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Research Article

Phenomenology of Trimaximal Mixing with One Texture Equality

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We study neutrino mass matrices with one texture equality and the neutrino mixing matrix having either its first (TM_1) or second (TM_2) column identical to that of the tribimaximal mixing matrix. We found that out of total fifteen possible neutrino mass matrices with one texture equality, only six textures are compatible with TM_1 mixing and six textures are compatible with TM_2 mixing in the light of the current neutrino oscillation data. These textures have interesting implications for the presently unknown parameters such as the neutrino mass scale, effective Majorana neutrino mass, effective neutrino mass, the atmospheric mixing, and the Dirac- and Majorana-type CP violating phases. We, also, present the S_3 group motivation for some of these textures.

1. Introduction

In the last two decades, significant advances have been made by various neutrino oscillation experiments in determining the neutrino masses and mixings. Various neutrino parameters like three mixing (solar, atmospheric, and reactor) angles and the two mass squared differences (Δm^2_{21} and $|\Delta m^2_{31}|$) have been measured by various neutrino oscillation experiments with fairly good precision. In addition, the recent neutrino oscillation data hint towards a nonmaximal atmospheric mixing angle (θ_{23}) [1] and Dirac-type CP-violating phase (δ) near 270° [2, 3]. However, many other attributes like leptonic CP-violation, neutrino mass ordering (normal mass ordering (NO) or inverted mass ordering (IO)), nature of neutrinos (Dirac or Majorana), and absolute neutrino mass scale are still unknown. Furthermore, the origin of the lepton flavor structure still remains an open issue. The neutrino mass matrix which encodes the neutrino properties contains several unknown physical parameters. The phenomenological approaches based on Abelian or non-Abelian flavor symmetries can play a significant role in determining the specific texture structure of the neutrino mass matrix with reduced number of independent parameters.

Several predictive models such as texture zeros [4–25], vanishing cofactors [26–37], equalities among elements/cofactors [38, 39], and hybrid textures [40–46] amongst others can explain the presently available neutrino oscillation data, since the presence of texture equalities, just like texture zeros or vanishing cofactors, reduces the number of free parameters in the neutrino mass matrix and, hence, must have a similar predictability as that of texture zeros or vanishing cofactors. In the flavor basis, neutrino mass matrices with one texture equality and two texture equalities have been studied in the literature [38, 39]. The hybrid textures which combine a texture equality with a texture zero or a vanishing cofactor have been studied in the literature [40–46].

In addition, discrete non-Abelian symmetries leading to the Tri-Bi-Maximal (TBM) [47, 48] neutrino mixing pattern have been widely studied in the literature. The TBM mixing matrix given by

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1)$$

predicts a vanishing reactor mixing angle ($\theta_{13} = 0$) and maximal atmospheric mixing angle ($\theta_{23} = \pi/4$), and the solar mixing angle is predicted to be (θ_{12}) which is $\sin^{-1}(1/\sqrt{3})$. However, the nonzero value of θ_{13} confirmed by various neutrino oscillation experiments underlines the need for necessary modifications to the TBM mixing pattern to make it compatible with the present experimental data [49–53]. One of the simplest possibilities is to keep one of the columns of the TBM mixing matrix unchanged while modifying its remaining two columns to within the unitarity constraints. This gives rise to three mixing patterns, viz., TM_1, TM_2, and TM_3 which have their first, second, and third columns identical to the TBM mixing matrix, respectively. The TM_3 mixing scheme predicts $\theta_{13} = 0$ and is, hence, phenomenologically unviable. The TM_1 and TM_2 mixing schemes have been successfully employed to explain the pattern of lepton mixing and have been extensively studied in the literature [54–71]. The TM_1 mixing, in particular, gives a very good fit to the present neutrino oscillation data. Recently, neutrino mass matrices with texture zero(s) in combination with TM_1 and TM_2 mixing have been studied [72–74].

In the present work, we study a class of neutrino mass matrices having one texture equality with TM_1 or TM_2 of the TBM in the neutrino mixing matrix. Neutrino mass matrices having one texture equality along with TM_1 or T M_2 of the TBM have a total of six free parameters and, hence, lead to very predictive textures for the neutrino mass matrices.

There are a total of fifteen possible structures with one texture equality in the neutrino mass matrix, and they are listed in Table 1.

There exists a μ-τ permutation symmetry between different structures of neutrino mass matrices, and the corresponding permutation matrix has the following form:

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

Neutrino mass matrices with one texture equality, therefore, are related to each other as

$$M'_\nu = P_{23} M_\nu P_{23}^T, \quad (3)$$

leading to the following relations between the neutrino oscillation parameters:

$$\theta'_{12} = \theta_{12}, \theta'_{13} = \theta_{13}, \theta'_{23} = \frac{\pi}{2} - \theta_{23}, \delta' = \pi - \delta. \quad (4)$$

Table 1: Fifteen possible texture structures with one equality between two nonzero elements.

Textures	Constraints on elements
$M_{\nu 1}$	$M_{ee}=M_{e\mu}$
$M_{\nu 2}$	$M_{ee}=M_{e\tau}$
$M_{\nu 3}$	$M_{e\mu}=M_{\mu\mu}$
$M_{\nu 4}$	$M_{\mu\mu}=M_{\mu\tau}$
$M_{\nu 5}$	$M_{e\tau}=M_{\tau\tau}$
$M_{\nu 6}$	$M_{\mu\tau}=M_{\tau\tau}$
$M_{\nu 7}$	$M_{ee}=M_{\mu\tau}$
$M_{\nu 8}$	$M_{e\tau}=M_{\mu\mu}$
$M_{\nu 9}$	$M_{e\mu}=M_{\tau\tau}$
$M_{\nu 10}$	$M_{ee}=M_{\mu\mu}$
$M_{\nu 11}$	$M_{ee}=M_{\tau\tau}$
$M_{\nu 12}$	$M_{\mu\mu}=M_{\tau\tau}$
$M_{\nu 13}$	$M_{e\mu}=M_{e\tau}$
$M_{\nu 14}$	$M_{e\mu}=M_{\mu\tau}$
$M_{\nu 15}$	$M_{e\tau}=M_{\mu\tau}$

Neutrino mass matrices with one texture equality related by the μ-τ permutation operation are

$$\begin{aligned} &M_{\nu 1} \leftrightarrow M_{\nu 2}, M_{\nu 3} \leftrightarrow M_{\nu 5}, M_{\nu 4} \leftrightarrow M_{\nu 6}, \\ &M_{\nu 7} \leftrightarrow M_{\nu 7}, M_{\nu 8} \leftrightarrow M_{\nu 9}, M_{\nu 10} \leftrightarrow M_{\nu 11}, \\ &M_{\nu 12} \leftrightarrow M_{\nu 12}, M_{\nu 13} \leftrightarrow M_{\nu 13}, M_{\nu 14} \leftrightarrow M_{\nu 15}. \end{aligned} \quad (5)$$

In the flavor basis, where the charged lepton mass matrix M_l is diagonal, the complex symmetric Majorana neutrino mass matrix M_ν can be diagonalized by a unitary matrix V':

$$M_\nu = V' M_\nu^{\mathrm{diag}} V'^T, \quad (6)$$

where $M_\nu^{\mathrm{diag}} = \mathrm{diag}\,(m_1, m_2, m_3)$. The unitary matrix V' can be parametrized as

$$V' = P_l V \text{ with } V = U P_\nu, \quad (7)$$

where

$$U = \begin{pmatrix} c12c13 & c13s12 & e^{-i\delta}s13 \\ -c23s12 - ei\delta c12s13s23 & c12c23 - ei\delta s12s13s23 & c13s23 \\ s12s23 - ei\delta c12c23s13 & -ei\delta c23s12s13 - c12s23 & c13c23 \end{pmatrix},$$

$$P_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}, \quad P_l = \begin{pmatrix} e^{i\phi_e} & 0 & 0 \\ 0 & e^{i\phi_\mu} & 0 \\ 0 & 0 & e^{i\phi_\tau} \end{pmatrix}, \quad (8)$$

with $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. P_ν is the diagonal phase matrix containing the two Majorana-type CP-violating phases α and β. δ is the Dirac-type CP-violating phase. The phase matrix P_l is physically unobservable. The matrix V is called the neutrino mixing matrix or the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) [75–78] matrix. The effective Majorana neutrino mass matrix can be written as

$$M_\nu = P_l U P_\nu M_\nu^{\text{diag}} P_\nu^T U^T P_l^T. \tag{9}$$

The Dirac-type CP-violation in neutrino oscillation experiments can be described in terms of the Jarlskog rephasing invariant quantity J_{CP} [79] with

$$\begin{aligned} J_{CP} &= \operatorname{Im}\{U_{11}U_{22}U_{12}^*U_{21}^*\} \\ &= \sin\theta_{12}\sin\theta_{23}\sin\theta_{13}\cos\theta_{12}\cos\theta_{23}\cos^2\theta_{13}\sin\delta. \end{aligned} \tag{10}$$

The effective Majorana neutrino mass $|M_{ee}|$, which determines the rate of neutrinoless double beta decay, is given by

$$|M_{ee}| = \left|m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2\right|. \tag{11}$$

There are many experiments such as CUORICINO [80], CUORE [81], MAJORANA [82], SuperNEMO [83], and EXO [84] which aim to achieve a sensitivity up to 0.01 eV for $|M_{ee}|$. The KamLAND-Zen experiment [85] provides the upper limits on the effective Majorana neutrino mass which is given by

$$|M_{ee}| < (0.36 - 0.156)\ \text{eV}, \tag{12}$$

at 90% confidence level (C.L.).

The measurement of the absolute neutrino mass scale via the decay kinematics is usually described by the effective neutrino mass [86]

$$m_\beta \equiv \sqrt{m_1^2|U_{e1}|^2 + m_2^2|U_{e2}|^2 + m_3^2|U_{e3}|^2}. \tag{13}$$

Recently, the KATRIN [87] experiment has reported the upper limit of $m_\beta < 0.8$ eV at 90% C.L.

Further, cosmological observations provide more stringent constraints on absolute neutrino mass scale by putting an upper bound on the sum of neutrino masses:

$$\sum = \sum_{i=1}^{3} m_i. \tag{14}$$

Recent Planck data [88] in combination with baryon acoustic oscillation (BAO) measurements provide a tight bound on the sum of neutrino masses $\sum m_i \leq 0.12$ eV at 95% C.L.

2. TM_2 Mixing and One Texture Equality

A neutrino mass matrix with TM_2 mixing can be written as

$$M_{\text{TM}_2} = P_l U_{\text{TM}_2} P_\nu M_\nu^{\text{diag}} P_\nu^T U_{\text{TM}_2}^T P_l^T, \tag{15}$$

where the mixing matrix TM_2, also known as trimaximal mixing, can be parametrized [68–71] as

$$U_{TM_2} = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\theta & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\sin\theta \\ \frac{e^{i\phi}\sin\theta}{\sqrt{2}} - \frac{\cos\theta}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{e^{i\phi}\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{6}} \\ -\frac{\cos\theta}{\sqrt{6}} - \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{e^{i\phi}\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{6}} \end{pmatrix}. \tag{16}$$

The mass matrix M_{TM_2} is invariant under the transformation $G_2^T M_{\text{TM}_2} G_2 = M_{\text{TM}_2}$ with $G_2 = U_{\text{TM}_2}\,\text{diag}\,(-1, 1, -1)\,U_{\text{TM}_2}^\dagger$, as the generator of Z_2 symmetry [90, 91]. Invariance of M_{TM_2} under G_2 when combined with one texture equality leads to the equality of three unphysical phases in M_{TM_2}, i.e., $\phi_e = \phi_\mu = \phi_\tau \equiv \phi_l$.

The most general neutrino mass matrix with TM_2 as the mixing matrix can be parametrized as

$$M_{\text{TM}_2} = \begin{pmatrix} u + \frac{2x}{3} & v - \frac{x}{3} & w - \frac{x}{3} \\ v - \frac{x}{3} & w + \frac{2x}{3} & u - \frac{x}{3} \\ w - \frac{x}{3} & u - \frac{x}{3} & v + \frac{2x}{3} \end{pmatrix} \approx \begin{pmatrix} a & b & c \\ b & c + d & a - d \\ c & a - d & b + d \end{pmatrix}. \tag{17}$$

The neutrino mass matrix M_{TM_2} can be realized within the framework of an A_4 model where the A_4 flavor symmetry is spontaneously broken by two real A_4 triplets ϕ, ϕ', and three real A_4 singlets, ξ, ξ', ξ'' which are $SU(2)_L$ gauge singlets [92–94]. Upon symmetry breaking, the VEVs of the flavone singlets and triplets take the alignments

$$\xi = u_a, \xi' = u_c, \xi'' = u_b, \phi = (v, v, v), \phi' = \left(v', 0, 0\right). \tag{18}$$

The neutrino mass matrix, in the flavor basis, is given by

$$\begin{pmatrix} u + \frac{2x}{3} & v - \frac{x}{3} & w - \frac{x}{3} \\ v - \frac{x}{3} & w + \frac{2x}{3} & u - \frac{x}{3} \\ w - \frac{x}{3} & u - \frac{x}{3} & v + \frac{2x}{3} \end{pmatrix}. \tag{19}$$

The above mass matrix leads to the TM_2 neutrino mixing matrix. For $v = w$, the above mass matrix leads to the TBM neutrino mixing matrix. The equality among the elements of mass matrix in Eq. (17) does not arise naturally, and hence, we assume additional constraints on the elements of mass matrix, e.g., $u = v - x$ which leads to one equality between the $(1, 1)$ and $(1, 2)$ elements of M_{TM_2} in Eq. (17). Therefore, all possible textures of neutrino mass matrices with TM_2 mixing and one texture equality are given by

$$M^1_{\mathrm{TM}_2} = \begin{pmatrix} a & a & b \\ a & b+d & a-d \\ b & a-d & a+d \end{pmatrix}, M^2_{\mathrm{TM}_2} = \begin{pmatrix} a & b & a \\ b & a+d & a-d \\ a & a-d & b+d \end{pmatrix}, \quad (20)$$

$$M^3_{\mathrm{TM}_2} = \begin{pmatrix} b+d & a & a-d \\ a & a & b \\ a-d & b & a+d \end{pmatrix}, M^5_{\mathrm{TM}_2} = \begin{pmatrix} b+d & a-d & a \\ a-d & a+d & b \\ a & b & a \end{pmatrix}, \quad (21)$$

$$M^4_{\mathrm{TM}_2} = \begin{pmatrix} a+d & b & a-d \\ b & a & a \\ a-d & a & b+d \end{pmatrix}, M^6_{\mathrm{TM}_2} = \begin{pmatrix} a+d & a-d & b \\ a-d & b+d & a \\ b & a & a \end{pmatrix}, \quad (22)$$

$$M^7_{\mathrm{TM}_2} = M^8_{\mathrm{TM}_2} = M^9_{\mathrm{TM}_2} = \begin{pmatrix} a & b & d \\ b & d & a \\ d & a & b \end{pmatrix}, \quad (23)$$

$$\begin{aligned} M^{10}_{\mathrm{TM}_2} = M^{15}_{\mathrm{TM}_2} &= \begin{pmatrix} a & b & b-d \\ b & a & b-d \\ b-d & b-d & a+d \end{pmatrix}, \\ M^{11}_{\mathrm{TM}_2} = M^{14}_{\mathrm{TM}_2} &= \begin{pmatrix} a & b & b+d \\ b & a+d & b \\ b+d & b & a \end{pmatrix}, \end{aligned} \quad (24)$$

$$M^{12}_{\mathrm{TM}_2} = M^{13}_{\mathrm{TM}_2} = \begin{pmatrix} a & b & b \\ b & b+d & a-d \\ b & a-d & b+d \end{pmatrix}, \quad (25)$$

where the neutrino mass matrices in each equation are related by μ-τ symmetry. The neutrino mixing angles can be calculated by using the following relations:

$$\sin^2\theta_{13} = |U_{13}|^2, \sin^2\theta_{12} = \frac{|U_{12}|^2}{1 - |U_{13}|^2}, \sin^2\theta_{23} = \frac{|U_{23}|^2}{1 - |U_{13}|^2}. \quad (26)$$

Substituting the elements of U form Eq. (16) into Eq. (26), we get

$$\begin{aligned} \sin^2\theta_{13} &= \frac{2}{3}\sin^2\theta, \sin^2\theta_{12} = \frac{1}{3 - 2\sin^2\theta}, \\ \sin^2\theta_{23} &= \frac{1}{2}\left(1 + \frac{\sqrt{3}\sin 2\theta \cos\phi}{3 - 2\sin^2\theta}\right). \end{aligned} \quad (27)$$

Using Eqs. (10) and (16), the Jarlskog rephasing invariant is given by

$$J_{CP} = \frac{1}{6\sqrt{3}} \sin 2\theta \sin\phi, \quad (28)$$

and the Dirac-type CP-violating phase can be calculated by using the equation [72–74]

$$\tan\delta = \frac{\cos 2\theta + 2}{2\cos 2\theta + 1} \tan\phi. \quad (29)$$

From Eqs. (11) and (16), the effective Majorana mass for TM_2 mixing is given by

$$|M_{ee}| = \frac{1}{3}\left|2m_1\cos^2\theta + m_2 e^{2i\alpha} + 2m_3\sin^2\theta e^{2i\beta}\right|, \quad (30)$$

and the effective neutrino mass for TM_2 mixing can be calculated by using Eqs. (12) and (16) as

$$m_\beta \equiv \sqrt{\frac{1}{3}\left(2m_1^2\cos^2\theta + m_2^2 + 2m_3^2\sin^2\theta\right)}. \quad (31)$$

The existence of one equality between the elements (a, b) and (c, d) of the neutrino mass matrix M_{TM_2} implies

$$M_{\mathrm{TM}_2(ab)} - M_{\mathrm{TM}_2(cd)} = 0, \quad (32)$$

which yields the complex equation

$$\sum (QV_{ai}V_{bi} - V_{ci}V_{di})m_i = 0, \quad (33)$$

where $Q = e^{i(\phi_a + \phi_b - (\phi_c + \phi_d))}$ and V is the PMNS matrix given in Eq. (7). The above equation can be rewritten as

$$m_1 A_1 + m_2 A_2 e^{2i\alpha} + m_3 A_3 e^{2i\beta} = 0, \quad (34)$$

where

$$A_i = (QU_{ai}U_{bi} - U_{ci}U_{di}), \quad (35)$$

with $(i = 1, 2, 3)$ and a, b, c, d can take values e, μ, and τ. Since the TM_2 mixing has equal elements in the second column, it leads to $A_2 \equiv (QU_{a2}U_{b2} - U_{c2}U_{d2}) = 0$. Therefore, using Eq. (16) in Eq. (34), we have

$$m_1 A_1 + m_3 A_3 e^{2i\beta} = 0. \quad (36)$$

Table 2: Current neutrino oscillation parameters from global fits [89] with $\Delta m^2_{3l} \equiv \Delta m^2_{31} > 0$ for NO and $\Delta m^2_{3l} \equiv \Delta m^2_{32} = -\Delta m^2_{23} < 0$ for IO.

Neutrino parameter	Normal ordering (best fit)		Inverted ordering ($\Delta\chi^2 = 2.6$)	
	Bfp ± 1σ	3σ range	Bfp ± 1σ	3σ range
θ°_{12}	$33.44^{+0.77}_{-0.74}$	$31.27 \longrightarrow 35.86$	$33.45^{+0.77}_{-0.74}$	$31.27 \longrightarrow 35.87$
θ°_{23}	$49.2^{+1.0}_{-1.3}$	$39.5 \longrightarrow 52.0$	$49.5^{+1.0}_{-1.2}$	$39.8 \longrightarrow 52.1$
θ°_{13}	$8.57^{+0.13}_{-0.12}$	$8.20 \longrightarrow 8.97$	$8.60^{+0.12}_{-0.12}$	$8.24 \longrightarrow 8.98$
δ°_{CP}	194^{+52}_{-25}	$105 \longrightarrow 405$	287^{+27}_{-32}	$192 \longrightarrow 361$
$\frac{\Delta m^2_{21}}{10^{-5} eV^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \longrightarrow 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \longrightarrow 8.04$
$\frac{\Delta m^2_{3l}}{10^{-3} eV^2}$	$+2.515^{+0.028}_{-0.028}$	$+2.431 \longrightarrow +2.599$	$-2.498^{+0.028}_{-0.029}$	$-2.584 \longrightarrow -2.413$

Table 3: Numerical predictions for viable textures having one equality in M_ν with TM_2 mixing at 3σ C.L. (only neutrino oscillation data are incorporated).

Texture	Ordering	m_{lowest} (eV)	$\lvert M_{ee}\rvert$ (eV)	$\sum$ (eV)	m_β	θ°_{23}	δ°	β°
$M^1_{TM_2}$	NO	0.003-0.0087	0.0-0.0083	0.062-0.073	0.009-0.0128	39.5-51.41	0-155 ⊕ 205-360	16-90 ⊕ 270-344
$M^2_{TM_2}$	NO	0.003-0.0083	0.0-0.0082	0.061-0.072	0.009-0.0124	39.5-51.4	0-160 ⊕ 202-360	0-81 ⊕ 279-360
$M^3_{TM_2}$	NO	0.025-0.5	0.006-0.5	0.10-1.51	0.026-0.5	39.5-45	90-160 ⊕ 200-270	0-68 ⊕ 292-360
	IO	0.02-0.48	0.015-0.5	0.12-1.45	0.05-0.5	45-51.4	0-90 ⊕ 270-360	0-90 ⊕ 270-360
$M^4_{TM_2}$	IO	0.003-0.008	0.014-0.05	0.1-0.11	0.048-0.051	39.8-51.4	0-150 ⊕ 210-360	0-77 ⊕ 283-360
$M^5_{TM_2}$	NO	0.02-0.5	0.005-0.5	0.09-1.5	0.02-0.5	45-51.4	0-90 ⊕ 270-360	0-90 ⊕ 270-360
	IO	0.027-0.5	0.017-0.47	0.13-1.48	0.05-0.5	39.8-45	90-150 ⊕ 209-270	0-59 ⊕ 303-360
$M^6_{TM_2}$	IO	0.003-0.009	0.014-0.05	0.101-0.111	0.048-0.051	39.8-51.4	0-151 ⊕ 211-360	19-90 ⊕ 270-340

Simultaneous solution of the real and imaginary parts of Eq. (36) leads to

$$\xi \equiv \frac{m_3}{m_1} = \frac{\text{Re}\,(A_1)}{\text{Im}\,(A_3)\sin 2\beta - \text{Re}\,(A_3)\cos 2\beta} = \frac{|A_1|}{|A_3|}, \quad (37)$$

$$\beta = \frac{1}{2}\tan^{-1}\frac{\text{Re}\,(A_3)\,\text{Im}\,(A_1) - \text{Re}\,(A_1)\,\text{Im}\,(A_3)}{\text{Re}\,(A_1)\,\text{Re}\,(A_3) + \text{Im}\,(A_1)\,\text{Im}\,(A_3)}. \quad (38)$$

Using experimentally available mass squared differences Δm^2_{21} and $\Delta m^2_{31}(\Delta m^2_{23})$ for NO (IO) with Eq. (37), the three neutrino mass eigenvalues are given by

$$m_1 = \sqrt{\frac{\Delta m^2_{31}}{\xi^2 - 1}}, m_2 = \sqrt{\Delta m^2_{21} + m_1^2}, m_3 = \xi m_1 \quad \text{for NO}, \quad (39)$$

$$m_1 = \sqrt{\frac{\Delta m^2_{23} - \Delta m^2_{21}}{1 - \xi^2}}, m_2 = \sqrt{\Delta m^2_{21} + m_1^2}, m_3 = \xi m_1 \quad \text{for IO}, \quad (40)$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$, $m_1 < m_2 < m_3$ for NO, and $m_3 < m_1 < m_2$ for IO.

For the numerical analysis, we generate ~10^7-10^9 points. The mass squared differences Δm^2_{21} and $\Delta m^2_{31}(\Delta m^2_{23})$ for NO (IO) are varied randomly within their 3σ experimental ranges given in Table 2. Parameters θ, ϕ, and α are, also, varied randomly within their full ranges (0-90°), (0-360°), and (0-360°), respectively. Equation (38), (39), and (40) are used to calculate the Majorana-type CP-violating phase β and three mass eigenvalues (m_1, m_2, and m_3) for both mass orderings. In addition, the mixing angles θ_{12}, θ_{13}, and θ_{23} are calculated by using Eq. (27) and must satisfy the experimental data given in Table 2. The Jarlskog invariant (J_{CP}), Dirac-type CP-violating phase (δ), effective Majorana mass ($|M_{ee}|$), effective neutrino mass (m_β), and the sum of neutrino masses ($\sum$) are calculated by using Eqs. (28), (29), (30), (31), and (14), respectively.

The numerical predictions for various neutrino parameters are given in Tables 3 and 4. Table 3 provides numerical predictions for viable textures under the constrains from neutrino oscillation data, whereas Table 4 provides numerical predictions for viable textures under the constrains from cosmological and neutrinoless double beta decay bounds along with neutrino oscillation data. The allowed range of parameter α is (0-360°) for all viable textures. J_{CP} which lies in the range (−0.037-0.037) for all viable textures except $M^3_{TM_2}(M^5_{TM_2})$ with NO (IO), and for these textures, the range of J_{CP} is ±(0.011–0.037). The parameter θ is constrained to lie within the ranges (10.0°-11.1°) for all viable textures. The solar mixing angle (θ_{12}) is constrained to lie in the ranges (35.68°-35.77°) for all allowed textures.

Table 4: Numerical predictions for viable textures having one equality in M_ν with TM_2 mixing at 3σ C.L. (cosmological and neutrinoless double beta decay bounds along with neutrino oscillation data are incorporated).

Texture	Ordering	m_{lowest} (eV)	$\|M_{ee}\|$ (eV)	$\sum$ (eV)	m_β	θ_{23}°	δ°	β°
$M^1_{TM_2}$	NO	0.003-0.0087	0.0-0.0083	0.062-0.073	0.009-0.0128	39.5-51.41	0-155 $\oplus$ 205-360	16-90 $\oplus$ 270-344
$M^2_{TM_2}$	NO	0.003-0.0083	0.0-0.0082	0.061-0.072	0.009-0.0124	39.5-51.4	0-160 $\oplus$ 202-360	0-81 $\oplus$ 279-360
$M^3_{TM_2}$	NO	0.025-0.031	0.006-0.03	0.10-0.12	0.026-0.032	39.5-40.34	137-160 $\oplus$ 200-222	45-68 $\oplus$ 292-316
$M^4_{TM_2}$	IO	0.003-0.008	0.014-0.05	0.1-0.111	0.048-0.051	39.8-51.4	0-150 $\oplus$ 210-360	0-77 $\oplus$ 283-360
$M^5_{TM_2}$	NO	0.021-0.031	0.005-0.03	0.098-0.12	0.023-0.032	49.7-51.4	0-42 $\oplus$ 317-360	45-90 $\oplus$ 270-316
$M^6_{TM_2}$	IO	0.003-0.009	0.014-0.05	0.101-0.111	0.048-0.051	39.8-51.4	0-151 $\oplus$ 211-360	19-90 $\oplus$ 270-340

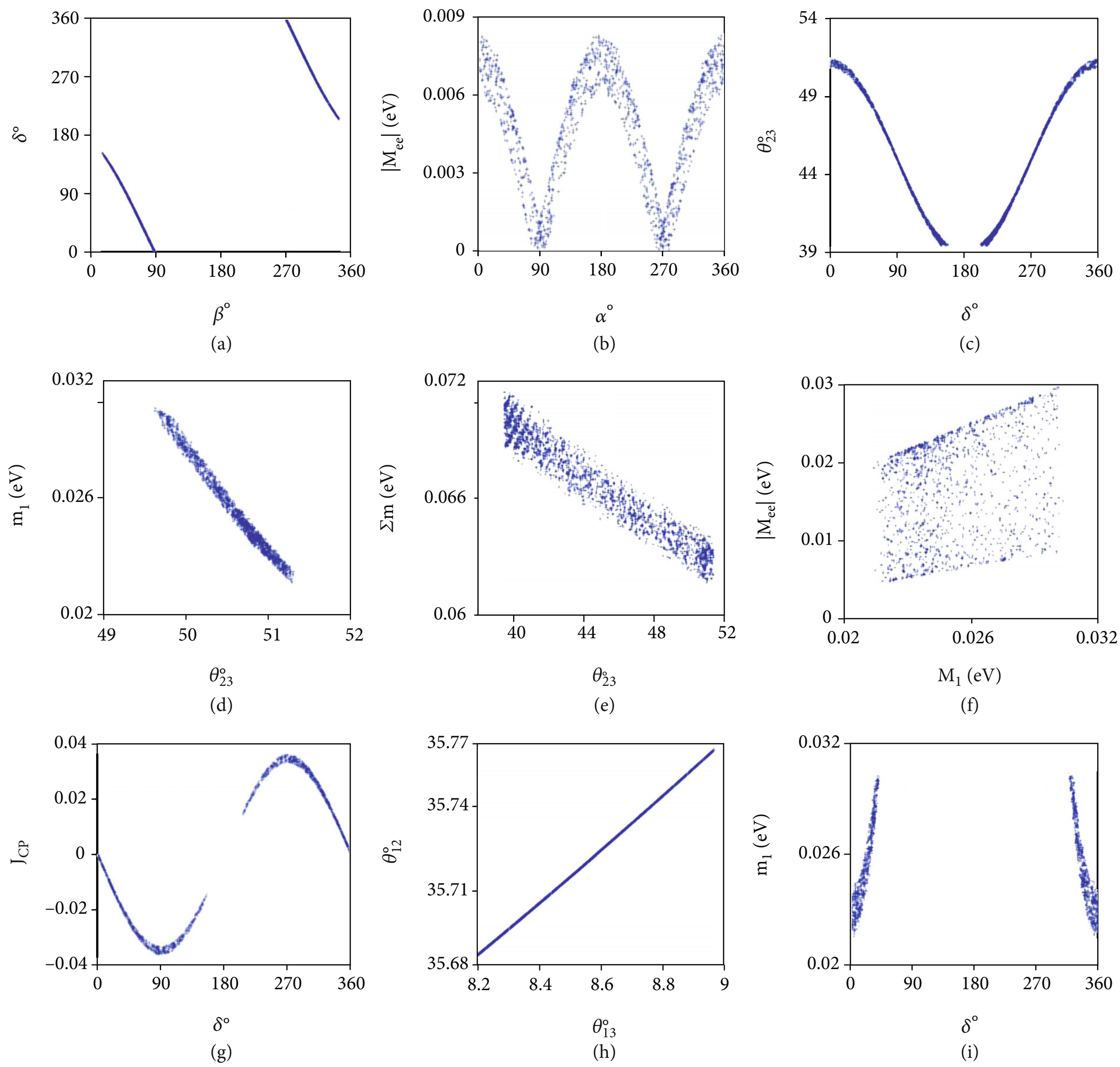

Figure 1: Correlation plots among various parameters for textures (a) $M^1_{TM_2}$, (b) $M^1_{TM_2}$, (c) $M^2_{TM_2}$, (d) $M^5_{TM_2}$, (e) $M^2_{TM_2}$, (f) $M^5_{TM_2}$, (g) $M^1_{TM_2}$, (h) $M^3_{TM_2}$, and (i) $M^5_{TM_2}$ with NO.

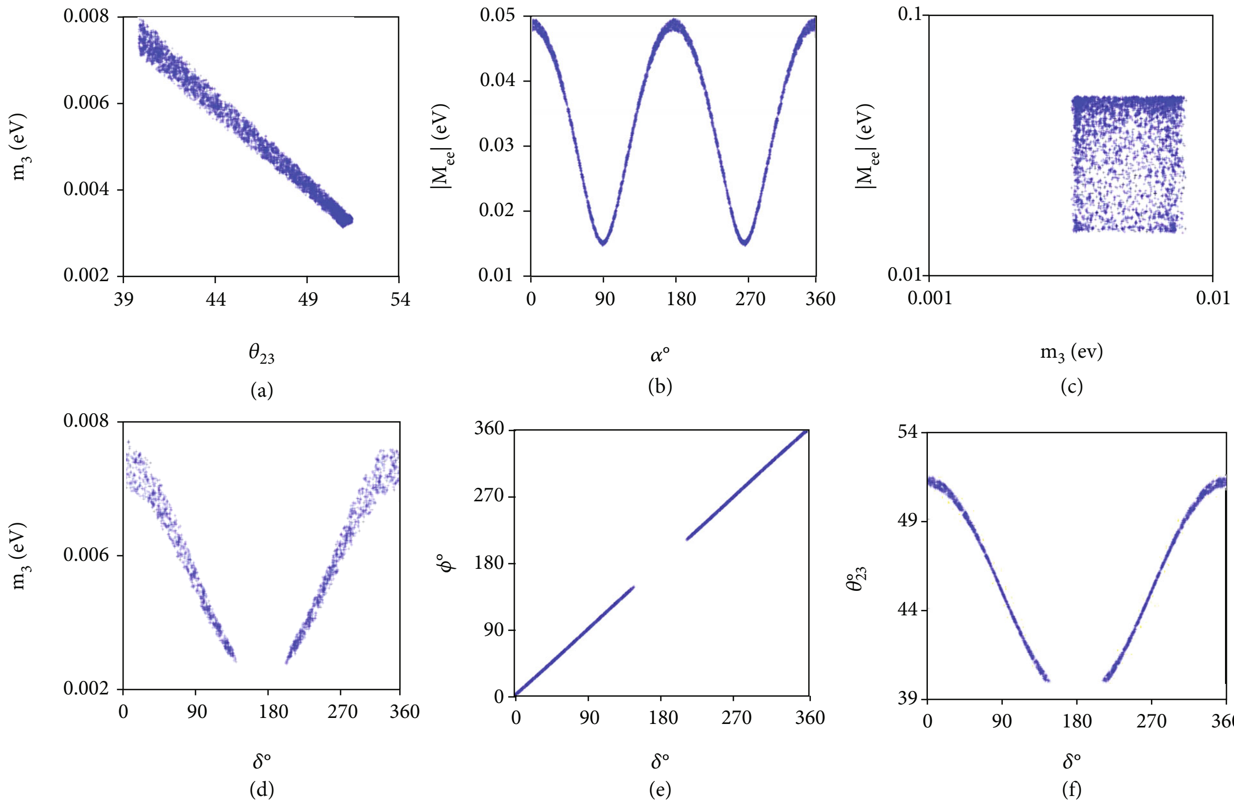

Figure 2: Correlation plots among various parameters for textures (a) $M^4_{TM_2}$, (b) $M^4_{TM_2}$, (c) $M^4_{TM_2}$, (d) $M^6_{TM_2}$, (e) $M^6_{TM_2}$, and (f) $M^6_{TM_2}$ with IO.

Figures 1 and 2 show correlations among various neutrino oscillation parameters. The Dirac CP-violating phase δ and phase ϕ are linearly correlated as shown in Figure 2(e). Figure 1(h) depicts the correlation between θ_{12} and θ_{13}. The Dirac-type CP-violating phase δ strongly depends on the Majorana-type CP-violating phase β as shown in Figure 1(a).

The main results for the neutrino mass matrices with one texture equality and TM_2 mixing are listed in the following:

(i) Textures $M^7_{TM_2}, M^8_{TM_2}$, and $M^9_{TM_2}$ lead to two degenerate eigenvalues and are, hence, experimentally ruled out at 3σ C.L.

(ii) Textures $M^{10}_{TM_2}, M^{11}_{TM_2}, M^{12}_{TM_2}, M^{13}_{TM_2}, M^{14}_{TM_2}$, and $M^{15}_{TM_2}$ lead to vanishing reactor mixing angle and, hence, are not viable at 3σ C.L.

(iii) Textures $M^3_{TM_2}$ and $M^5_{TM_2}$ for IO are not consistent with the experimental data if cosmological and neutrinoless double beta decay bounds along with neutrino oscillation data are incorporated

(iv) Textures $M^1_{TM_2}$ and $M^2_{TM_2}$ are consistent with NO only, whereas textures $M^4_{TM_2}$ and $M^6_{TM_2}$ are consistent with IO only

(v) For NO, textures $M^4_{TM_2}$ and $M^6_{TM_2}$ are not consistent with the experimental data as the mixing angles θ_{12} and θ_{13} are not within the 3σ range

(vi) All viable textures cannot have zero lowest mass eigenvalue for both mass orderings

(vii) The atmospheric mixing angle θ_{23} is below (above) maximal for textures $M^3_{TM_2}$ ($M^5_{TM_2}$) and $M^5_{TM_2}$ ($M^3_{TM_2}$) with NO and IO, respectively

(viii) θ_{23} is maximal for $\delta \sim \pi/2$ or $3\pi/2$ for textures $M^1_{TM_2}$ ($M^6_{TM_2}$) and $M^2_{TM_2}$ ($M^4_{TM_2}$) with NO (IO).

(ix) The parameter $|M_{ee}|$ is found to be nonzero for all viable textures except $M^1_{TM_2}$ and $M^2_{TM_2}$. $|M_{ee}|$ gets its largest value when $\delta \sim \pi/2$ or $3\pi/2$ for textures $M^3_{TM_2}$ and $M^5_{TM_2}$.

(x) For all viable textures, the effective neutrino mass (m_β) is well within the range provided by KATRIN experiment

(xi) The parameters m_1 (m_3), $|M_{ee}|$, and $\sum$ get their largest value when $\theta_{23} \sim 45°$ for textures $M^3_{TM_2}$ and $M^5_{TM_2}$ with NO (IO).

3. S_3 Group Motivation

The S_3, permutation group of three objects, is the smallest discrete non-Abelian group. The permutation matrices in the three dimensional reducible representation are

$$S^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S^{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, S^{(132)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$S^{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S^{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, S^{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (41)$$

where matrices in each equation belong to the same class of S_3. The most general neutrino mass matrix invariant under the S_3 group is proportional to the democratic matrix and is given by

$$M_\nu = aD \quad \text{with } D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (42)$$

where a is a complex number and D is called the Democratic matrix. The exact S_3 symmetry does not satisfy the current neutrino oscillation data, and hence, symmetry must be broken. Various models based on the S_3 symmetry have been presented in Refs. [95–122]. In Ref. [123, 124], the S_3 symmetry is broken by the linear combination of S_3 group matrices and successfully generates the nonzero θ_{13}.

The mass matrices M_{TM_2} in Eqs. (20)–(25) can be seen as the linear combination of a democratic part and a symmetry breaking part. The symmetry breaking matrix is the sum of two symmetric matrices out of which one is the S_3 group matrix which can be any of the $S^{(12)}, S^{(13)}, S^{(23)}$ matrices, and the other part is chosen in such a way that the resultant neutrino mass matrix still satisfies the magic symmetry [125–129] and remains invariant under Z_2 symmetry. The mass matrix $M^1_{\mathrm{TM}_2}$ can be rewritten as

$$M^1_{TM_2} = \begin{pmatrix} a & a & b \\ a & b+d & a-d \\ b & a-d & a+d \end{pmatrix} \equiv a\,D + c\,S^{(13)} + d\Delta, \quad (43)$$

where

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, S^{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad (44)$$

and a, c, d are arbitrary parameters with $b = c + a$.The S_3 symmetry of the neutrino mass matrix is broken, and the resultant neutrino mass matrix still satisfies S_3 invariant constraints

$$M_{\nu_{ii}} - M_{\nu_{jj}} = M_{\nu_{kj}} - M_{\nu_{ki}} \quad \text{with } i \neq j \neq k. \quad (45)$$

This leads to a trimaximal eigenvector for the resultant neutrino mass matrix. For example, a typical form of $M^1_{\mathrm{TM}_2}$ neutrino mass matrix is given by

$$M^1_{TM_2} = \begin{pmatrix} 0.004406 - 0.005578\,i & 0.004406 - 0.005578\,i & 0.002587 + 0.006938\,i \\ 0.004406 - 0.005578\,i & -0.003415 + 0.027066\,i & 0.010408 - 0.025705\,i \\ 0.002587 + 0.006938\,i & 0.010408 - 0.025705\,i & -0.001597 + 0.014550\,i \end{pmatrix}. \quad (46)$$

In this analysis, we take the charged lepton mass matrix to be diagonal. If a horizontal symmetry exists, it must, simultaneously, be a symmetry of the neutrinos as well as the charged leptons before the gauge symmetry breaking. After the symmetry breaking when the fermions acquire nonzero masses, the neutrino sector and the charged lepton sector should be governed by different subgroups of the symmetry group in order to have nonzero mixing. Here, we consider S_3 to be the residual symmetry in the neutrino sector and Z_3 symmetry as the residual symmetry in the charged lepton sector which yields nondegenerate diagonal charged lepton mass matrix [130].

Similarly, other viable textures in Eqs. (20)–(22) can, also, be decomposed into the democratic S_3 invariant part and the symmetry breaking part. The phenomenologically viable mass matrices in Eqs. (20)–(22) are related as follows by S_3 permutation symmetry:

$$S^{(123)} M^1_{TM_2} S^{(123)^T} = M^4_{TM_2}, S^{(132)} M^1_{TM_2} S^{(132)^T} = M^5_{TM_2}, S^{(12)} M^1_{TM_2} S^{(12)^T} = M^3_{TM_2},$$
$$S^{(13)} M^1_{TM_2} S^{(13)^T} = M^6_{TM_2}, S^{(23)} M^1_{TM_2} S^{(23)^T} = M^2_{TM_2}. \quad (47)$$

4. TM_1 Mixing and One Texture Equality

The neutrino mixing matrix with first column identical to TBM can be parametrized [60–67] as

$$U_{TM_1} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{\cos\theta}{\sqrt{3}} & \frac{\sin\theta}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}} + \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{3}} - \frac{e^{i\phi}\cos\theta}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}} - \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & \frac{e^{i\phi}\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{3}} \end{pmatrix}, \tag{48}$$

and the corresponding neutrino mass matrix is given by

$$M_{TM_1} = P_l U_{TM_1} P_\nu M_\nu^{\text{diag}} P_\nu^T U_{TM_1}^T P_l^T. \tag{49}$$

The most general neutrino mass matrix with TM_1 mixing can be written as

$$M_{\text{TM}_1} = \begin{pmatrix} a & 2b & 2c \\ 2b & 4b+d & a-b-c-d \\ 2c & a-b-c-d & 4c+d \end{pmatrix}. \tag{50}$$

The mass matrix M_{TM_1} is invariant under the transformation $G_1^T M_{\text{TM}_1} G_1 = M_{\text{TM}_1}$ where $G_1 = U_{\text{TM}_1}$ diag $(1,-1,-1)$ $U_{\text{TM}_1}^\dagger$ is the generator of Z_2 symmetry. This along with equality condition restricts the three unphysical phase angles to $\phi_e = \phi_\mu = \phi_\tau \equiv \phi_l$.

All possible textures of neutrino mass matrices with TM_1 mixing and one texture equality are given by

$$\begin{gathered}
M_{\text{TM}_1}^1 = \begin{pmatrix} 2b & 2b & 2c \\ 2b & 4b+d & b-c-d \\ 2c & b-c-d & 4c+d \end{pmatrix},
M_{\text{TM}_1}^2 = \begin{pmatrix} 2c & 2b & 2c \\ 2b & 4b+d & -b+c-d \\ 2c & -b+c-d & 4c+d \end{pmatrix}, \\
M_{\text{TM}_1}^3 = \begin{pmatrix} a & 2b & 2c \\ 2b & 2b & a+b-c \\ 2c & a+b-c & 4c-2b \end{pmatrix},
M_{\text{TM}_1}^5 = \begin{pmatrix} a & 2b & 2c \\ 2b & 4b-2c & a-b+c \\ 2c & a-b+c & 2c \end{pmatrix}, \\
M_{\text{TM}_1}^4 = \begin{pmatrix} a & 2b & 2c \\ 2b & \frac{1}{2}(a+3b-c) & \frac{1}{2}(a+3b-c) \\ 2c & \frac{1}{2}(a+3b-c) & \frac{1}{2}(a-5b+7c) \end{pmatrix},
M_{\text{TM}_1}^6 = \begin{pmatrix} a & 2b & 2c \\ 2b & \frac{1}{2}(a+7b-5c) & \frac{1}{2}(a-b+3c) \\ 2c & \frac{1}{2}(a-b+3c) & \frac{1}{2}(a-b+3c) \end{pmatrix}, \\
M_{\text{TM}_1}^7 = \begin{pmatrix} a & 2b & 2c \\ 2b & 3b-c & a \\ 2c & a & 3c-b \end{pmatrix}, \\
M_{\text{TM}_1}^8 = \begin{pmatrix} a & 2b & 2c \\ 2b & 2c & a+3b-3c \\ 2c & a+3b-3c & 6c-4b \end{pmatrix},
M_{\text{TM}_1}^9 = \begin{pmatrix} a & 2b & 2c \\ 2b & 6b-4c & a-3b+3c \\ 2c & a-3b+3c & 2b \end{pmatrix}, \\
M_{\text{TM}_1}^{10} = \begin{pmatrix} a & 2b & 2c \\ 2b & a & 3b-c \\ 2c & 3b-c & a-4b+4c \end{pmatrix},
M_{\text{TM}_1}^{11} = \begin{pmatrix} a & 2b & 2c \\ 2b & a+4b-4c & 3c-b \\ 2c & 3c-b & a \end{pmatrix}, \\
M_{\text{TM}_1}^{12} = M_{\text{TM}_1}^{13} = \begin{pmatrix} a & 2b & 2b \\ 2b & 4b+d & a-2b-d \\ 2b & a-2b-d & 4b+d \end{pmatrix}, \\
M_{\text{TM}_1}^{14} = \begin{pmatrix} a & 2b & 2c \\ 2b & a+b-c & 2b \\ 2c & 2b & a-3b+3c \end{pmatrix},
M_{\text{TM}_1}^{15} = \begin{pmatrix} a & 2b & 2c \\ 2b & a+3b-3c & 2c \\ 2c & 2c & a-b+c \end{pmatrix},
\end{gathered} \tag{51}$$

where textures represented in each equation are related by μ-τ permutation symmetry. The neutrino mixing angles

Table 5: Numerical predictions for viable textures having one equality in M_ν with TM_1 mixing at 3σ C.L. (only neutrino oscillation data are incorporated).

Texture	Ordering	m_{lowest} (eV)	$\lvert M_{ee}\rvert$ (eV)	$\sum$ (eV)	m_β	δ°	β°
$M^1_{TM_1}$	NO	0.0039-0.0066	0.0-0.0072	0.062-0.069	0.009-0.012	63-125 ⊕ 235-297	21-65 ⊕ 116-159 ⊕ 202-244 ⊕ 296-339
$M^2_{TM_1}$	NO	0.0039-0.0066	0.0-0.0072	0.062-0.069	0.009-0.012	64-125 ⊕ 235-296	26-68 ⊕ 112-154 ⊕ 206-249 ⊕ 291-334
$M^3_{TM_1}$	NO	0.046-0.67	0.027-0.55	0.16-2.0	0.002-0.003	64-122 ⊕ 235-297	0-43 ⊕ 137-222 ⊕ 318-360
	IO	0.05-0.42	0.06-0.41	0.21-1.25	0.077-0.42	67-125 ⊕ 235-294	0-44 ⊕ 136-227 ⊕ 314-360
$M^4_{TM_1}$	IO	0.007-0.016	0.017-0.051	0.107-0.121	0.048-0.052	65-124 ⊕ 235-295	0-324
$M^5_{TM_1}$	NO	0.04-0.92	0.02-0.88	0.15-2.8	0.045-0.92	64-124 ⊕ 235-296	0-36 ⊕ 143-218 ⊕ 322-360
	IO	0.062-0.36	0.064-0.29	0.22-1.1	0.079-0.37	66-125 ⊕ 236-293	0-35 ⊕ 144-213 ⊕ 324-360
$M^6_{TM_1}$	IO	0.008-0.017	0.017-0.052	0.108-0.123	0.049-0.524	65-125 ⊕ 234-294	0-324
$M^7_{TM_1}$	NO	0.085-0.72	0.032-0.7	0.26-2.2	0.085-0.72	63-125 ⊕ 236-297	0-37 ⊕ 143-218 ⊕ 323-360
$M^8_{TM_1}$	NO	0.03-0.32	0.02-0.31	0.118-0.98	0.031-0.32	63-124 ⊕ 236-297	0-53 ⊕ 128-235 ⊕ 308-360
	IO	0.014-0.23	0.03-0.22	0.116-0.7	0.05-0.24	66-125 ⊕ 235-294	0-65 ⊕ 115-244 ⊕ 296-360
$M^9_{TM_1}$	NO	0.025-0.31	0.018-0.19	0.107-0.93	0.025-0.32	67-125 ⊕ 235-297	0-50 ⊕ 130-224 ⊕ 312-360
	IO	0.0148-0.33	0.028-0.26	0.117-1.0	0.05-0.33	66-125 ⊕ 235-294	0-56 ⊕ 124-236 ⊕ 306-360
$M^{10}_{TM_1}$	NO	0.034-0.7	0.014-0.65	0.13-2.1	0.034-0.7	63-124 ⊕ 237-297	43-141 ⊕ 221-314
	IO	0.028-0.4	0.021-0.27	0.14-1.2	0.05-0.41	66-125 ⊕ 235-295	34-145 ⊕ 215-325
$M^{11}_{TM_1}$	NO	0.031-0.3	0.011-0.27	0.122-0.86	0.032-0.29	65-125 ⊕ 235–295	47-128 ⊕ 229-313
	IO	0.03-0.51	0.023-0.48	0.14-1.52	0.05-0.51	65-124 ⊕ 236-295	44-136 ⊕ 224-316
$M^{14}_{TM_1}$	NO	0.04-0.48	0.02-0.36	0.16-1.43	0.04-0.48	63-123 ⊕ 237-297	47-132 ⊕ 230-314
	IO	0.06-0.4	0.028-0.38	0.21-1.2	0.07-0.41	65-126 ⊕ 234-294	43-135 ⊕ 225-316
$M^{15}_{TM_1}$	NO	0.04-0.55	0.01-0.5	0.15-1.7	0.04-0.55	64-123 ⊕ 236-295	53-125 ⊕ 237-307
	IO	0.06-0.52	0.03-0.5	0.22-1.6	0.07-0.52	65-125 ⊕ 235-295	54-126 ⊕ 234-305

for TM_1 mixing in terms of parameters θ and ϕ are [72–74] given by

$$\sin^2\theta_{13} = \frac{1}{3}\sin^2\theta, \sin^2\theta_{12} = 1 - \frac{2}{3-\sin^2\theta},$$
$$\sin^2\theta_{23} = \frac{1}{2}\left(1 + \frac{\sqrt{6}\sin 2\theta\cos\phi}{3-\sin^2\theta}\right). \quad (52)$$

For TM_1 mixing, the Jarlskog rephasing invariant [72–74] is

$$J_{CP} = \frac{1}{6\sqrt{6}}\sin 2\theta\sin\phi, \quad (53)$$

and the Dirac-type CP-violating phase [72–74] is given by

$$\tan\delta = \frac{\cos 2\theta + 5}{5\cos 2\theta + 1}\tan\phi. \quad (54)$$

The effective Majorana mass for TM_1 mixing can be calculated by using Eqs. (11) and (48) as

$$\lvert M_{ee}\rvert = \frac{1}{3}\left|2m_1 + m_2\cos^2\theta e^{2i\alpha} + m_3\sin^2\theta e^{2i\beta}\right|, \quad (55)$$

and the effective neutrino mass for TM_1 by using Eqs.(12) and (48) is given by

Table 6: Numerical predictions for viable textures having one equality in M_ν with TM_1 mixing at 3σ C.L. (cosmological and neutrinoless double beta decay bounds along with neutrino oscillation data are incorporated).

Texture	Ordering	m_{lowest} (eV)	$\lvert M_{ee}\rvert$ (eV)	$\sum$ (eV)	m_β	δ°	β°
$M^1_{TM_1}$	NO	0.0039-0.0066	0.0-0.0072	0.062-0.069	0.009-0.012	64-125 ⊕ 235-296 ⊕ 201-245 ⊕ 295-339	21-65 ⊕ 116-158
$M^2_{TM_1}$	NO	0.0039-0.0066	0.0-0.0072	0.062-0.069	0.009-0.012	64-125 ⊕ 235-296 ⊕ 206-249 ⊕ 291-334	26-68 ⊕ 112-154
$M^4_{TM_1}$	IO	0.007-0.016	0.017-0.051	0.107-0.12	0.048-0.052	66-125 ⊕ 235-295	0-322
$M^6_{TM_1}$	IO	0.008-0.017	0.017-0.051	0.108-0.12	0.049-0.052	65-125 ⊕ 235-295	0-320
$M^8_{TM_1}$	IO	0.014-0.017	0.038-0.049	0.116-0.12	0.05-0.052	84-124 ⊕ 236-273	11-53 ⊕ 125-170 ⊕ 193-235 ⊕ 307-347
$M^9_{TM_1}$	NO	0.025-0.031	0.018-0.031	0.107-0.12	0.026-0.32	112-125 ⊕ 235-247	1-29 ⊕ 152-208 ⊕ 332-359
	IO	0.0148-0.0167	0.037-0.047	0.117-0.12	0.05-0.052	67-95 ⊕ 269-293	16-44 ⊕ 135-168 ⊕ 191-225 ⊕ 316-348

$$m_\beta = \sqrt{\frac{1}{3}\left|2m_1^2 + m_2^2\cos^2\theta + m_3^2\sin^2\theta\right.}. \tag{56}$$

The existence of one equality between the elements of the neutrino mass matrix implies

$$M_{TM_1(ab)} - M_{TM_1(cd)} = 0, \tag{57}$$

which yields the following complex equation:

$$\sum (QV_{ai}V_{bi} - V_{ci}V_{di})m_i = 0, \tag{58}$$

where $Q = e^{i(\phi_a+\phi_b-(\phi_c+\phi_d))}$. The above equation can be rewritten as

$$m_1A_1 + m_2A_2e^{2i\alpha} + m_3A_3e^{2i\beta} = 0, \tag{59}$$

where

$$A_i = (QU_{ai}U_{bi} - U_{ci}U_{di}), \tag{60}$$

with $(i = 1, 2, 3)$ and a, b can take values e, μ, and τ. Solving the real and imaginary parts of Eq. (59) simultaneously, we obtain the following two mass ratios:

$$\begin{aligned}\zeta \equiv \frac{m_2}{m_1} &= \frac{(\mathrm{Re}\,(A_1)\,\mathrm{Im}\,(A_3) - \mathrm{Re}\,(A_3)\,\mathrm{Im}\,(A_1))\cos 2\beta + (\mathrm{Re}\,(A_1)\,\mathrm{Re}\,(A_3) + \mathrm{Im}\,(A_1)\,\mathrm{Im}\,(A_3))\sin 2\beta}{(\mathrm{Re}\,(A_3)\,\mathrm{Im}\,(A_2) - \mathrm{Re}\,(A_2)\,\mathrm{Im}\,(A_3))\cos 2(\alpha-\beta) + (\mathrm{Re}\,(A_2)\,\mathrm{Re}\,(A_3) + \mathrm{Im}\,(A_2)\,\mathrm{Im}\,(A_3))\sin 2(\alpha-\beta)},\\ \xi \equiv \frac{m_3}{m_1} &= \frac{(\mathrm{Re}\,(A_2)\,\mathrm{Im}\,(A_1) - \mathrm{Re}\,(A_1)\,\mathrm{Im}\,(A_2))\cos 2\alpha - (\mathrm{Re}\,(A_1)\,\mathrm{Re}\,(A_2) + \mathrm{Im}\,(A_1)\,\mathrm{Im}\,(A_2))\sin 2\alpha}{(\mathrm{Re}\,(A_3)\,\mathrm{Im}\,(A_2) - \mathrm{Re}\,(A_2)\,\mathrm{Im}\,(A_3))\cos 2(\alpha-\beta) + (\mathrm{Re}\,(A_2)\,\mathrm{Re}\,(A_3) + \mathrm{Im}\,(A_2)\,\mathrm{Im}\,(A_3))\sin 2(\alpha-\beta)}.\end{aligned} \tag{61}$$

These mass ratios can be used to calculate the ratio of mass squared differences (R_ν) which is given by

$$R_\nu \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \frac{\zeta^2 - 1}{\xi^2 - 1} \quad R_\nu \equiv \frac{\Delta m_{21}^2}{\Delta m_{23}^2} = \frac{\zeta^2 - 1}{\zeta^2 - \xi^2}, \tag{62}$$

for NO and IO, respectively. Since, Δm_{21}^2 and $\Delta m_{31}^2 (\Delta m_{23}^2)$ for NO (IO) are experimentally known, the parameter R_ν should lie within its experimentally allowed range for a texture equality to be compatible with the current neutrino oscillation data. The neutrino mass eigenvalues can be calculated by using the relations

$$\begin{aligned} m_2 &= \sqrt{m_1^2 + \Delta m_{21}^2},\ m_3 = \sqrt{m_1^2 + \Delta m_{31}^2},\\ m_2 &= \sqrt{m_1^2 + \Delta m_{21}^2},\ m_3 = \sqrt{m_1^2 + \Delta m_{21}^2 - \Delta m_{23}^2},\end{aligned} \tag{63}$$

for NO and IO, respectively.

For numerical analysis, we follow the same procedure as in TM_2 mixing except that the parameters β and m_1 are generated randomly within their allowed ranges. The mass eigenvalues are calculated by using Eq. (63), and texture equality is imposed by requiring the parameter R_ν in Eq. (62) to lie within its 3σ experimental range.

The numerical predictions for unknown parameters are summarized in Table 5 (where constrains only from

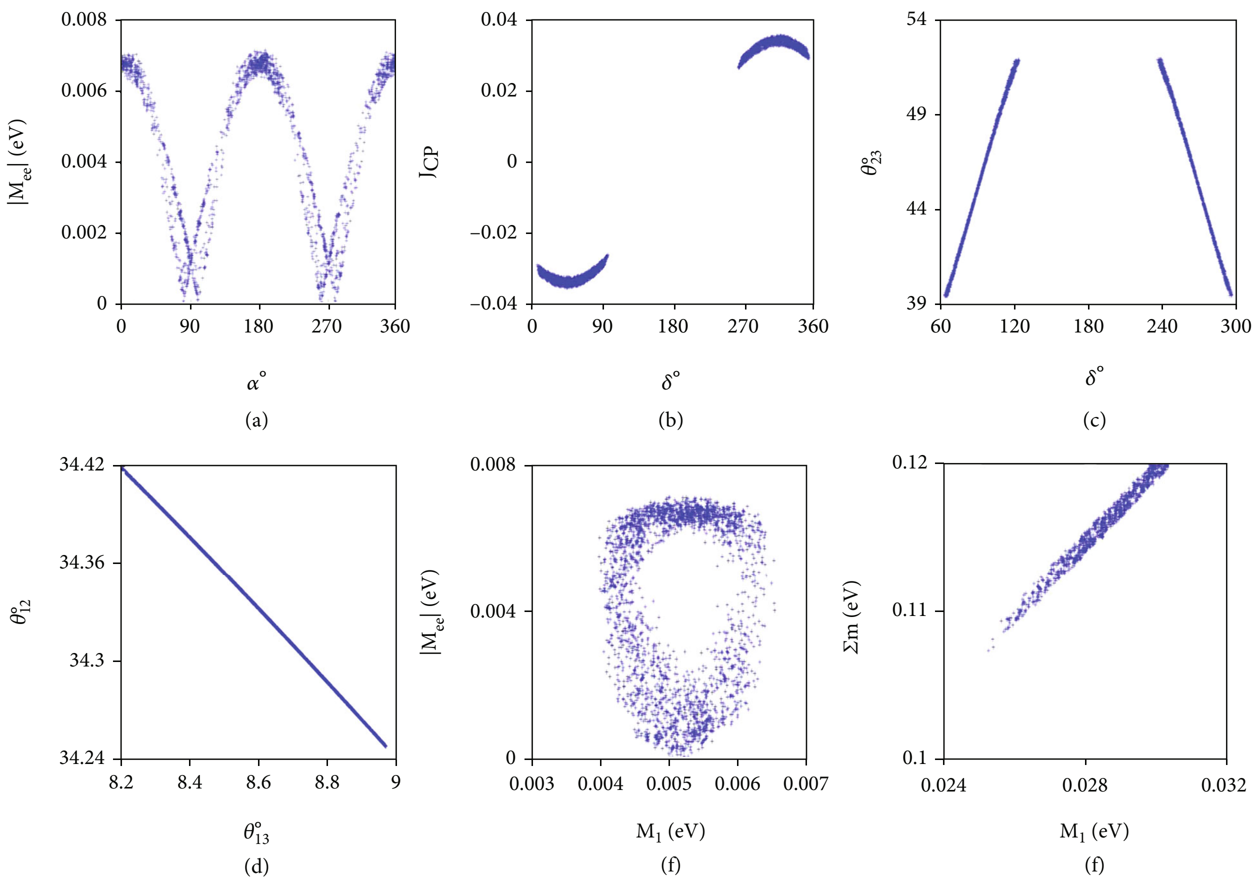

FIGURE 3: Correlation plots among various parameters for textures (a) $M^{1}_{\mathrm{TM}_1}$, (b)$M^{1}_{\mathrm{TM}_1}$, (c) $M^{2}_{\mathrm{TM}_1}$, (d) $M^{2}_{\mathrm{TM}_1}$, (e) $M^{2}_{\mathrm{TM}_1}$, and (f) $M^{9}_{\mathrm{TM}_1}$ with NO.

neutrino oscillation data are used) and Table 6 (where the constrains from cosmological and neutrinoless double beta decay bounds along with neutrino oscillation data are used). The allowed ranges of the parameter θ_{12} are (34.24°-34.42°) for all viable textures. The parameter θ is constrained to lie in the ranges (14.3°-15.7°), whereas J_{CP} lies in the ranges $\pm(0.026-0.036)$ for all viable textures. The Majorana phase α varies in the range (0-360°) for all viable textures with NO only. Correlation plots among various neutrino oscillating parameters are shown in Figures 3 and 4 for NO and IO, respectively. $|M_{ee}|$ strongly depends on the Majorana phase α as shown in Figure 3(a) for NO and Figure 4(b) for IO. As shown in Figure 3(d), θ_{12} is inversely proportional to θ_{13} which is the classical prediction of TM_1 mixing. The Dirac-type CP-violating phase is constrained to lie in the regions around 90° and 270° which is consistent with the recent observations in the long-baseline neutrino oscillation experiments such as T2K and NOvA [2, 3] which shows a preference for the Dirac-type CP-violating phase δ to lie around $\delta \sim 270°$. The main implications for textures having TM_1 mixing with one texture equality are summarized in the following:

(i) For IO, textures $M^{1}_{\mathrm{TM}_1}$, $M^{2}_{\mathrm{TM}_1}$, and $M^{7}_{TM_1}$ are not consistent with the neutrino oscillation data at 3σ C.L.

(ii) For NO, textures $M^{4}_{\mathrm{TM}_1}$ and $M^{6}_{\mathrm{TM}_1}$ predict large θ_{13} and small θ_{12} and are, hence, experimentally ruled out at 3σ C.L.

(iii) Textures $M^{12}_{\mathrm{TM}_1}$ and $M^{13}_{\mathrm{TM}_1}$ predict a vanishing reactor mixing angle and degenerate mass eigenvalues and are, hence, not viable for both mass orderings

(iv) Textures $M^{3}_{\mathrm{TM}_1}$, $M^{5}_{\mathrm{TM}_1}$, $M^{10}_{\mathrm{TM}_1}$, $M^{14}_{\mathrm{TM}_1}$, $M^{15}_{\mathrm{TM}_1}$, and $M^{11}_{\mathrm{TM}_1}$ for both mass orderings and textures $M^{7}_{\mathrm{TM}_1}$, $M^{8}_{\mathrm{TM}_1}$ for NO predict large $\sum m_i$ and are, hence, not viable with experimental data when cosmological and neutrinoless double beta decay bounds along with neutrino oscillation data are incorporated

(v) All viable textures cannot have zero lowest mass eigenvalue for both mass orderings

(vi) The atmospheric mixing angle θ_{23} is maximal for $\delta \sim \pi/2$ or $3\pi/2$ for all viable textures

(vii) The parameter $|M_{ee}|$ is found to be bounded from below for all viable textures except $M^{1}_{\mathrm{TM}_1}$ and $M^{2}_{\mathrm{TM}_1}$ with NO

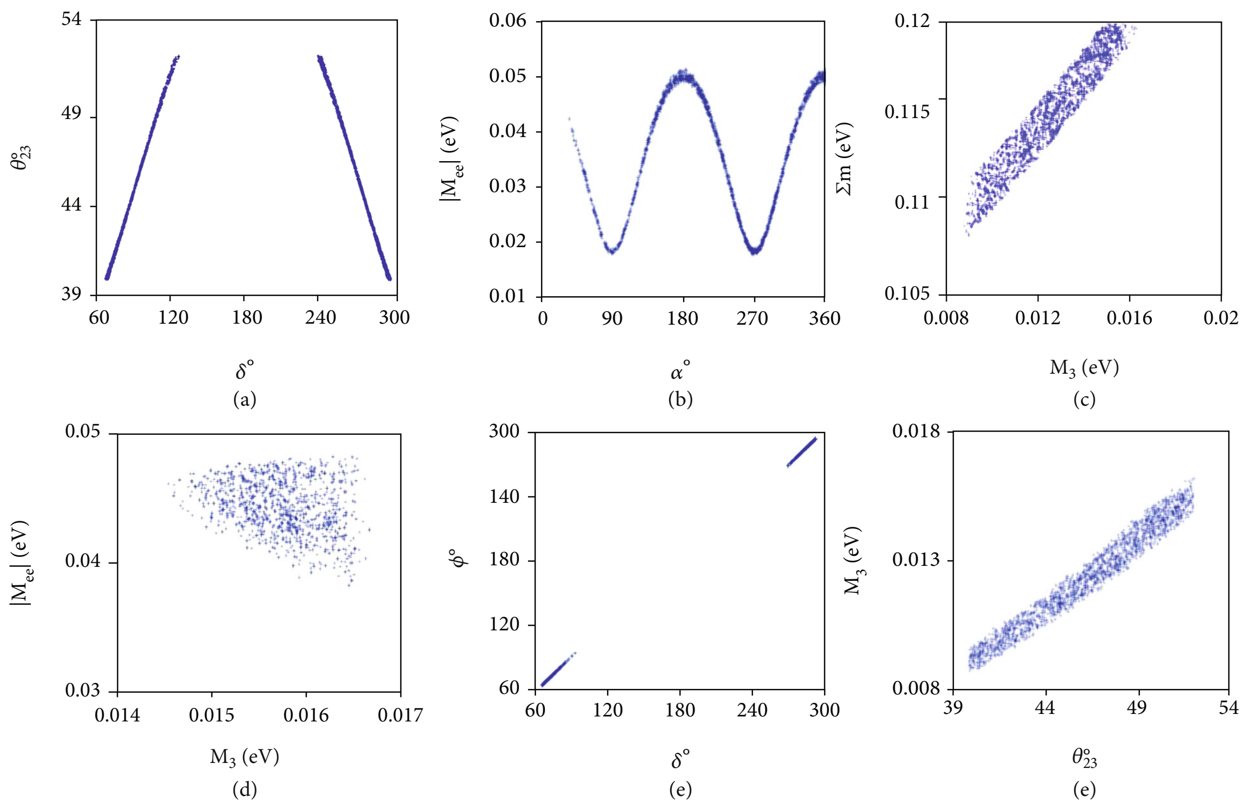

Figure 4: Correlation plots among various parameters for textures (a) $M^4_{\mathrm{TM}_1}$, (b) $M^4_{\mathrm{TM}_1}$, (c) $M^6_{\mathrm{TM}_1}$, (d) $M^8_{\mathrm{TM}_1}$, (e) $M^9_{\mathrm{TM}_1}$, and (f) $M^6_{\mathrm{TM}_1}$ with IO.

(viii) The parameter m_β is found to lie within the current experimental range for all viable textures

(ix) The Dirac-type CP-violating phase δ is directly proportional to the parameter ϕ for all viable textures

5. Summary

We studied the phenomenological implications of one texture equality in the neutrino mass matrix with TM_1 or T M_2 mixing. The presence of one texture equality in M_ν with TM_1 or TM_2 as the mixing matrix reduces the number of free parameters significantly and, hence, leads to very predictive neutrino mass matrices. Out of total fifteen possible textures of M_ν, thirteen textures are phenomenologically allowed with TM_1 mixing, and only six textures are allowed with TM_2 mixing in the light of current neutrino oscillation data at 3σ C.L. However, the number of viable textures reduced to six for TM_1 mixing if the constrains from cosmology and neutrinoless double beta decay experiments along with neutrino oscillation data are used. Since, the T M_2 mixing predicts a value of θ_{12} away from its best fit value, TM_1 mixing is phenomenologically more appealing. In this analysis, we have obtained interesting predictions for unknown parameters such as the Dirac- and Majorana-type CP-violating phases, effective Majorana neutrino mass, effective neutrino mass, Jarlskog rephasing invariant, neutrino mass scale, and the sum of neutrino masses. For TM_1 mixing, the Dirac-type CP-violating phase (δ) is restricted to the regions around $\pi/2$ and $3\pi/2$, the atmospheric mixing angle (θ_{23}) is maximal for $\delta \sim \pi/2$ or $3\pi/2$, and the lowest neutrino mass eigenvalue cannot be zero for all viable textures. For TM_2 mixing, the CP-violating phases δ and β are strongly correlated, θ_{23} is below (above) maximal for textures $M^3_{\mathrm{TM}_2}(M^5_{\mathrm{TM}_2})$ and $M^5_{\mathrm{TM}_2}(M^3_{\mathrm{TM}_2})$ with NO and IO, respectively, and the lowest mass eigenvalue cannot be zero for all viable textures. For M_{TM_2} mass matrices with one texture equality, the residual S_3 symmetry is broken, and the resulting neutrino mass matrix is invariant under Z_2 64.

Data Availability

The numerical prediction data used to support the findings of this study are included within the article. The raw data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

A previous version of this manuscript was presented in http://arXiv.org with identifier 2202.13070.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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