/k_B T$, T is the temperature, and k_B is the Boltzmann constant.

By taking an arbitrary operator A and A in the Hilbert space S and in tilde space S, respectively, the Bogoliubov transformation is

$$\begin{pmatrix} A(\alpha) \\ \xi \tilde{A}^{\dagger}(\alpha) \end{pmatrix} = U(\alpha) \begin{pmatrix} A(k) \\ \xi \tilde{A}^{\dagger}(k) \end{pmatrix}, \tag{1}$$

where $\xi = -1$ for bosons and $\xi = +1$ for fermions. The Bogoliubov transformation, $U(\alpha)$, is defined as

$$U(\alpha) = \begin{pmatrix} u(\alpha) & -w(\alpha) \\ \xi w(\alpha) & u(\alpha) \end{pmatrix}, \tag{2}$$

with $u^2(\alpha) + \xi w^2(\alpha) = 1$. Here, the α parameter is the compactification parameter defined by $\alpha = (\alpha_0, \alpha_1, \cdots \alpha_{D-1})$. The temperature effect is described by the choice $\alpha_0 \equiv \beta$ and $\alpha_1, \cdots \alpha_{D-1} = 0$, where $\beta = 1/k_B T$ with k_B being the Boltzmann constant.

Any field in the TFD formalism can be written in terms of the α-parameter. As an example, consider the scalar field. Using the Bogoliubov transformation, the scalar field dependent of α-parameter becomes

$$\begin{aligned} \phi(x;\alpha) &= U(\alpha)\phi(x)U^{-1}(\alpha), \\ \tilde{\phi}(x;\alpha) &= U(\alpha)\tilde{\phi}(x)U^{-1}(\alpha). \end{aligned} \tag{3}$$

Then, the propagator for the scalar field in terms of α -parameter is written as

$$\begin{aligned} G_0^{(AB)}\left(x - x';\alpha\right) &= i\left\langle 0, \tilde{0} \right| \tau\left[\phi^A(x;\alpha)\phi^B\left(x';\alpha\right)\right] \left| 0, \tilde{0} \right\rangle \\ &= i \int \frac{d^4k}{(2\pi)^4} e^{-ik\left(x - x'\right)} G_0^{(AB)}(k;\alpha), \end{aligned} \tag{4}$$

where A and B represent the duplicate notation with A and $B = 1, 2$ and τ is the time ordering operator. Here

$$G_0^{(AB)}(k;\alpha) = U^{-1}(\alpha) G_0^{(AB)}(k) U(\alpha). \tag{5}$$

It is important to note that the physical quantities are given by the nontilde variables. Using the Bogoliubov transformation in equation (5), the Green function becomes

$$G_0^{(11)}(k;\alpha) = G_0(k;\alpha) = G_0(k) + v^2(k;\alpha)\left[G_0(k) - G_0^*(k)\right], \tag{6}$$

with

$$G_0(k) = \frac{1}{k^2 - m^2 + i\varepsilon}, \tag{7}$$

and $v^2(k;\alpha)$ being the generalized Bogoliubov transformation [16] which is given as

$$\begin{aligned} v^2(k;\alpha) = \sum_{s=1}^{d} \sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1}, \cdots, l_{\sigma_s} = 1}^{\infty} (-\eta)^{s + \sum_{r=1}^{s} l_{\sigma_r}} \exp \\ \cdot \left[-\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} k^{\sigma_j} \right], \end{aligned} \tag{8}$$

where d is the number of compactified dimensions, $\eta = 1(-1)$ for fermions (bosons), $\{\sigma_s\}$ denotes the set of all combinations with s elements, and k is the 4-momentum.

3. Teleparallelism Equivalent to General Relativity (TEGR)

General relativity which is the standard approach for gravitation is based on the Riemann geometry in which the field variables are the components of the metric tensor. Such a formulation does not lead to gravitational conserved quantities, partly because of the inclusion of the local Lorentz symmetry and partly due to the difficulty to formally establish a reference frame. Those problems are solved by TEGR which is formulated in terms of the tetrad field in a Weitzenböck manifold. In the 1930 decade, Einstein tried to establish a unified field theory using the concept of distant teleparallelism [17] which led to the formulation of a New General

Relativity by Hyashi and Shirafuji [18]. Then a Hamiltonian formulation was successfully established which yielded conserved quantities such as the gravitational energy-momentum tensor and angular momentum [6].

Let us consider a Weitzenböck manifold endowed with the Cartan connection [19]

$$\Gamma_{\mu\lambda\nu} = e^{a}{}_{\mu}\partial_{\lambda}e_{a\nu}, \tag{9}$$

then the associated torsion tensor is

$$T^{a}{}_{\lambda\nu} = \partial_{\lambda}e^{a}{}_{\nu} - \partial_{\nu}e^{a}{}_{\lambda}. \tag{10}$$

The Cartan connection is curvature free. On the other hand, it is identically related to the Christoffel symbols ${}^{0}\Gamma_{\mu\lambda\nu}$, which exist in the realm of the Riemannian geometry, by

$$\Gamma_{\mu\lambda\nu} = {}^{0}\Gamma_{\mu\lambda\nu} + K_{\mu\lambda\nu}, \tag{11}$$

where $K_{\mu\lambda\nu}$ is the contortion tensor which is defined by

$$K_{\mu\lambda\nu} = \frac{1}{2}\left(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu}\right), \tag{12}$$

with $T_{\mu\lambda\nu} = e_{a\mu}T^{a}{}_{\lambda\nu}$.

In order to establish a Lagrangian density for TEGR, we firstly note that the scalar curvature constructed out of the Christoffel symbols is written in terms of the torsion tensor, due to the identity in equation (12), as

$$eR(e) \equiv -e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^{a}T_{a}\right) + 2\partial_{\mu}(eT^{\mu}). \tag{13}$$

Then, getting rid of the total divergency which does not alters the field equations, we have

$$\mathfrak{L}\left(e_{a\mu}\right) = -\kappa\, e\Sigma^{abc}T_{abc} - \mathfrak{L}_{M}, \tag{14}$$

where $\kappa = 1/(16\pi)$, $\mathfrak{L}_{M}$ is the Lagrangian density of matter fields, and Σ^{abc} is defined by

$$\Sigma^{abc} = \frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right) + \frac{1}{2}\left(\eta^{ac}T^{b} - \eta^{ab}T^{c}\right), \tag{15}$$

with $T^{a} = e^{a}{}_{\mu}T^{\mu}$. If a derivative with respect to tetrad field is performed in the Lagrangian density, it yields

$$\partial_{\nu}\left(e\Sigma^{a\lambda\nu}\right) = \frac{1}{4}\kappa e\, e^{a}{}_{\mu}\left(t^{\lambda\mu} + T^{\lambda\mu}\right), \tag{16}$$

where

$$t^{\lambda\mu} = \kappa\left[4\Sigma^{bc\lambda}T_{bc}{}^{\mu} - g^{\lambda\mu}\Sigma^{abc}T_{abc}\right], \tag{17}$$

is the gravitational energy-momentum tensor [20]. The symmetry of $\Sigma^{a\lambda\nu}$ leads to

$$\partial_{\lambda}\partial_{\nu}\left(e\Sigma^{a\lambda\nu}\right) \equiv 0. \tag{18}$$

This allows one to define the total energy-momentum vector. It reads

$$P^{a} = \int_{V} d^{3}x\, e\, e^{a}{}_{\mu}\left(t^{0\mu} + T^{0\mu}\right), \tag{19}$$

which may be written in the following form

$$P^{a} = 4\kappa\int_{V} d^{3}x\partial_{\nu}\left(e\Sigma^{a0\nu}\right). \tag{20}$$

It is important to point out that the energy-momentum verctor respects the Lorentz symmetry, and it is invariant under coordinate transformation.

It is possible to use the Lagrangian density of TEGR above to establish a two-point Green function considering the tetrads as the observable fields on space-time. Thus, from

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{21}$$

and expression (14), the graviton propagator is [7]

$$\left\langle e_{b\lambda}, e_{d\gamma}\right\rangle = \Delta_{bd\lambda\gamma} = \frac{\eta_{bd}}{\kappa q^{\lambda}q^{\gamma}}. \tag{22}$$

Then, the Green function reads

$$G_{0}\left(x, x'\right) = -i\Delta_{bd\lambda\gamma}\, g^{\lambda\gamma}\eta^{bd}. \tag{23}$$

Explicitly, it is

$$G_{0}\left(x, x'\right) = -\frac{i64\pi}{q^{2}}, \tag{24}$$

with $q = x - x'$, where x and x' are four vectors. With the weak field approximation, the gravitational energy-momentum tensor $t^{\lambda\mu}$ becomes

$$\begin{aligned} t^{\lambda\mu}(x) = \kappa\Big[& g^{\mu\alpha}\partial^{\gamma}e^{b\lambda}\partial_{\gamma}e_{b\alpha} - g^{\mu\gamma}\partial^{\alpha}e^{b\lambda}\partial_{\gamma}e_{b\alpha} \\ & - g^{\mu\alpha}\left(\partial^{\lambda}e^{b\gamma}\partial_{\gamma}e_{b\alpha} - \partial^{\lambda}e^{b\gamma}\partial_{\alpha}e_{b\gamma}\right) \\ & - 2g^{\lambda\mu}\partial^{\gamma}e^{b\alpha}\left(\partial_{\gamma}e_{b\alpha} - \partial_{\alpha}e_{b\gamma}\right)\Big]. \end{aligned} \tag{25}$$

In order to avoid divergences, we adopt the usual procedure to write the energy-momentum tensor at different points in space-time and then taking the proper limit. Hence

$$\begin{aligned} \left\langle t^{\lambda\mu}(x)\right\rangle = \left\langle 0\left|t^{\lambda\mu}(x)\right|0\right\rangle, = \lim_{x^{\mu}\to x'^{\mu}} 4i\kappa\Big(& -5g^{\lambda\mu}\partial'^{\gamma}\partial_{\gamma} \\ & + 2g^{\mu\alpha}\partial'^{\lambda}\partial_{\alpha}\Big)G_{0}\left(x - x'\right), \end{aligned} \tag{26}$$

where $\langle e_c{}^\lambda(x), e_{b\alpha}(x')\rangle = i\eta_{cb}\,\delta^\lambda_\alpha\, G_0(x-x')$. In this sense, it is possible to use the metric of a regular black hole to introduce thermal effects via TFD as explained in the last section. It worths to notice that in the weak field approximation, TEGR becomes a usual field which is very different from the metric formulation that cannot dissociate metric from space-time.

4. Gravitational Casimir Effect and Stefan-Boltzmann Law at Finite Temperature for Regular Black Holes

In this section, the framework of TFD is used to calculate the mean value of gravitational energy-momentum (26) which is obtained in the weak field approximation of TEGR. Thus, we have

$$\left\langle t^{\lambda\mu(AB)}(x;\alpha)\right\rangle = \lim_{x\to x'} 4i\kappa\left(-5g^{\lambda\mu}\partial'^{\gamma}\partial_\gamma + 2g^{\mu\alpha}\partial'^{\lambda}\partial_\alpha\right)G_0^{(AB)}\left(x-x';\alpha\right). \quad (27)$$

If we use the Casimir prescription

$$T^{\lambda\mu(AB)}(x;\alpha) = \left\langle t^{\lambda\mu(AB)}(x;\alpha)\right\rangle - \left\langle t^{\lambda\mu(AB)}(x)\right\rangle, \quad (28)$$

then

$$T^{\lambda\mu(AB)}(x;\alpha) = \lim_{x\to x'} 4i\kappa\left(-5g^{\lambda\mu}\partial'^{\gamma}\partial_\gamma + 2g^{\mu\alpha}\partial'^{\lambda}\partial_\alpha\right)\bar{G}_0^{(AB)}\left(x-x';\alpha\right), \quad (29)$$

where

$$\bar{G}_0^{(AB)}\left(x-x';\alpha\right) = G_0^{(AB)}\left(x-x';\alpha\right) - G_0^{(AB)}\left(x-x'\right). \quad (30)$$

It is possible to describe a class of regular black holes by the following line element [21]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (31)$$

with

$$f(r) = 1 - \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}, \quad (32)$$

where M_0 is the the mass of the regular black hole; in fact, it coincides with the ADM mass in the limit $r \longrightarrow \infty$. The parameter r_0 can be seen as a fundamental length of the regular black hole. Such a line element reproduces known solutions for a suitable choice of the parameters p and q which has to be integers. For instance, the Bardeen solution is achieved for $p=3$ and $q=2$, while the Hayward solution for $p=q=3$. Thus, it should be noted that the metric in equation (31) represents a class of solutions. Such regular black holes arose in order to deal with an open problem in gravitation, the existence of singularities. In fact, the Bardeen solution was the first class of regular black holes which can be understood as a magnetic monopole coupled to the Einstein equation [22]. It is important to point out that although the metric in equation (31) has no singularity at $r=0$, it does have an event horizon given by the solution of $f(R_H)=0$. On the other hand, the stability of such solutions need to be investigated.

4.1. Gravitational Stefan-Boltzmann Law. In order to analyze the gravitational Stefan-Boltzmann law, we have to choose $\alpha = (\beta, 0, 0, 0)$ in the TFD formalism. Then, the Bogoliubov transformation is

$$v^2(\beta) = \sum_{j_0=1}^{\infty} e^{-\beta k^0 j_0}. \quad (33)$$

Hence, we have to use the following Green function:

$$\bar{G}_0^{(11)}\left(x-x';\beta\right) = 2\sum_{j_0=1}^{\infty} G_0^{(11)}\left(x-x'-i\beta j_0 n_0\right), \quad (34)$$

where $n_0 = (1,0,0,0)$. Thus, from (29), we can calculate the energy with $(AB)=(11)$ which is the physical component in the matrix obtained in TFD. It reads

$$\begin{aligned} T^{00(11)}(x;\alpha) = \varepsilon(r,T) = \lim_{x\to x'} \sum_{j_0=1}^{\infty} 4\kappa i \cdot \Bigg\{ & -3\left[1 + \frac{4M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_0\partial_0 + 5\partial'_1\partial_1 \\ & + \frac{5}{r^2}\left[1 + \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\left(\partial'_2\partial_2 + \frac{1}{\sin^2\theta}\partial'_3\partial_3\right)\Bigg\} G_0^{(11)}\left(x-x'-i\beta j_0 n_0\right). \end{aligned} \quad (35)$$

Once the Riemann zeta function is defined by $\zeta(4) = \sum_{j_0=1}^{\infty} 1/j_0^4 = \pi^4/90$, then the gravitational Stefan-Boltzmann energy is

$$\varepsilon(r,T) = \frac{32\pi^4}{15}\left\{1 + \frac{6M_0}{r[1+(r_0/r)^q]^{p/q}}\right\}T^4. \quad (36)$$

Here, we have to notice that for the vanishing of the physical parameter of the regular black hole M_0, the energy does not goes to zero. Therefore, we need to regularize such an expression by requiring $E(r,T) = \varepsilon(r,T) - 32\pi^4 T^4/15$. That leads to

$$E(r,T) = \left\{\frac{192\pi^4 M_0}{15\, r[1+(r_0/r)^q]^{p/q}}\right\}T^4, \quad (37)$$

which is the regularized energy.

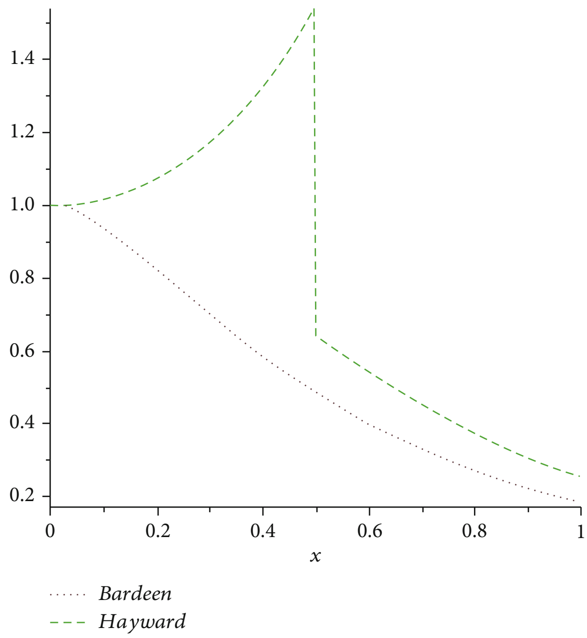

Figure 1: Gravitational entropy. The y axis is $90\, S/3072 M_0 \pi^5 R^2 T^3$ and $x = r_0/R$.

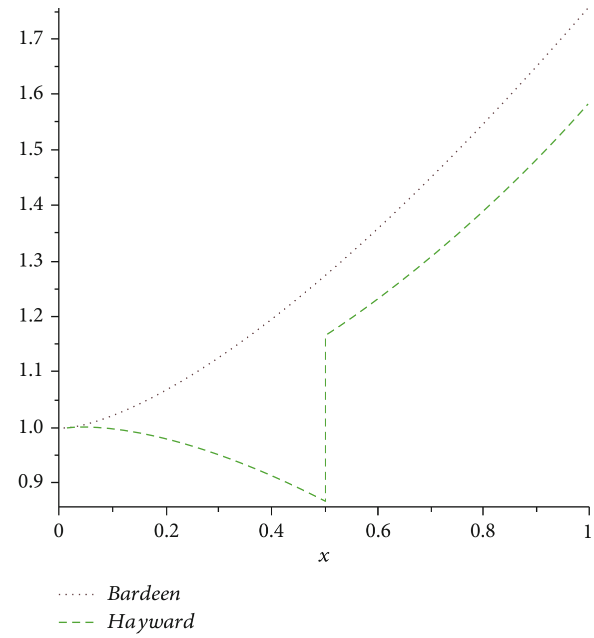

Figure 2: The horizon temperature over $[90\, R^2/3072 M_0 \pi^4]^{1/3}$ is shown as a function of $x = r_0/R$.

The first law of thermodynamics states

$$E + P = T\left(\frac{\partial P}{\partial T}\right)_V. \qquad (38)$$

Then, if we use the regularized energy above, such first-order differential equation has the solution $P = E/3$. This is very interesting to note that it is equal to the photon state equation. In terms of the temperature, the pressure is explicitly

$$P(r, T) = \left\{\frac{192\pi^4 M_0}{45\, r[1 + (r_0/r)^q]^{p/q}}\right\} T^4. \qquad (39)$$

In order to calculate the gravitational entropy, we recall the definition $P = -\partial F/\partial V$, where F is the free energy and $S = -\partial F/\partial T$. Therefore, from $(\partial P/\partial T)_V = (\partial S/\partial V)_T$, the entropy is

$$S = \left(\frac{3072 M_0 \pi^5}{45}\right) T^3 \int_0^R \left\{\frac{r}{[1 + (r_0/r)^q]^{p/q}}\right\} dr, \qquad (40)$$

which for the Bardeen and Hayward solutions may be represented graphically by Figure 1.

We choose to analyze the dependency with respect to r_0 for a constant R, instead of the temperature whose dependence is quite simple. Thus, the role of black hole geometry is better understood. It is interesting to note that there is a discontinuity for the point around $x = 0.5$ for the Hayward solution. Such a feature reflects a natural limit for the r_0 scale; after all, it is not expected that the geometric structure of the regular black hole coincides with the integration surface itself. The integration hyper-surface is a sphere of radius R. Thus, the gravitational entropy exists on an arbitrary portion of space which is a different approach from the usual black hole thermodynamics. Usually, a black hole has a fixed entropy written in terms of its event horizon area. Here, the following principles are assumed: (i) the entropy is a function of macroscopic parameters. If the "no hair" theorems are valid [23], then the entropy is a function of mass, angular momentum, and charge. (ii) The Penrose process leads to an arbitrary manipulation of these parameters provided that the event horizon area remains unchanged. That holds for regular black holes [24]. It implies that the entropy has to be a function of area. It should be noted that the metric of the regular black hole tends to the Schwarzschild metric for a position far from the event horizon. Therefore, the entropy tends to be proportional to the event horizon area; hence, it is reasonable to admit that $S_H = A_H/4$ for a regular black hole, where $A_H = 4\pi R_H^2$, with R_H being the solution of

$$\frac{2 M_0}{R_H[1 + (r_0/R_H)^q]^{p/q}} = 1. \qquad (41)$$

Therefore, the temperature of the event horizon of a class of regular black holes defined by (31) is

$$T_H = \left[\frac{90\, R^2}{3072 M_0 \pi^4 \left\{\int_0^R \left(r dr/[1 + (r_0/r)^q]^{p/q}\right)\right\}}\right]^{1/3}. \qquad (42)$$

Thus, for the Bardeen and Hayward solutions, the horizon temperature assumes the form as that in Figure 2.

It should be noted that the same discontinuity of entropy in Hayward solution also appears in the temperature. This discontinuity can also be interpreted as an improbability of the Hayward solution to be an experimental reality. On the other hand, the marked difference between the solutions for small r_0 indicates an immediate choice when an experiment to measure the temperature of the event horizon of a regular black hole can be performed. Such an expression is a unique prediction of TFD applied to TEGR.

4.2. Gravitational Casimir Effect. If the Casimir effect description is desired, then the choice $\alpha = (0, i2d, 0, 0)$ has to be made which leads to the following Bogoliubov transformation

$$v^2(d) = \sum_{l_1=1}^{\infty} e^{-i2dk^1 l_1}. \tag{43}$$

If the Green function is given by

$$\bar{G}_0^{(11)}\left(x - x'\,; d\right) = 2\sum_{l_1=1}^{\infty} G_0^{(11)}\left(x - x' - 2dl_1 r\right), \tag{44}$$

then

$$\begin{aligned} T^{00(11)}(d,r) = \varepsilon_c(d,r) = \lim_{x\to x'} \sum_{l_1=1}^{\infty} 4\kappa i & \\ \cdot \Bigg\{ -3\left[1 + \frac{4M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_0\partial_0 & \\ + 5\partial'_1\partial_1 + \frac{5}{r^2}\left[1 + \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\Bigg(\partial'_2\partial_2 & \\ + \frac{1}{\sin^2\theta}\partial'_3\partial_3\Bigg)\Bigg\} G_0^{(11)}\left(x - x' - 2dl_1 n_1\right), & \end{aligned} \tag{45}$$

where $n_1 = (0, 1, 0, 0)$. Hence, the energy associated to the gravitational Casimir effect for regular black holes is

$$\begin{aligned} \varepsilon_c(d,r) = \sum_{l_1} -\frac{4}{d^4 l_1^4}\Bigg\{1 - \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}} & \\ + \left[\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\left(\frac{5dl_1}{r}\right)\Bigg\}, & \end{aligned} \tag{46}$$

which for the approximation $d \ll r$ becomes

$$\varepsilon_c(d,r) = -\frac{2\pi^4}{45d^4}\left\{1 - \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right\}. \tag{47}$$

It should be noted that the vacuum contribution is negative and has a dependency of d^{-4}. In order to take into account only the regular black hole contribution, a regularization procedure is necessary. Thus, subtracting the vacuum energy, we have

$$E_c(d,r) = \left\{\frac{4\pi^4 M_0}{45d^4\, r[1+(r_0/r)^q]^{p/q}}\right\}, \tag{48}$$

where $E_c(d,r)$ is the regularized Casimir energy. It is interesting to note that on the event horizon, it is exactly minus the vacuum energy.

Similarly, the Casimir pressure is

$$\begin{aligned} T^{33(11)}(d,r) = \rho_c(d,r) = \lim_{x\to x'} \sum_{l_1=1}^{\infty} 4\kappa i\left[1 - \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right] & \\ \cdot \Bigg\{5\left[1 + \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_0\partial_0 & \\ - 3\left[1 - \frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_1\partial_1 & \\ - \frac{5}{r^2}\left(\partial'_2\partial_2 + \frac{1}{\sin^2\theta}\partial'_3\partial_3\right)\Bigg\} G_0^{(11)} & \\ \cdot \left(x - x' - 2dl_1 n_1\right). & \end{aligned} \tag{49}$$

This yields

$$\begin{aligned} \rho_c(d,r) = -\frac{4}{d^4}\sum_{l_1}\frac{1}{l_1^4}\Bigg\{3 - \frac{18M_0}{r[1+(r_0/r)^q]^{p/q}} & \\ - \frac{6dl_1 M_0}{r^2[1+(r_0/r)^q]^{p/q}}\Bigg\}, & \end{aligned} \tag{50}$$

which, after the limit $d/r \ll 1$ is taken, reads

$$\rho_c(d,r) = -\frac{2\pi^4}{15d^4}\left\{1 - \frac{6M_0}{r[1+(r_0/r)^q]^{p/q}}\right\}. \tag{51}$$

Again, in order to consider the nonvacuum contribution, a regularized pressure is necessary. Thus, the regularized gravitational Casimir pressure is

$$P_c(d,r) = \left\{\frac{12\pi^4 M_0}{15d^4\, r[1+(r_0/r)^q]^{p/q}}\right\}. \tag{52}$$

It should be noted that both the regularized Casimir energy and pressure are very small due to the weak field approximation. We would like to point out that in this formalism, the vacuum itself has some gravitational features which explain why a regularization is necessary.

4.3. Gravitational Casimir Effect at Finite Temperature. The choice $\alpha = (\beta, i2d, 0, 0)$ is suitable to describe the Casimir effect at finite temperature. As a consequence, following the TFD prescription, the Bogoliubov transformation is given by

$$\begin{aligned} v^2\left(k^0,k^3;\beta,d\right) &= v^2\left(k^0;\beta\right)+v^2\left(k^1;d\right) \\ &\quad + 2v^2\left(k^0;\beta\right)v^2\left(k^1;d\right), \\ &= \sum_{j_0=1}^{\infty} e^{-\beta k^0 j_0} + \sum_{l_1=1}^{\infty} e^{-i2dk^1 l_1} \\ &\quad + 2\sum_{j_0,l_1=1}^{\infty} e^{-\beta k^0 j_0 - i2dk^1 l_1}, \end{aligned} \tag{53}$$

where the first term takes into account temperature effects, the second term stands for the Casimir effect only, and the last term the interaction between both. The Green function is then

$$\bar{G}_0^{(11)}\left(x-x';\beta,d\right) = 4\sum_{j_0,l_1=1}^{\infty} G_0^{(11)}\left(x-x'-i\beta j_0 n - 2dl_1 r\right). \tag{54}$$

As before, the gravitational Casimir energy is obtained from expression (29) which reads

$$\begin{aligned} \varepsilon_c(\beta,d) &= \lim_{x\to x'} \sum_{j_0,l_1=1}^{\infty} 4\kappa i \left\{ -3\left[1+\frac{4M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_0\partial_0 \right. \\ &\quad + 5\partial'_1\partial_1 + \frac{5}{r^2}\left[1+\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right] \\ &\quad \cdot \left.\left(\partial'_2\partial_2 + \frac{1}{\sin^2\theta}\partial'_3\partial_3\right)\right\} G_0^{(11)} \\ &\quad \cdot \left(x-x'-i\beta j_0 n - 2dl_1 r\right), \end{aligned} \tag{55}$$

where $\varepsilon_c(\beta,d) = T^{00(11)}(\beta;d)$. It worths to obtain the regularized Casimir energy at finite temperature which is achieved by subtracting the vacuum energy from $\varepsilon_c(\beta,d)$; explicitly, it is

$$\begin{aligned} E_c(\beta,d) &= -64\sum_{j_0,l_1=1}^{\infty}\left[\frac{1}{4d^2l_1^2\left(1+\left(2M_0/r[1+(r_0/r)^q]^{p/q}\right)\right)+j_0^2\left(1-2M_0/r[1+(r_0/r)^q]^{p/q}\right)\beta^2}\right]^3 \left\{4d^2l_1^2\left[1\right.\right. \\ &\quad + 12\left(\frac{M_0}{r[1+(r_0/r)^q]^{p/q}}\right)^2 + \left[\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\left(1+\frac{5dl_1}{r}\right)\left.\right]\left(1+\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right) - j_0^2\left[3-72\left(\frac{M_0}{r[1+(r_0/r)^q]^{p/q}}\right)^3\right. \\ &\quad + 30dl_1\left(\frac{M_0}{r^2[1+(r_0/r)^q]^{p/q}}\right) + 4\left(\frac{M_0}{r[1+(r_0/r)^q]^{p/q}}\right)^2\left(6+\frac{5dl_1}{r}\right)]\beta^2\left.\right\} + 64\sum_{j_0,l_1=1}^{\infty}\frac{4d^2l_1^2-3j_0^2\beta^2}{\left(4d^2l_1^2+j_0^2\beta^2\right)^3}, \end{aligned} \tag{56}$$

where $E_c(\beta,d)$ is the regularized expression. Similarly, the gravitational Casimir pressure, $\rho_c(\beta,d)$, at finite temperature is given by

$$\begin{aligned} \rho_c(\beta,d) &= \lim_{x\to x'}\sum_{j_0,l_1=1}^{\infty} 4\kappa i\left[1-\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right] \\ &\quad \cdot \left\{5\left[1+\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_0\partial_0\right. \\ &\quad - 3\left[1-\frac{2M_0}{r[1+(r_0/r)^q]^{p/q}}\right]\partial'_1\partial_1 \\ &\quad - \frac{5}{r^2}\left.\left(\partial'_2\partial_2+\frac{1}{\sin^2\theta}\partial'_3\partial_3\right)\right\}G_0^{(11)} \\ &\quad \cdot \left(x-x'-i\beta j_0 n - 2dl_1 r\right), \end{aligned} \tag{57}$$

where $\rho_c(\beta,d) = T^{11(11)}(\beta;d)$. As the regularized energy, the regularized Casimir pressure is

$$\begin{aligned} P_c(\beta,d) &= -64\sum_{j_0,l_1=1}^{\infty}\left[\frac{1}{4d^2l_1^2\left(1+\left(2M_0/r[1+(r_0/r)^q]^{p/q}\right)\right)+j_0^2\left(1-2M_0/r[1+(r_0/r)^q]^{p/q}\right)\beta^2}\right]^3 \\ &\quad \cdot\left\{4d^2l_1^2\left[3+8\left(\frac{M_0}{r[1+(r_0/r)^q]}\right)^2 - 6dl_1\left(\frac{M_0}{r^2[1+(r_0/r)^q]}\right)\left(1-\left(\frac{2M_0}{r[1+(r_0/r)^q]}\right)\right)\right]\left[1-\left(\frac{2M_0}{r[1+(r_0/r)^q]}\right)^2\right]\right. \\ &\quad + j_0^2\left[24\left(\frac{M}{r}\right)^2 + 6dl_1\left(\frac{2M}{r^2}\right)\left(3+\left(\frac{2M}{r}\right)\right)-1\right]\left[1-\left(\frac{2M}{r}\right)\right]^2\beta^2\left.\right\} + 64\sum_{j_0,l_1=1}^{\infty}\frac{4d^2l_1^2-j_0^2\beta^2}{\left(4d^2l_1^2+j_0^2\beta^2\right)^3}, \end{aligned} \tag{58}$$

where $P_c(\beta, d) = \rho_c(\beta, d) + 64\sum_{j_0,l_1=1}^{\infty} 4d^2l_1^2 - j_0^2\beta^2/(4d^2l_1^2 + j_0^2\beta^2)^3$. The regularized expressions take into account only the contributions of the regular black holes. They are small corrections to the vacuum quantities which have the known limits for $\beta \longrightarrow \infty$. It should be noted that the gravitational Casimir effect is a very controversial idea due to the energy problem in general relativity. As a matter of fact, the lack of a gravitational energy-momentum tensor in this approach prevents one from analyzing such effect. On the other hand in the framework of TEGR, the gravitational Casimir effect can be explored.

5. Conclusion

The regular black holes are studied at finite temperature. The temperature effects are introduced using the TFD formalism. TFD is a tool that allows to analyze temperature effects in addition to the time dependence. Using the Teleparalelism Equivalent to General Relativity, the gravitational thermodynamics to the regular black holes is investigated. This gravitational theory has a well defined energy-momentum tensor that allows to calculated the gravitational Stefan-Boltzmann law and Casimir effect associated to the regular black holes. A regularized gravitational Stefan-Boltzmann law for the regular black hole is obtained. Using the first law of thermodynamics, the gravitational pressure and the gravitational entropy are determined. The relation between gravitational energy and pressure is equal the relation that describes the photon. The gravitational entropy obtained here exists on an arbitrary portion of space. Then, it is a different approach from the usual black hole thermodynamics, since the usual black hole has a fixed entropy given in terms of its event horizon area. In addition, the temperature of the event horizon for regular black holes has been calculated. Furthermore, the gravitational Casimir energy and Casimir pressure at zero and finite temperature for this class of regular black holes are determined. It is interesting to note that such results can be experimentally verified; once confirmed, it suggests that the torsion tensor is the true quantity responsible by gravitation instead of curvature as the mainstream approach for the gravitational field.

Data Availability

No data were used to support this study. Our study is completely theoretical.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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