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Research Article

Masses of Single, Double, and Triple Heavy Baryons in the Hyper-Central Quark Model by Using GF-AEIM

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By using the generalized fractional analytical iteration method (GF-AEIM), the single, the double, and the triple heavy baryons masses are calculated in the hyper-central model in the two cases. In the first case, the potential is a combination of Coulombic potential, the linear confining potential, and the harmonic oscillator potential. In the second case, we add the hyperfine interaction. The energy eigenvalues and the baryonic wave function are obtained in the fractional forms. The present results are a good agreement with experimental data and are improved with other recent works.

1. Introduction

The constituent quark model (CQM), which is based on a hyper-central approach, has lately become popular for describing baryon internal structure [1–5]. Although the various theories are somewhat distinct, the heavy baryon spectrum is usually well described. Understanding the dynamics of QCD at the hadronic scale necessitates research into hadrons containing heavy quarks [6–10]. Due to the experimental observation of several heavy flavor baryons, heavy baryon characteristics have become a subject of considerable attention in recent years. All single charm quark-carrying spatial-ground-state baryons have been detected, and their masses have been calculated. Many spin-1/2b-baryons, Ξb, and Ωb, as well as spin-3/2 baryons, have been identified [11–15]. The doubly heavy baryons, which are made up of two heavy quarks and one light quark, are particularly intriguing because they offer a new platform for simultaneously exploring heavy quark symmetry and chiral dynamics [16–18]. There are a lot of theoretical models, and the mass of the doubly heavy baryon Ξ^{++}cc is predicted to be in the range of $3.5\sim3.7$ GeV. The mass splitting between Ξ^{++}cc and Ξ^{+}cc is predicted to be several MeV due to the mass difference of the light quarks u, d. The predicted mass in lattice QCD is about 3.6 GeV, which is quite close to the LHCb observation. The lifetimes of Ξ^{++}cc and Ξ^{+}cc are predicted to be quite long, $50\sim250$ and $200\sim700$ fs [19–28], respectively. In Refs. [29–31], the authors calculated heavy flavor baryons containing single and double charm (beauty) quarks with light flavor combinations and considered the confinement potential as hyper-central Coulomb plus power potential with power index ν. In Ref. [32], the author calculated baryons using Feynman–Hellmann theorem and semi-empirical mass formula within the framework of a nonrelativistic constituent quark model. In Ref. [33], the author studied heavy flavor baryons by using the Bethe-Salpeter equation in the heavy quark limit and calculated the Isgur-Wise function. In Ref. [34], the author calculated different properties of single heavy flavor baryons using heavy quark symmetry in the nonrelativistic quark model. In Ref. [35], the author investigated charmed baryons and spin splittings in quenched lattice QCD. In Ref. [7], the author evaluated ground-state magnetic moments of heavy baryons in the relativistic quark model using heavy-hadron chiral perturbation theory. In Refs. [36, 37], the authors solved the Schrodinger equation using the iteration method to obtain masses of heavy baryons containing single, double, and triple in a hyper-central approach with confining interaction and hyperfine interaction. In Ref. [38], the authors solved the Schrodinger equation using a variational method to obtain masses of single, double, and triple in the hyper-central approach with confining interaction and hyperfine interaction. In Ref. [39], the authors use two potentials, the first potential is Cornell potential, and the second potential is the same as Ref. [36, 37], and they solved the Schrodinger equation numerically to obtain single, double, and triple baryon masses. In Ref. [40], the authors obtained the masses of heavy flavor baryon masses by using the nonrelativistic quark model with hyper-central Coulomb plus linear potential and Coulomb plus harmonic oscillator potential. In Ref. [41], the authors obtained mass spectra of the doubly heavy baryons that the two heavy quarks inside a baryon form a compact heavy "diquark core" in a color antitriplet and bind with the remaining light quark into a colorless baryon. In Ref. [42], the author calculated masses of the ground-state baryons consisting of three or two heavy and one light quarks in the framework of the relativistic quark model, and masses of the triply and doubly heavy baryons are obtained by using the perturbation theory for the spin-independent and spin-dependent parts of the three-quark Hamiltonian. In Ref. [43], the author studied heavy baryons within Isgur-Wise formalism by using the extended Cornell potential and solved the Schrodinger equation using iteration to obtain eigenvalues of energy and baryonic wave function.

In the present work, we employ the generalized fractional iteration method, and we calculate the masses of heavy flavor baryons containing single, double, and triple in the ground state in two cases: the first case, in the absence of hyperfine interaction, and the second case, in the presence of hyperfine interaction. To the best of our knowledge, the masses of heavy flavor baryons are not considered in the fractional quark models.

This paper is arranged as follows. In Section 2, we display interaction potential. In Section 3, the theoretical method is explained. In Section 4, the results and discussion are written. In Section 5, the conclusion is written.

2. Interaction Potential

The potentials could take any confining form (e.g., linear, log, power law). In many practical applications, a harmonic oscillator potential produces spectra not much different from those obtained from potentials like the Coulombic plus linear that QCD prejudice would favor. Since harmonic oscillator models have nice mathematical properties, they have often been employed as the confining potential. On the other hand, the Coulombic term alone is not sufficient because it would allow free quarks to ionize from the system. As used by QCD [44, 45], the potential being studied in this work consists of a Coulombic-like term combined with a linear confining term $(ax - c/x)$, and the harmonic oscillator potential, which has the form of x^2, has been added [36].

$$V(x) = \mathrm{ax}^2 + \mathrm{bx} - \frac{c}{x}, \quad (1)$$

where x is the relative quark pair coordinate and a, b, and c are constants. In Equation (1), two features the Coulomb potential (CP) and confinement potential, the short distance is described by CP, and the long distance is described by the confinement part and supported confinement force by adding harmonic potential x^2. We consider the present potential hyperfine interaction potentials. In the second case, we have added hyperfine interaction potentials ($H_s(x)$, H_I (x), and H_{SI} (x)). The nonperturbative confining interaction potential is the potential as defined in Equation (1). The nonconfining potential due to the exchange interactions contains a δ -like term, an illegal operator term [46]. We have changed it by a Gaussian of the quark pair relative distance where the non-confining spin-spin interaction potential is proportional to a δ-function which is an illegal operator term. We modify it to a Gaussian function of the relative distance of the quark pair

$$H_s = A_S \frac{S_1.S_2}{\left(\sqrt{\pi}\,\sigma_s\right)^3} \mathrm{Exp}\left(\frac{-x^2}{{\sigma_s}^2}\right), \quad (2)$$

where s_i is the spin operator of the ith quark ($s_i = \sigma_i/2$), with $\sigma_i$being the vector of Pauli matrices) and A_S and σ_s are constants. Another spin as well as isospin-dependent interaction potentials can arise from quark-exchange interactions. We conclude that two additional terms should be added to the Hamiltonian for quark pairs which result in hyperfine interactions similar to Equation (3). The first one depends on isospin only and has the form [36, 46].

$$H_I = A_I \frac{t_1.t_2}{\left(\sqrt{\pi}\,\sigma_I\right)^3} \mathrm{Exp}\left(\frac{-x^2}{{\sigma_I}^2}\right), \quad (3)$$

where t_i is the isospin operator of the ith quarks and A_I and σ_I are constants. The second one is a spin-isospin interaction given by [36, 46]

$$H_{\mathrm{SI}} = A_{\mathrm{SI}} \frac{(S_1.S_2)(t_1.t_2)}{\left(\sqrt{\pi}\sigma_{\mathrm{SI}}\right)^3} \mathrm{Exp}\left(\frac{-x^2}{{\sigma_{\mathrm{SI}}}^2}\right), \quad (4)$$

where s_i and t_i are the spin and isospin operators of the ith quark, respectively, and A_{SI} and σ_{SI} are constants. Then, from Equations (2), (3), and (4), the hyperfine interaction (a nonconfining potential) is given by

$$H_{\mathrm{int}} = H_s(x) + H_I(x) + H_{SI}(x). \quad (5)$$

The parameters of the hyperfine interaction (5) are given in Table 1.

3. Theoretical Method

3.1. Generalized Fractional Derivative. Fractional derivative plays an important role in applied science. Riemann-Liouville and Riesz and Caputo give a good formula that allows applying boundary and initial conditions as in Ref.

Table 1: Constituent hyperfine–potential parameters used in cases I and II [46, 47].

Parameter	Value
A_S	67.4 $(fm)^2$
σ_s	4.76 (fm)
A_I	51.7 $(fm)^2$
σ_I	1.57 (fm)
A_{SI}	-106.2 $(fm)^2$
σ_{SI}	2.31 (fm)

[48].

$$D_t^{\alpha}(r) = \int_{r_0}^{r} K_a(r-s) f^{(n)}(s) d(s), \quad r > r_0, \tag{6}$$

with

$$K_a(r-s) = \frac{(r-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}, \tag{7}$$

where $f^{(n)}$ is the n, the derivative of the function $f(t)$, and $K_a(r-s)$ is the kernel, which is fixed for a given real number α. The kernel $K_a(r-s)$ has a singularity at $r = s$. Caputo and Fabrizio [49] suggested a new formula for the fractional derivative with a smooth exponential kernel of the form to avoid the difficulties found in Equation (6)

$$D_t^{\alpha} = \frac{M(a)}{1-\alpha} \int_{r_0}^{r} \exp\left(\frac{\alpha(r-s)}{1-\alpha}\right) \dot{y}(s)\, d(s), \tag{8}$$

where $M(a)$ is a normalization function with $M(0) = M(1) = 1$.

A new formula for a fractional derivative called generalized fractional derivative (CFD) is proposed [50]. Generalized fractional derivative has been suggested to provide more advantages than other classical Caputo and Riemann–Liouville fractional derivative definitions, which gives a new direction for simply solving fractional differential equations (see Ref. [50]). Secondly, the fractional quark model recently takes more attention to well-reproducing meson properties (Refs. [51–53]).

$$D^{\alpha}[f_{\rm nl}(r)] = {\rm kr}^{1-\alpha} f_{\rm nl}(r), \tag{9}$$

$$D^{\alpha}[D^{\alpha} f(r)] = k^2 \left[(1-\alpha) r^{1-2\alpha} f_{\rm nl}(r) + r^{2-2\alpha} f_{\rm nl}''(r)\right], \tag{10}$$

where $k = (\Gamma[\beta]/\Gamma[\alpha-\beta+1])$ with $0 < \alpha \leq 1$, $0 < \beta \leq 1$.

3.2. Generalized Fractional Exact Solution Method of the Radial Schrödinger Equation for the Confining Potential. The baryon as a bound state of three constituent quarks, we define the configuration of three particles by two the Jacobi coordinates ρ and λ as [29, 36, 37, 46, 54–56]

$$\vec{\rho} = \frac{1}{\sqrt{2}}\left(\vec{r_1} - \vec{r_2}\right), \qquad \vec{\lambda} = \frac{1}{\sqrt{6}}\left(\vec{r_1} + \vec{r_2} - 2\vec{r_3}\right), \tag{11}$$

where

$$m_{\rho} = \frac{2\, m_1 m_2}{(m_1 + {\rm m}_2)}, \qquad m_{\rho} = \frac{3m_3(m_1 + m_2)}{2(m_1 + m_2 + m_3)}, \tag{12}$$

where m_1, m_2, and m_3 are the constituent quark masses. Instead of ρ and λ, one can introduce the hyperspherical coordinates, which are given by the angles $\Omega\rho = (\theta_\rho, \varphi_\rho)$ and $\Omega_\lambda = (\theta_\lambda, \varphi_\lambda)$ together with the hyperradius x and the hyperangle, ξ defined, respectively, by [46]

$$x = \sqrt{\rho^2 + \lambda^2}, \quad \xi = \tan^{-1}\left(\frac{\rho}{\lambda}\right). \tag{13}$$

Therefore, the Hamiltonian will be

$$H = \frac{{p_\rho}^2}{2\, m_\rho} + \frac{{p_\lambda}^2}{2\, m_\lambda} + v(\rho, \lambda) = \frac{p^2}{2\, m} + v(x). \tag{14}$$

In the hyper-central constituent quark model (hCQM), the quark potential, V, is assumed to depend on the hyperradius x only, which is to be hypercentral. Therefore, $v = v(x)$ is in general a three-body potential, since the hyperradius x depends on the coordinates of all the three quarks. Since the potential depends on x only, in the three-quark wave function, one can factor out the hyperangular part, which is given by hyperspherical harmonics. The remaining hyperradial part of the wave function is determined by the hypercentral Schrödinger equation [47, 57].

$$\left[\frac{d^2}{{\rm dx}^2} + \frac{5}{x}\frac{d}{\rm dx} - \frac{\gamma(\gamma+4)}{x^2}\right]\Psi_{v,\gamma}(x) = -2m(E - V(x))]\Psi_{v,\gamma}(x), \tag{15}$$

where $\Psi_{v,\gamma}(x)$ is the hyperradial wave function and γ is the grand angular quantum number given by $\gamma = 2{\rm n} + l_\rho + l_\lambda$; l_ρ and l_λ are the angular momenta associated with the ρ and λ variables; and n is a nonnegative integer number. v determines the number of the nodes of the wave function, and m is the reduced mass [47].

$$m = \frac{2\, m_\rho\, m_\lambda}{m_\rho + m_\lambda}. \tag{16}$$

Now, we want to solve the hyperradial Schrödinger equation for the three-body potential interaction (1). The wave function is factorized similarly to the central potential case. The transformation

$$\Psi_{v,\gamma}(x) = x^{-5/2}\phi_{v,\gamma}(x), \tag{17}$$

reduces Equation (15) to the form

$$\left[\frac{d^2}{dx^2} + 2m(E - V(x)) - \frac{(2\gamma+3)(2\gamma+5)}{4x^2}\right]\phi_{v,\gamma}(x) = 0. \tag{18}$$

Assume $z = xA$, $x = z/A$
where $A = 1$ GeV.
Then, Equation (1) becomes

$$V(z) = \frac{a z^2}{A^2} + \frac{bz}{A} - \frac{cA}{z}. \tag{19}$$

We note that Equation (18) becomes

$$\left[\frac{d^2}{dz^2} + \frac{2m}{A^2}(E - V(z)) - \frac{(2\gamma+3)(2\gamma+5)}{4z^2}\right]\phi_{v,\gamma}(z) = 0, \tag{20}$$

to put the present model that defined Equation (20) in the fractional quark model, and we used the fractional definition defined in Equations (9) and (10) as in Ref. [50] in which we replaced the classical derivative by fractional derivative

$$D^\alpha\left[D^\alpha\phi_{v,\gamma}(z^\alpha)\right] = \left[-\frac{2m}{A^2}(E - V(z^\alpha)) + \frac{(2\gamma+3)(2\gamma+5)}{4z^2}\right]\phi_{v,\gamma}(z^\alpha), \tag{21}$$

where

$$V(z^\alpha) = \frac{a z^{2\alpha}}{A^2} + \frac{bz^\alpha}{A} - \frac{cA}{z^\alpha}, \tag{22}$$

and we assume that

$$\phi_{v,\gamma}(z^\alpha) = z^{-\left(\frac{1-\alpha}{2}\right)}R_{v,\gamma}(z). \tag{23}$$

By substituting Equations (9), (10), (22), and (23) into Equation (21), we obtain the following equation

$$\left[\frac{d^2}{dz^2} + \varepsilon z^{2\alpha-2} - a_1 z^{4\alpha-2} - b_1 z^{3\alpha-2} + c_1 z^{\alpha-2} - \frac{(2\gamma+3)(2\gamma+5)}{4k^2 z^2} + \frac{((-\alpha^2/4)/(1/4))}{z^2}\right]R_{v,\gamma}(z^\alpha) = 0, \tag{24}$$

where

$$\varepsilon = \frac{2mE}{A^2k^2}, a_1 = \frac{2ma}{A^4k^2}, b_1 = \frac{2mb}{A^3k^2}, c_1 = \frac{2mc}{Ak^2}. \tag{25}$$

The analytical exact iteration method (AEIM) requires making the following ansatz [36] as follows

Table 2: The values of the used quark masses in two cases in GeV [59].

m_u	m_d	m_s	m_c	m_b
0.330	0.335	0.310	1.6	4.980

$$R_{v,\gamma}(z^\alpha) = f(z^\alpha)\exp[g(z^\alpha)], \tag{26}$$

where

$$f_{(z^\alpha)=}\begin{cases}1, & n=0,\\ \prod_{i=1}^{n}\left(z^\alpha - \alpha_i^{(n)}\right) & n = 1, 2, \cdots\end{cases}, \tag{27}$$
$$g(z^\alpha) = -\frac{1}{2}\alpha_1 z^{2\alpha} - \beta_1 z^\alpha + \delta_1 \ln z^\alpha, \alpha_1 > 0, \beta_1 > 0.$$

It is clear that $f(z)$ are equivalent to the Laguere polynomials at $\alpha = 1$. From Equation (21), we obtain

$$R_{v,\gamma}{}''(z^\alpha) = \left[g_l{}''(z^\alpha) + g_l{}'^2(z^\alpha) + \frac{f''(z^\alpha) + 2f'(z^\alpha)g'(z^\alpha)}{f(z^\alpha)}\right]R_{v,\gamma}(z^\alpha), \tag{28}$$

$$\begin{aligned} &a_1 z^{4\alpha-2} + b_1 z^{3\alpha-2} - \varepsilon z^{2\alpha-2} - c_1 z^{\alpha-2} \\ &+ \left(\frac{(2\gamma+3)(2\gamma+5)}{4k^2} + \frac{\alpha^2}{4} - \frac{1}{4}\right)z^{-2} = \alpha_1{}^2\alpha^2 z^{4\alpha-2} \\ &+ 2\alpha_1\alpha^2\beta_1 z^{3\alpha-2} + \left(-\alpha(2\alpha-1)\alpha_1 - 2\alpha_1\alpha^2\delta + \beta_1{}^2\alpha^2\right)z^{2\alpha-2} \\ &+ \left(-\beta_1\alpha(\alpha-1) - 2\beta_1\alpha^2\delta\right)z^{\alpha-2} + \left(-\delta\alpha + \delta^2\alpha^2\right)z^{-2}. \end{aligned} \tag{29}$$

Now, comparing the coefficient of z both sides of Equation (29)

$$\alpha_1 = \frac{\sqrt{a_1}}{\alpha}, \tag{30}$$

$$\beta_1 = \frac{b_1}{2\alpha\sqrt{a_1}}, \tag{31}$$

$$c_1 = \beta_1\alpha(\alpha-1) + 2\beta_1\alpha^2\delta, \tag{32}$$

$$\varepsilon = \alpha(2\alpha-1)\alpha_1 + 2\alpha_1\alpha^2\delta - \beta_1{}^2\alpha^2, \tag{33}$$

$$\delta = \frac{1}{2\alpha}\left[1 \pm \alpha\sqrt{\alpha^2 + \frac{1}{k^2}\left(4\gamma^2 + 16\gamma + 15\right)}\right]. \tag{34}$$

Let us assume $\omega^2 = 3k/m$ and then $\omega = \sqrt{2a/m}$ as in Ref. [36].

Equations (30), (31), and (32) become

$$\alpha_1 = \frac{m\omega}{A^2\alpha k}, \tag{35}$$

Table 3: Single charm baryon masses in the ground state (masses are in GeV) at ($\alpha = \beta$ =0.665). The last column shows the relative error in comparison to experimental data.

Baryon	Present work	Exp.	Ref. [36]	Ref. [38]	Ref. [30]	Ref. [39]	Ref. [29]	Relative error
$\sum_c^{++}(uuc)$	2.448	2.454	2.452	2.318	2.443	2.459	2.425	0.2%
$\sum_c^{+}(udc)$	2.453	2.453	2.457	2.323	2.460	2.461	—	0.0%
$\sum_c^{0}(ddc)$	2.458	2.454	2.461	2.328	2.477	2.462	2.460	0.2%
Total error	0.13%	—	0.19%	5.3%	0.53%	0.26%	0.46%	

Table 4: Single beauty baryon masses in ground state (masses are in GeV) at ($\alpha = \beta$ =0.56). The last column shows the relative error in comparison to experimental data.

Baryon	P.W	Exp.	Ref. [29]	Ref. [36]	Ref. [37]	Ref. [39]	Relative error
$\sum_b^{+}(uub)$	5.806	5.807	5.772	5.807	5.816	5.834	0.02%
$\sum_b^{-}(ddb)$	5.817	5.815	5.816	5.818	5.821	5.844	0.03%
$\Xi_b^0(usb)$	5.784	5.787	5.880	5.821	5.886	5.956	0.05%
$\Xi_b^-(dsb)$	5.789	5.792	5.903	5.826	5.887	5.961	0.05%
Total error	0.0375%	—	1.0275%	0.42%	0.9%	1.7%	

Table 5: Double charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha = \beta$ =0.1).

Baryon	P.W	Ref. [30]	Ref. [36]	Ref. [39]	Ref. [40]	Ref. [41]
$\Xi_{cc}^{++}(ucc)$	3.622	3.730	3.583	3.703	3.676	3.601
$\Xi_c^{+}(dcc)$	3.627	3.755	3.588	3.708	3.676	—
$\Omega_{cc}^{+}(scc)$	3.600	3.857	3.592	3.846	3.815	3.592
$\Xi_{bb}^{0}(ubb)$	10.395	—	10.284	10.467	10.340	10.182
$\Omega_{bb}^{-}(sbb)$	10.373	—	10.239	10.606	10.454	10.276

Table 6: charm and beauty baryon masses in ground state (masses are in GeV) at ($\alpha = \beta$ =0.2).

Baryon	P.W	Ref. [39]	Ref. [39]	Ref. [41]	Ref. [42]
$\Omega_{cb}^{+}(ucb)$	7.027	7.087	6.988	6.931	6.792
$\Omega_{cb}^{0}(scb)$	7.01	7.226	7.103	7.033	6.999
$\Omega_{ccb}^{+}(ccb)$	8.321	8.357	8.190	—	8.018
$\Omega_{cbb}^{0}(cbb)$	11.706	11.737	11.542	—	11.280

$$\beta_1 = \frac{b}{A\,\alpha\,\omega\,k}, \tag{36}$$

$$c = \frac{b\,k((\alpha-1)+2\alpha\,\delta)}{2\,m\,\omega}. \tag{37}$$

The energy eigenvalue for the mode $\nu = 0$ and grand angular momentum γ from Equations (24), (30), (31), (34), and (35), (36) is

$$E_{0,\gamma} = \frac{\omega}{2}k(2\alpha-1) + \omega k\alpha\delta - \frac{b^2}{2\,m\,\omega^2}, \tag{38}$$

then, from Equations (23), (26), and (34), (35), and (36), the normalized eigenfunctions are given as

$$\Psi_{0,\gamma} = N_{0,\gamma}x^{\alpha/2+\delta\,\alpha-3}\mathrm{Exp}\left(\frac{-m\omega}{2\,\alpha\,A^2\,k}x^{2\alpha} - \frac{2mc}{k^2A\,(\alpha(\alpha-1)+2\alpha^2\delta)}x^{\alpha}\right). \tag{39}$$

4. Results and Discussion

We calculate the baryon masses given by three-quark masses and the energy $E_{\nu\gamma}$ which is a function of a, b, and m_q in two cases, the first case without the hyperfine interaction masses and the second case, with the hyperfine interaction potential $\langle H_{\mathrm{int}}\rangle$ treated as a perturbation. The first-order energy correction from the nonconfining potential $\langle H_{\mathrm{int}}\rangle$ can be obtained by using the unperturbed wave function [36].

4.1. The Interaction Potential without Hyperfine Interaction. In the first case, the Baryon mass then becomes the sum of quarks mass and energy, thus [58]

$$M = m_{q1} + m_{q2} + m_{q3} + E_{\nu\gamma}. \tag{40}$$

Table 7: Single charm and beauty baryon masses in ground state (masses are in GeV) at ($\alpha = \beta = 0.678$) and ($\alpha = \beta = 0.54$). The last column shows the relative error in comparison to experimental data.

Baryon	$I(j^p)$	$<H_{int}>$	P.W	Exp	Ref. [36]	Ref. [38]	Relative error
$\sum_c^{++}(uuc)$	$1(1/2^+)$	0.00292501	2.454	2.454	2.452	2.318	0.0%
	$1(3/2^+)$	0.0157481	2.467	2.518	2.581	2.446	2%
$\sum_c^{+}(udc)$	$1(1/2^+)$	0.00292876	2.460	2.453	2.457	2.323	0.3%
	$1(3/2^+)$	0.0157452	2.472	2.518	2.586	2.451	1.8%
$\sum_c^{0}(ddc)$	$1(1/2^+)$	0.00293254	2.465	2.454	2.461	2.328	0.4%
	$1(1/2^+)$	0.0157422	2.478	2.518	2.591	2.456	1.6%
$\sum_b^{+}(uub)$	$1(1/2^+)$	0.00636099	5.807	5.807	5.807	5.700	0.0%
	$1(3/2^+)$	0.0135397	5.815	5.829	5.936	5.826	0.2%
$\sum_b^{-}(ddb)$	$1(1/2^+)$	0.0063621	5.818	5.815	5.818	5.708	0.05%
	$1(3/2^+)$	0.013539	5.826	5.836	5.946	5.836	0.2%
Total error	—	—	0.655%	—	0.953%	2.765%	

Table 8: Double and triple charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha = \beta = 0.39$).

Baryon	$I(j^p)$	P.W	Ref. [36]	Ref. [39]	Ref. [39]	Ref. [42]	Ref. [38]	Ref. [41]
$\Xi_{cc}^{++}(ucc)$	$1/2(1/2^+)$	3.608	3.583	3.703	3.532	3.510	3.597	3.601
	$1/2(3/2^+)$	3.760	3.722	3.765	3.623	3.548	3.708	3.703
$\Omega_{cc}^{+}(scc)$	$0(1/2^+)$	3.586	3.592	3.846	3.667	3.719	3.718	3.710
	$0(3/2^+)$	3.738	3.731	3.904	3.758	3.746	3.847	3.814
$\Xi_{c}^{+}(dcc)$	$1/2(1/2^+)$	3.613	3.588	3.708	3.537	3.510	3.584	3.606
	$1/2(3/2^+)$	3.765	3.726	3.770	3.629	3.548	3.713	3.706
$\Omega_{ccc}^{++}(ccc)$	$0(3/2^+)$	5.053	4.842	5.035	4.880	4.803	4.978	—
$\Xi_{bb}^{0}(ubb)$	$1/2(1/2^+)$	10.380	10.284	10.467	10.334	10.130	10.339	10.182
	$1/2(3/2^+)$	10.532	10.427	10.525	10.431	10.144	10.468	10.214
$\Omega_{bb}^{0}(dbb)$	$1/2(1/2^+)$	10.386	10.289	—	—	10.130	10.344	—
	$1/2(3/2^+)$	10.538	10.432	—	—	10.144	10.473	—
$\Omega_{bb}^{-}(sbb)$	$0(1/2^+)$	10.359	10.293	10.606	10.397	10.424	10.478	10.276
	$0(3/2^+)$	10.510	10.436	10.664	10.495	10.432	10.607	10.309
$\Omega_{bbb}^{-}(bbb)$	$0(3/2^+)$	15.208	14.810	15.175	15.023	14.569	15.118	—

We use the same potential of Refs. [36, 38, 39], but we solve the Schrodinger equation by using the generalized fractional analytical iteration method. The quark masses and potential parameters are listed in Tables 1 and 2 and the constants a, b, and c of the potential and ω as in Ref. [39]. We note that the generalized fractional analytical iteration method plays an important role. In Table 3, we calculate single charm baryon masses in the ground state (masses are in GeV) at ($\alpha = \beta = 0.665$). The present results are close with experimental data such that $\sum_c^+(udc)$ and are good compared with other works such that in Ref. [36] total error is 0.19%. In Ref. [38], the total error is 5.3%. In Ref. [30], the total error is 0.53%. In Ref. [39], the total error is 0.26%, and in Ref. [29], the total error is 0.46%, but the total error of the present work is 0.13%.

In Table 4, we calculate single beauty baryon masses in the ground state (masses are in GeV) at (α =0.56). The present results are close to experimental data and are a good agreement compared with other works such that in Ref. [36] the total error is 0.42%. In Ref. [39], the total error is 1.7%. In Ref. [5], the total error is 1.0275%. In Ref. [37], the total error is 0.26%, and in Ref. [29], the total error is 0.9%, but the total error of the present work is 0.0375%.

In Table 5, we calculate double charm and beauty baryon masses in the ground state at ($\alpha = \beta = 0.1$), and we note that the present results are a good agreement with recent works such that [30, 36, 39–41]. In Table 6, we calculate charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha = \beta = 0.2$), and our results are a good agreement with recent works such that [39, 41, 42].

4.2. The Interaction Potential with Hyperfine Interaction. In the second case, the Baryon mass then becomes the sum of quarks mass and energy with the hyperfine interaction

Table 9: charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha\beta = 0.2$).

Baryon	$I(j^P)$	P.W	Ref. [36]	Ref. [39]	Ref. [41]	Ref. [42]
$\Omega^{+}_{cb}(ucb)$	1/2 (1/2$^+$)	6.951	6.935	7.078	6.931	6.792
	1/2 (3/2$^+$)	7.103	7.076	7.145	6.997	6.827
$\Omega^{0}_{cb}(scb)$	1/2 (1/2$^+$)	6.929	6.945	7.226	7.033	6.999
	1/2 (3/2$^+$)	7.081	7.085	7.284	7.101	7.024
$\Omega^{+}_{ccb}(ccb)$	1/2 (1/2$^+$)	8.244	8.038	8.357	—	8.018
	1/2 (3/2$^+$)	8.397	8.186	8.415	—	8.025
$\Omega^{0}_{cbb}(cbb)$	0(1/2$^+$)	11.631	11.363	11.737	—	11.280
	0(3/2$^+$)	11.782	11.512	11.795	—	11.287

potential $<H_{\text{int}}>$ treated as a perturbation, thus as in Refs. [36, 37, 39]

$$<H_{\text{int}}> = \int \Psi H_{\text{int}} \Psi \, \text{dx},$$
$$M = m_{q1} + m_{q2} + m_{q3} + E_{\nu\gamma} + < H_{\text{int}} > . \quad (41)$$

In this case, we also get good results with experiment and theoretical works as in Table 7, and we calculate single charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha = 0.678$) in case charm and ($\alpha = 0.54$) in case beauty. The present results are a good agreement with experimental data such as $\sum_c^{++}(uuc)$ and are good compared with other works such that in Ref. [36] the total error is 0.953%. In Ref. [38], the total error is 2.765%, but the total error of the present work is 0.655%. In Table 8, we calculate double and triple charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha = 0.39$). The present result is good with recent works such that Refs. [36, 37, 39, 41, 42]. In Table 9, we calculate charm and beauty baryon masses in the ground state (masses are in GeV) at ($\alpha = 0.2$). The present result is a good agreement with recent works such that Refs. [36, 39, 41, 42].

In Refs. [36, 37], the authors solved the Schrodinger equation using the iteration method, and the considered potential is a combination of Coulombic, linear confining, and harmonic oscillator terms to obtain masses of heavy baryons containing single, double, and triple in the hyper-central approach with confining interaction and hyperfine interaction in the first case, and the total error in Ref. [36] is 0.19% and 0.42% when they calculated single charm and beauty baryon masses in the ground state, respectively; in the second case, the total error is 0.953% when they calculated single charm and beauty baryon masses in the ground state. In Ref. [37], the total error is 0.9% when they calculated single beauty baryon masses in the ground state. In Ref. [38], the authors solved the Schrodinger equation using a variational method, and the considered potential is Coulomb as well as linear confining terms and the spin–isospin dependent potential to obtain masses of single, double, and triple in the hyper-central approach with confining interaction and hyperfine interaction, in the first case total error is 5.3% when they calculated single charm baryon masses in the ground state and the second case total error is 2.765%% when calculated single charm and beauty baryon masses in the ground state. In Ref. [30], the authors obtained the masses of the baryons containing a single charm, and beauty quark in the presence confinement potential is assumed in the hyper-central coordinates of the Coulomb plus power potential form; in the first case, the total error is 0.53% when they calculated single charm baryon masses in the ground state. In Ref. [39], the authors use two potentials: the second potential is the same as the hyper-central approach of Ref. [36, 37] and uses the Cornell potential and the hyper-central approach, but they solved the Schrodinger equation numerically to obtain single, double, and triple baryon masses, and in the first case, the total error is 0.26% and 1.7% when they calculated single charm and beauty baryon masses in the ground state. In Ref. [29], in the first case, the total error is 0.46% and 1.0275% when calculating single charm and beauty baryon masses in the ground state. The authors obtained the masses of heavy flavor baryon masses by using the nonrelativistic quark model with hyper-central Coulomb plus linear potential and Coulomb plus harmonic oscillator potential in Ref. [40]. In Ref. [41], the authors obtain that the mass spectra of the doubly heavy baryons are computed assuming that the two heavy quarks inside a baryon form a compact heavy "diquark core" in a color antitriplet and bind with the remaining light quark into a colorless baryon. The two reduced two-body problems are described by the relativistic Bethe-Salpeter equations (BSEs) with the relevant QCD-inspired kernels. In Ref. [42], the author calculates the masses of the ground-state baryons consisting of three or two heavy and one light quark in the framework of the relativistic quark model and the hyperspherical expansion. The masses of the triply and doubly heavy baryons are obtained by using the perturbation theory for the spin-independent and spin-dependent parts of the three-quark Hamiltonian.

5. Conclusion

In this paper, we employ the generalized fractional iteration method and calculate the masses of heavy flavor baryons containing single, double, and triple in the ground state in the two cases. In the first case, the hyperfine interaction is not included, and the second case is the presence of hyperfine interaction. We calculate the three-body analytical solution of the hyper-central Schrodinger equation using the generalized fractional analytical iteration method. The present method plays an important role in improving the results in comparison with experimental data and other works.

In the case of the interaction potential without hyperfine interaction, we calculate single charm baryon masses that are close to experimental data and are a good agreement compared with other works because in Ref. [36], the total error is 0.19%; in Ref. [38], the total error is 5.3%; in Ref. [30], the total error is 0.53%; in Ref. [39], the total error is 0.26%; and in Ref. [29], the total error is 0.46%, but the total error of present work is 0.13%.

We have calculated a single beauty baryon mass as $\sum_c^{++}$ (uuc) which closes with experimental data and is improved in comparing with other works as in Refs. [29, 36, 37, 39] in which the total error is 0.42%, 1.7%, 1.0275%, and 0.26%, respectively, but the total error of the present work is 0.0375%.

In the case of the interaction potential with hyperfine interaction, we calculate single charm and beauty baryon masses that are close with experimental data such that $\sum_c^{++}$(uuc) and are a good agreement compared with other works because in Ref. [36], the total error is 0.953%. In Ref. [38], the total error is 2.765%, but the total error of the present work is 0.655%. We conclude that the interaction potential without and with hyperfine interaction in the framework of GF-AEIM gives a good description of the heavy flavor baryons in comparison with experimental data and other works.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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