

Research Article

Supersymmetric Dyons, Superstrings, and Rotating Wormholes

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We construct supersymmetric dyon solutions based on the 't Hooft/Polyakov monopole. We show that these solutions satisfy κ -symmetry constraints and can therefore be generalized to supersymmetric solutions of type I $SO(32)$ string theory. After applying a T -duality transformation to these solutions, we obtain two $D3$ -branes connected by a wormhole, embedded in an $M5$ -brane. We analyze the geometries of each $D3$ -brane for two cases: one corresponding to a dyon with vanishing spin and the other corresponding to a magnetic monopole with nonvanishing spin. In the case of the vanishing spin, the scalar curvature is finite everywhere. In the case of the nonvanishing spin, we find a frame dragging effect due to the spin. We also find that the scalar curvature diverges along the spin quantization axis as $1/\rho^2$, ρ being the cylindrical, radial coordinate defined with respect to the spin axis. These solutions demonstrate the subtle relationship between the Yang-Mills and gravitational interactions, i.e., gauge/gravity duality.

1. Introduction

In a previous study, we have investigated spin 0 dyons within the context of type I $SO(32)$ superstring theory in 10 dimensions [1]. Based on the 't Hooft/Polyakov monopole, we have constructed dyon solutions which are exact solutions of the non-Abelian Dirac-Born-Infeld action and the Wess-Zumino-like action. After applying a T -duality transformation to the solutions, we have obtained solutions corresponding to electrically and magnetically charged wormholes (for additional information about wormholes and their physical properties, please consult the following references [2–6]) which connect two $D3$ -branes.

In this study, we extend our previous work to include solutions with nonvanishing spin. Specifically, we have applied supersymmetry transformations to the solutions obtained previously, yielding spin 1/2 and spin 1 dyons. We then show that the solutions also preserve a combined κ -symmetry and supersymmetry so that they are also solutions of superstring theory. After applying a suitable coordinate/gauge transformation, followed by a T -duality transformation, we obtain rotating wormhole solutions which are both magnetically and electrically charged.

We now outline the steps in our analysis. In Section 2, we review dimensional reduction of $D = 10, N = 1$ supersymmetry to $D = 6, N = 2$ and then to $D = 4, N = 4$ supersymmetry. This reduction is carried out with the purpose of showing, explicitly, the connection between dyons in four dimensions and dyons derived from superstrings in ten dimensions. In Section 3, we use the results of Section 2 to reinterpret the spin 0 dyon solutions in four spacetime dimensions [7] as a gauge field dimensionally reduced from ten to six spacetime dimensions. We then apply supersymmetry transformations to the gauge fields, thereby recasting the supersymmetric dyon solutions in four dimensions as a $D = 6, N = 2$ supersymmetric gauge theory. As a corollary of our analysis, we extend the work of Kastor and Na [8], which applies to supersymmetric magnetic monopoles to include supersymmetric dyons. In Section 4, we show that the solutions obtained in Section 3 preserve combined κ -symmetry and supersymmetry and are therefore solutions of type IIB superstring theory, which we then recast as solutions of type I $SO(32)$ superstring theory, residing on an $M5$ -brane. In Section 5, we apply a T -duality transformation to the superstring solutions obtained in Section 4, reducing the theory from $D = 1 + 4$ to $D = 1 + 3$. The result is two rotating

dyons of equal but opposite charge, each residing on a curved $D3$ -brane, connected to one another by a wormhole. Finally, we present numerical and graphical examples, depicting the scalar curvature and frame dragging effect.

Concerning the system of units and sign conventions, we adhere to the same conventions as in our previous work [1]. Specifically, in D dimensions, the Levi-Civita symbol is $\varepsilon_{012\dots D} = 1$. Greek letters denote spacetime indices, i.e., 0, 1, 2, and 3. Uncapitalized Roman letters denote either the spatial indices 1, 2, 3 (alternatively, 3-space coordinates are denoted as x, y , and z , where $x \equiv x^1$, $y \equiv x^2$, and $z \equiv x^3$) or the indices of the generators of the gauge group. Capitalized Roman indices denote indices of ten spacetime dimensions, i.e., 0, 1, 2, ... 9. The signature of the metric, η_{MN} , is mostly positive. The gamma matrices satisfy the following relations: $\Gamma^{M\dagger} = \Gamma_M$. Also, we employ the Lorentz-Heaviside units of electromagnetism so that $c = \hbar = \varepsilon_0 = \mu_0 = 1$. As a consequence, the Dirac quantization condition is $g_e g_m = (4\pi)n_m/2$, g_e (g_m) being the electric (magnetic) charge and n_m being an integer.

2. Dimensional Reduction of $D = 10, N = 1$ Supersymmetry

In this section, we describe the dimensional reduction of the $D = 10, N = 1$ supersymmetric Yang Mills theory, first to the $D = 6, N = 2$ theory, then to the $D = 4, N = 4$ theory. This reduction is performed, specifically, with the purpose of demonstrating how dyons in $D = 4$ can be naturally described as evolving from this dimensional reduction process.

We begin with the $D = 10, N = 1$ supersymmetric Lagrangian density [9]

$$\mathcal{L} = -\frac{1}{4} F_{MN}^a F^{aMN} - i\frac{1}{2} \bar{\lambda}^a \Gamma^M \mathcal{D}_M \lambda^a, \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - ig_{D9} f^{abc} [A_\mu^b, A_\nu^c]. \quad (2)$$

The quantity g_{D9} is the Yang-Mills coupling constant in ten dimensions (note that $g_{D9}^2 = g_{D3}^2 (2\pi)^6 \alpha'^3$, where g_{D3} is the Yang-Mills coupling constant in four dimensions and α' is the string coupling constant. See Appendix B of reference [1]), and f^{abc} are the structure constants of the gauge group. Here, the gaugino field, λ , is the supersymmetric partner of the gauge field. The action is invariant under the supersymmetric transformations

$$\delta A_M^a = -i\bar{\zeta} \Gamma_M \lambda^a, \quad (3a)$$

$$\delta \lambda^a = \frac{1}{2} F_{MN}^a \Gamma^{MN} \zeta, \quad (3b)$$

where $\Gamma^{MN} = \Gamma^M \Gamma^N - \Gamma^N \Gamma^M$. The gaugino field λ^a and supersymmetric parameter ζ are the 32 component Major-

ana spinors with positive chirality, i.e., $\bar{\lambda} = (\lambda^a)^T C$, where C is the charge conjugation matrix and $\Gamma^{(10)} \lambda^a = (+1) \lambda^a$, where the chirality matrix $\Gamma^{(10)} = i^{-4} \varepsilon_{01\dots 9} \Gamma^0 \Gamma^1 \dots \Gamma^9$ (the chirality matrix in D dimensions is $\Gamma^{(D)} \equiv K \varepsilon_{01\dots (D-1)} \Gamma^0 \Gamma^1 \dots \Gamma^{D-1}$, where $D = 2k + 2$ and $K = i^{-k}$ for the Minkowski signature and $K = i^{-(k+1)}$ for the Euclidean signature).

Using Noether's theorem, we obtain the supercurrent by varying the Lagrangian density with respect to the fields X ($X = A_M^a$ or λ^a) [10],

$$\zeta J^M + \zeta^\dagger J^{\dagger M} \equiv \sum_X \frac{\delta \mathcal{L}}{\delta (\partial_M X)} - K^M, \quad (4)$$

where K^M is a function whose divergence is the variation of the Lagrangian density under supersymmetry transformations, i.e., $\partial_M K^M = \delta \mathcal{L}$. The supercharges, Q_α , are obtained from the supercurrents

$$Q_\alpha = \int d^9 x J_\alpha^0, \quad (5a)$$

$$Q_\alpha^\dagger = \int d^9 x J_\alpha^{\dagger 0}. \quad (5b)$$

The supercharges, which are the generators of supersymmetry transformations,

$$[\zeta^\dagger Q^\dagger + \zeta Q, X] = \delta X, \quad (6)$$

can be obtained from equation (4). Alternatively, we can compare equation (6) directly to equations (3a) and (3b) and obtain

$$Q_\alpha = -\frac{1}{2} \int dx^9 F_{MN}^a (\lambda^{a\dagger} \Gamma^0 \Gamma^{MN})_\alpha. \quad (7)$$

In deriving equation (7), we have used the equal-time, canonical anticommutation and commutation relations

$$\{\lambda_\alpha^a(\vec{x}, t), \lambda_\beta^{\dagger b}(\vec{y}, t)\} = \delta^{ab} \delta_{\alpha\beta} \delta^{(9)}(\vec{x} - \vec{y}), \quad (8a)$$

$$[A_M^a(\vec{x}, t), E_N^b(\vec{y}, t)] = -\eta_{MN} \delta^{ab} \delta^{(9)}(\vec{x} - \vec{y}). \quad (8b)$$

The field E_N^b is the canonical momentum conjugate to A_N^b ($E_N^b = F_{0N}^b$) and $i\bar{\lambda}_\alpha^a \Gamma^0$ is the canonical momentum conjugate to λ_α^a .

We now calculate the anticommutator $\{Q_\alpha, Q_\beta^\dagger\}$. This calculation, though similar to that of Witten and Olive [11], differs in that their calculation is based on monopole solutions resulting from the Higgs field embedded in $D = 4, N = 2$ supersymmetry, whereas this calculation is based on the sequential, dimensional reduction from $D = 10, N = 1$ supersymmetry to $D = 6, N = 2$ supersymmetry and finally to $D = 4, N = 4$ supersymmetry. Our reason for presenting the calculation is to demonstrate the relationship between

dyons in $D = 4$, $N = 4$ supersymmetry and superstrings in the type I $SO(32)$ theory. The anticommutator is evaluated as

$$\text{zeta}_2 \zeta_1^\dagger \{Q, Q^\dagger\} = [\zeta_2 Q, \zeta_1^\dagger Q^\dagger] = \zeta_2 \zeta_1^\dagger \left(-\frac{1}{2}\right)^2 \int dx^9 F_{MN}^a F_{KL}^a \Gamma^0 \Gamma^M \Gamma^N \Gamma^{K\dagger} \Gamma^{L\dagger}. \quad (9)$$

In evaluating equation (9), it is helpful to organize the terms as follows: in the first group, all terms where $\{M, N\}$ and $\{K, L\}$ assume different values; in the second group, both $\{M, N\}$ are contracted with $\{K, L\}$, resulting in terms with no Γ matrices; and in the third group, one of $\{M, N\}$ is contracted with one of $\{K, L\}$, resulting in terms with two Γ matrices. In the first group, terms which contain Γ^0 or $\Gamma^{0\dagger}$ vanish because $\Gamma^{0\dagger} = -\Gamma^0$. Each of the remaining terms can be expressed as a divergence. Such terms are typically assumed to vanish sufficiently fast at the boundary so that these terms make no contribution; however, these terms will become relevant when we consider dyon solutions and their associated central charges (see equation (21)). The second group evaluates P^0 , the energy, i.e.,

$$P_0 = - \int dx^9 \left(F_0^{aM} F_{0M}^a - \frac{1}{4} \eta_{00} F_{MN}^a F^{aMN} \right). \quad (10)$$

In obtaining this result, we have used the fact that $\Gamma^{0\dagger} = -\Gamma^0$ and assumed that all surface integrals vanish. The third group comprises of terms which contain the product $\Gamma^M \Gamma^N$. If both $M, N \neq 0$, the term vanishes by symmetry arguments and properties of the gamma matrices. The only terms which are nonvanishing from this group are those that contain $\Gamma^0 \Gamma^N$ (or $\Gamma^{0\dagger} \Gamma^N$), $N \neq 0$. Each of these terms evaluates to

$$P_N = \int dx^9 F_0^{aM} F_{NM}^a. \quad (11)$$

Thus,

$$\{Q, Q^\dagger\} = P_0 + \Gamma^0 \Gamma^N P_N. \quad (12)$$

In preparation for constructing dyon solutions in four dimensions, we constrain the Majorana spinors ζ and λ^a in $D = 10$, also, to be states of positive chirality in $D = 6$, i.e., $\Gamma^{(6)} \chi = (+1) \chi$, ($\chi = \zeta, \lambda^a$) (the chirality matrix in D dimensions is $\Gamma^{(D)} \equiv K \varepsilon_{01 \dots (D-1)} \Gamma^0 \Gamma^1 \dots \Gamma^{D-1}$, where $D = 2k + 2$ and $K = i^{-k}$ for the Minkowski signature and $K = i^{-(k+1)}$ for the Euclidean signature). We next reexpress the spinors χ in terms of projections, i.e.,

$$\chi = \chi_+ + \chi_-, \quad (13a)$$

$$\chi_+ = \chi_{+,+1} + \chi_{+,-1}, \quad (13b)$$

$$\chi_- = \chi_{-,+1} + \chi_{-,-1}, \quad (13c)$$

where

$$\begin{aligned} \chi_{+,\pm 1} &= \left(\frac{1 + \Gamma^{(10)}}{2} \frac{1 + \Gamma^{(6)}}{2} \right) \left(\frac{1 + \Gamma'^{(6)}}{2} \right) \frac{1 \pm \Gamma^0 \Gamma^4}{2} \chi, \\ \chi_{-,\pm 1} &= \left(\frac{1 + \Gamma^{(10)}}{2} \frac{1 + \Gamma^{(6)}}{2} \right) \left(\frac{1 - \Gamma'^{(6)}}{2} \right) \frac{1 \pm \Gamma^0 \Gamma^4}{2} \chi. \end{aligned} \quad (14)$$

Here, $\Gamma'^{(6)}$ is the chirality matrix for dimensions 0, 1, 4, 5, 6, and 7:

$$\Gamma'^{(6)} = \frac{1}{i^2} \varepsilon_{014567} \Gamma^0 \Gamma^1 \Gamma^4 \Gamma^5 \Gamma^6 \Gamma^7. \quad (15)$$

We note, in particular, that the $\zeta_{\pm,\pm 1}$ in the s -basis [9] are

$$\begin{aligned} \zeta_{+,\pm 1} &= a_{+,\pm 1}^* \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &\quad + a_{+,\pm 1} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right], \\ \zeta_{-,\pm 1} &= a_{-,\pm 1}^* \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &\quad + a_{-,\pm 1} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right], \end{aligned} \quad (16)$$

where $a_{+,\pm 1}$ and $a_{-,\pm 1}$ are the arbitrary complex constants.

With foresight, we make the following assumptions:

- (1) All potential functions $A_M^a = A_M^a(x^i)$, i.e., depend only on the three space coordinates and are time independent
- (2) $A_6^a = A_7^a = A_8^a = A_9^a = 0$
- (3) A_4 and A_5 may or may not commute
- (4) A_5^a asymptotically approaches a nonvanishing vacuum state, while A_4^a may vanish asymptotically, i.e.,

$$\lim_{r \rightarrow \infty} A_4^a A_4^a = v^2 \cos^2 \psi, \quad (17a)$$

$$\lim_{r \rightarrow \infty} A_5^a A_5^a = v^2 \sin^2 \psi, \quad (17b)$$

for $0 < \psi \leq \pi/2$ and v nonvanishing

The reduction from ten to six dimensions is trivial. Since A_6^a through A_9^a vanish, only the gamma matrices Γ^0 through Γ^5 appear in the supercharges. In reducing from ten to six dimensions, the ten dimensional gamma matrices may be represented as a direct product of six dimensional gamma matrices and a four dimensional identity matrix, i.e., $\Gamma^N \times I_4$, where $N = 0, \dots, 5$. The gamma matrices act on the first three component spinors of χ , while the four dimensional

identity matrix acts on the remaining two. The only significant consequence of the dimensional reduction is that the spinor χ is replaced by two spinors:

$$\chi_+ = \chi_{+,+} + \chi_{+,-}, \quad (18a)$$

$$\chi_- = \chi_{-,+} + \chi_{-,-}, \quad (18b)$$

and correspondingly the supercharge Q to two supercharges:

$$Q_+ = Q_{+,+} + Q_{+,-}, \quad (19a)$$

$$Q_- = Q_{-,+} + Q_{-,-}. \quad (19b)$$

Thus, dimensional reduction results in a transitioning from $D = 10$, $N = 1$ supersymmetry to $D = 6$, $N = 2$ supersymmetry with both supercharges being eigenstates of positive chirality in six dimensions, i.e., $\Gamma^{(6)} Q_{\pm} = +Q_{\pm}$. The central charges are derived from two groups of terms in the anticommutator, the first group and the third group. A typical nonvanishing boundary term from the first group is derived from

$$F_{ij}^a F_{kN}^a \Gamma^i \Gamma^j \Gamma^{k\ddagger} \Gamma^{N\ddagger}, \quad (20)$$

where $N = 4, 5$. Boundary terms derived from $F_{ij}^a F_{45}^a$ involve a curl integrated over a surface at infinity. Such terms, which can be expressed as a line integral, vanish asymptotically if F_{4i}^a and F_{5i}^a approach zero faster than $1/r$ as $r \rightarrow \infty$. This is the case for monopole or dyon solutions which asymptotically approach zero as $1/r^2$. The remaining terms can be expressed as a divergence which becomes a surface integral at the boundary. If F_{ij}^a approach zero as $1/r^2$ as $r \rightarrow \infty$ as is the case for monopole and dyon solutions, the surface integral is nonvanishing. Specifically, the contribution from the first group of terms is

$$\Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 g_m \nu \cos \psi + \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^5 g_m \nu \sin \psi, \quad (21)$$

where the magnetic charge g_m is obtained from the relationship

$$g_m \nu \sin \psi = \int_{S_{\infty}} B^{ai} A_5^a dS_i. \quad (22)$$

We have used the fact that the asymptotic behavior of A_4^a is given by equation (17a) and that the magnetic field is given by

$$B^{ak} = \frac{\epsilon^{kij}}{2!} F_{ij}^a. \quad (23)$$

In obtaining equation (22), we have used

$$\Gamma^i \Gamma^j \Gamma^k \Gamma^4 \partial_k (F_{ij}^a A_4^a) = \Gamma^i \Gamma^j \Gamma^k \Gamma^5 \left(F_{ij}^a F_{k4}^a + \frac{\epsilon^{kij}}{2!} \mathcal{D}_k F_{ij}^a A_4^a \right). \quad (24)$$

In equation (24), the second term to the right of the equal sign vanishes by virtue of the equations of motion, specifically that the divergence of the magnetic field vanishes.

The contribution to the central charges from the third group of terms corresponds to the momentum in the x^4 and x^5 directions. The relevant terms from equation (9)

$$\int dx^6 \int dx^3 F_{0K}^a F_{KN}^a \Gamma^0 \Gamma^N \Gamma^{0\ddagger}, \quad (25)$$

where $N = 4, 5$. The portion of the integral over the six dimensional space yields the volume of the six dimensional space which we normalize to one. The remaining part of the integral can be expressed as a divergence which by virtue of equation (17b) yields a nonvanishing surface contribution. Substituting the following expression

$$\Gamma^0 \Gamma^N \partial_k (F_{0K}^a A_N^a) = \Gamma^0 \Gamma^N (F_{0K}^a F_{KN}^a + \mathcal{D}_k F_0^{aK} A_N^a) \quad (26)$$

into equation (25) and using the fact the last term in equation (26) which vanishes by virtue of the equations of motion, i.e., the divergence of the electric field vanishes, we obtain the additional contributions to the central charges:

$$\Gamma^0 \Gamma^4 g_e \nu \cos \psi + \Gamma^0 \Gamma^5 g_e \nu \sin \psi. \quad (27)$$

Here, we have used the fact that the electric charge is obtained:

$$g_e \nu \cos \psi = \int_{S_{\infty}} E^{ai} A_4^a dS_i, \quad (28)$$

where $E^{ai} = E_i^a = F_{i0}^a$. Substituting equation (21) and equation (28) into equation (12), we obtain

$$\{Q_a, Q_b^{\ddagger}\} = \delta_{ab} \{P_0 + \Gamma^0 \Gamma^i P_i + \Gamma^0 \Gamma^4 g_e \nu \cos \psi + \Gamma^0 \Gamma^5 g_e \nu \sin \psi + \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 g_m \nu \cos \psi + \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^5 g_m \nu \sin \psi\}, \quad (29)$$

for $(\mathbf{a}, \mathbf{b} = +, -)$ (we use Fraktur font to denote “+” or “-”). Simplifying the terms involving central charges, we obtain

$$\{Q_a, Q_b^{\ddagger}\} = \delta_{ab} \left\{ P_0 + \Gamma^0 \Gamma^i P_i + g \nu \exp \left(i \Gamma^{(4)} \psi' \Gamma^0 \Gamma^4 \right) \cdot \left(\cos \psi + i \Gamma^{(4)} \sin \psi \Gamma^0 \Gamma^4 \Gamma^{(6)} \right) \Gamma^{(6)} \Gamma^0 \Gamma^4 \right\}. \quad (30)$$

The charge g and the angle ψ' are defined by (because of our choice of metric, i.e., $\eta_{00} = -1$, electromagnetic duality implies $*B^a \rightarrow E^a$ and $*E^a \rightarrow -B^a$)

$$g = \sqrt{g_m^2 + g_e^2}, \quad (31)$$

$$\tan \psi' = \frac{g_m}{g_e}.$$

Since $\Gamma^{(6)}\chi = 1\chi$, we can simplify equation (30).

$$\{Q_a, Q_b^\dagger\} = \delta_{ab}(P_0 + \Gamma^0 \Gamma^i P_i + g\nu \left(\exp \left\{ i\Gamma^{(4)}(\psi' + \psi) \right\} \Gamma^0 \Gamma^4 \right)) \Gamma^0 \Gamma^4. \quad (32)$$

Before reducing from six to four dimensions, we note that

$$Q_{a,\pm 1} = \frac{(1 \pm \Gamma^0 \Gamma^4)}{2} Q_a. \quad (33)$$

In the rest frame of the system, i.e., $P_i = 0$ and $P_0 = M$, M being the rest energy of the system, we can show

$$\begin{aligned} \{Q_{a,+1}, Q_{b,+1}^\dagger\} &= \delta_{ab} \left(M + g\nu \exp \left\{ i\Gamma^{(4)}(\psi' + \psi) \right\} \right), \\ \{Q_{a,-1}, Q_{b,-1}^\dagger\} &= \delta_{ab} \left(M - g\nu \exp \left\{ -i\Gamma^{(4)}(\psi' + \psi) \right\} \right), \\ \{Q_{a,+1}, Q_{b,-1}^\dagger\} &= 0. \end{aligned} \quad (34)$$

Alternatively, we define

$$\begin{aligned} Q_a^1 &= \frac{(1 + \Gamma^4)}{2} Q_{a,+1}, \\ Q_a^2 &= \frac{(1 - \Gamma^4)}{2} Q_{a,-1}. \end{aligned} \quad (35)$$

We can show by direct substitution of equation (35) into equation (34) that

$$\{Q_a^i, Q_b^{j\dagger}\} = \delta_{ab} \left(\delta_{ij} M + \Gamma^4 g\nu \exp \left\{ -i\Gamma^{(4)}(\psi' + \psi) \right\} \right), \quad (36)$$

for $i, j = 1, 2$. The reduction from six to four dimensions is relatively straightforward. In reducing from ten to six to four dimensions, the requisite ten dimensional gamma matrices are represented.

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \times I_2 \times I_4, \\ \Gamma^{(4)} &= \gamma^5 \times I_2 \times I_4, \\ \Gamma^4 &= \gamma^5 \times \sigma_1 \times I_4, \\ \Gamma^5 &= \gamma^5 \times \sigma_2 \times I_4. \end{aligned} \quad (37)$$

Here, γ^μ and γ^5 are the four dimensional gamma matrices, σ_1 and σ_2 are Pauli matrices, and I_2 and I_4 are the identity matrices in two and four dimensions, respectively. Finally, the reduction from six dimensions to four dimensions requires that Γ^4 and $\Gamma^{(4)}$ from equation (37) be substituted into equation (36). In reducing from $D=6$, $N=2$ to $D=4$, $N=4$ supersymmetry, each supercharge Q_a is replaced by two supercharges $Q_{a,\pm 1}$.

The supersymmetry algebra, equation (36), obtained from dimensional reduction of $D=10$, $N=1$ supersymme-

try, differs from that of Witten and Olive [11] which is based on $D=4$, $N=2$ supersymmetry. The most obvious distinction is that there are two sets of supercharges, i.e., ($\alpha = +, -$). In our construction of dyons with spin in Section 3, the second set of supercharges generates spin 1 dyon solutions in addition to spin 1/2 and spin 0 solutions. The other distinction is derived from the fact that the components of the vector potential A_4^a and A_5^a in our analysis replace the components of the Higgs field, in Witten and Olive's analysis. Witten and Olive removes one of these components of the Higgs field by performing a chiral rotation, which would, in a certain sense, be equivalent to setting $\psi = 0$ in our analysis. In our subsequent analysis of dyons with spin, Section 3, we do not eliminate one of A_4^a and A_5^a by a coordinate rotation, analogous to the chiral rotation. The reason is that our analysis is complicated because A_4^a and A_5^a , in general, do not commute. Instead, we are able to set $\psi' = \psi$, which is a direct consequence of the dyon solutions being BPS states.

3. Dyons with Spin

In this section, we review the construction of dyons with spin. One method of incorporating spin is to construct dyon solutions from the $D=6$ supersymmetric extension of the Yang-Mills-Higgs action. This methodology shows, implicitly, the relationship between dyons with spin in $D=4$ and superstrings. We begin the analysis with a discussion of the 't Hooft/Polyakov monopole which is derived from the Yang-Mills-Higgs Lagrangian density. 't Hooft [12] and Polyakov [13] have shown that within the context of the spontaneously broken, the Yang-Mills gauge theory $SO(3)$ magnetic monopole solutions of finite mass must necessarily exist and furthermore possess an internal structure. These solutions, which possess zero spin, are derived from the Yang-Mills-Higgs Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} \mathcal{D}_\mu \Phi^a \mathcal{D}^\mu \Phi^a - V(\Phi^a \Phi^a), \quad (38)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - ig_{D3} f^{abc} [A_\mu^b, A_\nu^c]. \quad (39)$$

The Higgs field Φ^a is scalar transforming according to the adjoint representation of the gauge group, and consequently, its covariant derivative is

$$\mathcal{D}_\mu \Phi^a = \partial_\mu \Phi^a - ig_{D3} f^{abc} [A_\mu^b, \Phi^c]. \quad (40)$$

The quantity g_{D3} is the Yang-Mills coupling constant in four dimensions, and f^{abc} are the structure constants of the gauge group. For our purposes, we assume that the gauge group is $SU(2)$ (or a group which contains $SU(2)$ as a subgroup). In addition, we require that the potential $V(\Phi^a \Phi^a)$ vanishes so that the magnetic monopole solutions are BPS states, which are solvable in closed form [1, 7, 8, 14]. Straightforwardly, one can also show that these solutions

can be modified to be electrically charged as well as magnetically charged. As a consequence of the solutions being BPS states, one can show that the electric and magnetic components of the fields are related to Φ^a .

$$E_i^a = \cos \psi \mathcal{D}_i \Phi^a, \quad (41a)$$

$$B_i^a = \sin \psi \mathcal{D}_i \Phi^a, \quad (41b)$$

where

$$E_i^a = F_{i0}^a, \quad (42a)$$

$$B_i^a = \epsilon_i^{jk} F_{jk}^a. \quad (42b)$$

The electric and magnetic fields are obtained from E_i^a and B_i^a :

$$E_i = E_i^a \frac{\Phi^a}{v}, \quad (43a)$$

$$B_i = B_i^a \frac{\Phi^a}{v}. \quad (43b)$$

Here,

$$v^2 = \lim_{r \rightarrow \infty} \Phi^a \Phi^a. \quad (44)$$

See equation (50) below.

In equations (41a) and (41b), the electric q_e and magnetic q_m charges are

$$q_e = q \cos \psi, \quad (45a)$$

$$q_m = q \sin \psi, \quad (45b)$$

where $q = \sqrt{q_e^2 + q_m^2}$. For these solutions $\psi' = \psi$, see equation (31).

From the perspective of six dimensions, the function Φ^a can be reinterpreted as gauge fields

$$A_4^a = \Phi^a \cos \psi, \quad (46a)$$

$$A_5^a = \Phi^a \sin \psi. \quad (46b)$$

This follows because the Higgs field Φ^a does not depend on the coordinates of dimensions four and five so that under gauge transformations, the components A_4^a and A_5^a transform in the same manner as Φ^a . In six dimensions, the dyon is described in terms of the potential function

$$\begin{aligned} A &= A_\mu dx^\mu + A_4 dx^4 + A_5 dx^5 \\ &= \cos \psi v Q(r) T^r dt + \frac{W(r)}{g_{D3}} \left[T^\theta \sin \theta n_m d\phi - T^\phi d\theta \right] \\ &\quad + \cos \psi v Q(r) T^r dx^4 + \sin \psi v Q(r) T^r dx^5, \end{aligned} \quad (47)$$

where v is vacuum expectation value of Φ^a in the asymptotic

limit of large r (see equation (17a)). The magnetic charge of the dyon is $g_m = 4\pi n_m / g_{D3}$, for n_m an integer, which is the Higgs field winding number. T^r , T^θ , and T^ϕ constitute a representation of the $SU(2)$ algebra. The quantities r , θ , and ϕ are the spherical polar coordinates in three dimensions (in the transformation to spherical polar coordinates, we have chosen the x -axis, rather than the z -axis, to be the azimuthal axis. The motivation for this choice is to provide consistency with our choice of Γ matrices. Specifically, spin states are chosen to be eigenvalues of the spin operator S^x . See equation (69)). The elements T^r , T^θ , and T^ϕ are related to T^a , ($a = 1, 2, 3$):

$$T^r \equiv T \cdot e_r = T^a e_r^a = T^y \sin \theta \cos n_m \phi + T^z \sin \theta \sin n_m \phi + T^x \cos \theta, \quad (48a)$$

$$T^\theta \equiv T \cdot e_\theta = T^a e_\theta^a = T^y \cos \theta \cos n_m \phi + T^z \cos \theta \sin n_m \phi - T^x \sin \theta, \quad (48b)$$

$$T^\phi \equiv T \cdot e_\phi = T^a e_\phi^a = -T^y \sin n_m \phi + T^z \cos n_m \phi, \quad (48c)$$

where the T^a are generators of an $SU(2)$ subalgebra of $SO(32)$ (the gauge group $SO(32)$ is relevant for our discussion of superstrings in Section 4).

$$e_r = e_r^a \hat{e}_{x^a} = \cos \theta \hat{e}_{x^1} + \sin \theta \cos n_m \phi \hat{e}_{x^2} + \sin \theta \sin n_m \phi \hat{e}_{x^3}, \quad (49a)$$

$$e_\theta = e_\theta^a \hat{e}_{x^a} = -\sin \theta \hat{e}_{x^1} + \cos \theta \cos n_m \phi \hat{e}_{x^2} + \cos \theta \sin n_m \phi \hat{e}_{x^3}, \quad (49b)$$

$$e_\phi = e_\phi^a \hat{e}_{x^a} = -\sin n_m \phi \hat{e}_{x^2} + \cos n_m \phi \hat{e}_{x^3}. \quad (49c)$$

Here, the \hat{e}_{x^a} , ($a = 1, 2, 3$) are unit vectors in the x , y , and z directions, respectively.

The Higgs field is

$$\Phi^a T^a = v Q(r) e_r^a T^a = v Q(r) T^r. \quad (50)$$

Using equations (48a), (48b), and (48c), we can express the $\mathcal{D}(\Phi^a T^a)$ in spherical polar coordinates

$$\begin{aligned} \mathcal{D}_r(\Phi^a T^a) &= v Q(r)' T^r, \\ \mathcal{D}_\theta(\Phi^a T^a) &= v [1 - W(r)] Q(r) T^\theta, \\ \mathcal{D}_\phi(\Phi^a T^a) &= v [1 - W(r)] Q(r) n_m \sin \theta T^\phi. \end{aligned} \quad (51)$$

The solutions $W(r)$ and $Q(r)$ are obtained as in reference [14].

$$W(r) = w(u) = 1 - \frac{u}{\sinh u}, \quad (52a)$$

$$Q(r) = q(u) = \coth u - \frac{1}{u}, \quad (52b)$$

where the dimensionless variable u is related to the radial coordinate r :

$$u = \frac{r}{L_{\text{dyon}}}. \quad (53)$$

The quantity L_{dyon} characterizes the size of the dyon, i.e., the region of space in which it exhibits internal structure:

$$L_{\text{dyon}} = \frac{1}{\sin \psi M_{\text{gluon}}}, \quad (54)$$

where the mass of the gluon, resulting from spontaneous symmetry breaking, is

$$M_{\text{gluon}} = g_{D_3} v. \quad (55)$$

In addition, the mass of the dyon is related to the mass of a gluon

$$M_{\text{dyon}} = g v = \frac{g}{g_{D_3}} M_{\text{gluon}}. \quad (56)$$

For our purposes, we also require that solutions be invariant under $SL(2, Z)$ transformations, weak/strong duality, so that we include in the Lagrangian density Witten's θ term [15].

$$\mathcal{L}_\theta = -\frac{\theta g_{D_3}^2}{32\pi^2} F_{\mu\nu}^a * F^{a\mu\nu}. \quad (57)$$

This term contributes only a surface term to the action and therefore does not affect the classical equations of motion. In the monopole sector of the theory, however, the term does have a nontrivial effect in that it shifts the allowed values of the electric charge [7]. The electric charge, q_e , is given as

$$q_e = n_e g_{D_3} - n_m \frac{g_{D_3} \theta}{2\pi}, \quad (58)$$

where n_e is an integer.

The dyon solutions, equation (47), also satisfy the equations of motion derived from the supersymmetric Lagrangian density, equation (1) with a gaugino field set equal to zero. The solutions, equations (46a), (46b), and (47), satisfy the assumptions placed on the $D = 10$ supersymmetric solutions discussed in Section 2, with the additional property that the solutions are also BPS states.

In order to construct dyon solutions with spin, we begin with the $D = 6$, $N = 2$ supersymmetric Yang-Mills theory, obtained from the dimensional reduction of the $D = 10$, $N = 1$ theory, presented in Section 2. The $D = 6$, $N = 2$ theory comprises two supercharges of positive chirality in six dimensions, Q_a , ($a = +, -$). The theory is invariant under supersymmetry transformations generated by supercharges Q_a (the gamma matrices in $D = 10$ are represented as $\Gamma^N \times I_4$, where Γ^N are six dimensional gamma matrices. See Sec-

tion 2).

$$\delta A_0^a = \sum_a \delta A_{a0}^a = -i\bar{\zeta}_a \Gamma_0 \lambda_a^a = -i\zeta_a^\dagger \lambda_a^a, \quad (59a)$$

$$\delta A_i^a = \sum_a \delta A_{ai}^a = -i\bar{\zeta}_a \Gamma^i \lambda_a^a = -i\zeta_a^\dagger \Gamma^0 \Gamma_i \lambda_a^a, \quad (59b)$$

$$\delta A_4^a = \sum_a \delta A_{a4}^a = -i\bar{\zeta}_a \Gamma_4 \lambda_a^a = -i\zeta_a^\dagger \Gamma^0 \Gamma^4 \lambda_a^a, \quad (59c)$$

$$\delta A_5^a = \sum_a \delta A_{a5}^a = -i\bar{\zeta}_a \Gamma_5 \lambda_a^a = -\zeta_a^\dagger \Gamma^{(6)} \Gamma^0 \Gamma^4 \Gamma^{(4)} \lambda_a^a, \quad (59d)$$

$$\begin{aligned} \delta \lambda^a &= \sum_a \delta \lambda_a^a = \sum_a \frac{1}{2} F_{MN}^a \Gamma^{MN} \zeta_a \\ &= \sum_a (E^a \Gamma^0 - \Gamma^i F_{i5}^a \Gamma^0 (\Gamma^0 \Gamma^5) \\ &\quad - iB^a \Gamma^0 \Gamma^{(4)} - i\Gamma^i F_{i4}^a \Gamma^0 \Gamma^{(4)} \Gamma^{(6)} (\Gamma^0 \Gamma^5)) \zeta_a. \end{aligned} \quad (59e)$$

Supersymmetry is broken by a part of ζ_a which is an eigenstate of $\Gamma^0 \Gamma^4$ with eigenvalue -1, i.e., $\zeta_{a,-1}$. Substituting equations (41a) and (41b) and equations (42a) and (42b) into equation (59a) and equation (59b), we obtain

$$\delta A_{aM}^a = 0, \quad (60a)$$

$$\delta \lambda_{a,-1}^a = 2(E^a \Gamma^0 - iB^a \Gamma^0 \Gamma^{(4)}) \zeta_{a,-1}, \quad (60b)$$

$$\delta \lambda_{a,+1}^a = 0. \quad (60c)$$

As a characteristic of BPS states, half of the supersymmetries are broken, i.e., for $\zeta_{a,-1}$, and half are unbroken, i.e., for $\zeta_{a,+1}$. The dimensional reduction to $D = 4$ is trivial. The six dimensional gamma matrices are replaced by those given in equation (37). It is notable that in our analysis, there are two broken supercharges, a result which differs from those of others. See Harvey, for example, [7]. The difference is a consequence of the fact these other analyses begin with the $D = 4$, $N = 2$ supersymmetric Yang-Mills-Higgs theory. In contrast, we begin with the $D = 10$, $N = 1$ supersymmetric Yang-Mills theory with only gauge fields, and through dimensional reduction, we obtain a second supercharge. For these dyon solutions, the gaugino field has been explicitly set to zero. The broken supersymmetry transformations, which are generated by the two supercharges, each result in a nonvanishing contribution to the fermion (gaugino) field. Furthermore, these transformations which break supersymmetry do not change the energy of the system so that these nonvanishing fermionic "zero" modes can be considered as deformations of the dyon background which keep the energy of the dyon fixed [7]. Since each of these fermionic modes carries spin 1/2, it is possible to construct dyon states, i.e., deformed dyon backgrounds, with either spin 1/2 or spin 1.

To first order the supersymmetry transformation, equation (59a) leaves the potential function, A_M^a , unchanged. In reference [8], Kastor and Na have shown that, because of the nonlinearity inherent in the supersymmetry transformations, there are nonvanishing contributions to A_M^a when

higher order corrections to the supersymmetry transformations are taken into account. Their methodology utilizes an iterative procedure to calculate higher order corrections to the supersymmetry transformations. They perform their analysis using magnetic monopole solutions, i.e., dyons with vanishing electric charge or $\psi = \pi/2$. Since the changes resulting from the inclusion of electric charge are not immediately obvious, we review their methodology when electric charge is included in the analysis.

They begin with an iterative expansion of the supersymmetry transformations

$$\Psi = \exp(\delta)\bar{\Psi} = \bar{\Psi} + \delta\bar{\Psi} + \frac{1}{2!}\delta^2\bar{\Psi} + \frac{1}{3!}\delta^3\bar{\Psi} + \frac{1}{4!}\delta^4\bar{\Psi}, \quad (61)$$

where Ψ represents both bosonic and fermionic fields after the transformation and $\bar{\Psi}$ the bosonic fields before the transformation. This series can be interpreted as follows: the second term to the right of the second equal sign is obtained directly from equations (60a), (60b), and (60c). The third term is obtained by substituting the second term into equations (60a), (60b), and (60c). The series terminates after the fourth term because of the Grassman nature of $\zeta_{a,-1}$. Substituting equation (60a) and equation (60b) in equation (61), we obtain

$$\delta^2 A_0^a = \sum_a \delta^2 A_{a0}^a = 2\zeta_{a,-1}^\dagger \Gamma^0 \Gamma^{(4)} \Gamma^j \zeta_{a,-1} B_j^a, \quad (62a)$$

$$\delta^2 A_i^a = \sum_a \delta^2 A_{ai}^a = -i2\zeta_{a,-1}^\dagger \Gamma_i \Gamma^j \zeta_{a,-1} E_j^a, \quad (62b)$$

$$\delta^2 A_4^a = \sum_a \delta^2 A_{a4}^a = 2\zeta_{a,-1}^\dagger \Gamma^0 \Gamma^{(4)} \Gamma^j \zeta_{a,-1} B_j^a, \quad (62c)$$

$$\delta^2 A_5^a = \sum_a \delta^2 A_{a5}^a = 2\zeta_{a,-1}^\dagger \Gamma^0 \Gamma^{(4)} \Gamma^j \zeta_{a,-1} E_j^a, \quad (62d)$$

$$\delta\lambda^a = \sum_a \delta\lambda_{a,-1}^a = \sum_a 2\left(E^a \Gamma^0 - iB^a \Gamma^0 \Gamma^{(4)}\right) \zeta_{a,-1}. \quad (62e)$$

Following Kastor and Na [8], we evaluate the matrix elements in equations (62a), (62b), (62c), (62d), and (62e). We first quantize the fermionic zero modes. This involves replacing the complex constants, $a_{-,\pm 1}^*$ in $\zeta_{-,\pm 1}$, equation (16), by the operators $a_{a,-1}$ and $a_{a,-1}^\dagger$ and then integrating the anticommutator of the fermionic zero modes, equation (62e),

$$\delta_{ab} \int dx^3 dy^3 \{\delta\lambda_{a,-1}^a, \delta\lambda_{b,-1}^{b\dagger}\}. \quad (63)$$

Using equations (8a), (8b), and (16), we obtain

$$\{a_{a,-1}, a_{b,-1}^\dagger\} = \frac{1}{4M} \delta_{ab}, \quad (64a)$$

$$\{a_{a,-1}^\dagger, a_{b,-1}^\dagger\} = 0, \quad (64b)$$

$$\{a_{a,-1}, a_{b,-1}\} = 0, \quad (64c)$$

where we have used the fact that the mass of the dyon is

$$M = \int dx^3 \left(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a \right) = gv. \quad (65)$$

Applying equation (64a), (64b), and (64c) in the evaluation of equation (62a), (62b), (62c), (62d), and (62e), we obtain

$$\delta^2 A_0^a = -2\vec{\mu}_m \cdot \frac{1}{g} \vec{\mathcal{D}}\Phi^a, \quad (66a)$$

$$\delta^2 A^{\leftarrow a} = 2\vec{\mu}_e \times \frac{1}{g} \vec{\mathcal{D}}\Phi^a, \quad (66b)$$

$$\delta^2 A_4^a = 2\vec{\mu}_m \cdot \frac{1}{g} \vec{\mathcal{D}}\Phi^a, \quad (66c)$$

$$\delta^2 A_5^a = 2\vec{\mu}_e \cdot \frac{1}{g} \vec{\mathcal{D}}\Phi^a, \quad (66d)$$

$$\delta\lambda_{-1}^a = \sum_a 2 \mathcal{D}\Phi \Gamma^0 \left[\cos \psi + \sin \psi \left(-i\Gamma^{(4)} \right) \right] \zeta_{a,-1}. \quad (66e)$$

Here, the electric dipole moment, due to the spinning magnetic charge, is

$$\vec{\mu}_m \equiv \frac{q_m}{2M_{\text{dyon}}} \zeta_{a,-1}^\dagger \vec{S} \zeta_{a,-1}, \quad (67)$$

and the magnetic dipole moment, due spinning electric charge, is

$$\vec{\mu}_e \equiv \frac{q_e}{2M_{\text{dyon}}} \zeta_{a,-1}^\dagger \vec{S} \zeta_{a,-1}. \quad (68)$$

The spin operator is defined in terms of the Lorentz generators of the rotation group, i.e.,

$$S^l \equiv \epsilon_{jk}^l \left(-\frac{i}{4} \right) \left[\Gamma^j, \Gamma^k \right]. \quad (69)$$

Because the supersymmetric spinors, $\zeta_{a,-1}$, are eigenstates of S^x (with eigenvalue 1/2), then

$$\zeta_{a,-1}^\dagger \vec{S} \zeta_{a,-1} = \zeta_{a,-1}^\dagger S^x \zeta_{a,-1} \hat{e}_x = \frac{1}{2} \hat{e}_x. \quad (70)$$

The complex constants, $a_{-,\pm 1}$ in $\zeta_{-,\pm 1}$, equation (16), are arbitrary, and consequently, different sets of dyon solutions are obtained when quantizing the fermionic modes. Specifically, choosing both $a_{-,+1} = 0$ and $a_{-,-1} = 0$ results in spin 0 dyon solutions. Choosing either $a_{-,+1} = 0$ or $a_{-,-1} = 0$ yields two sets of spin 1/2 dyons with $S^x = +1/2$. Alternatively, interchanging $a_{-,\pm 1}^*$ with $a_{-,\pm 1}$ yields dyon solutions with $S^x = -1/2$. Setting both constants not equal to zero, simultaneously, we obtain spin 1 dyon solutions where $S^x = \pm 1, 0$. Considering all of these dyon solutions in total, we can evaluate $\vec{\mu}_m$ and $\vec{\mu}_e$, explicitly, where for the spin 0 dyon, $S^x = 0$,

for the two spin 1/2 dyons $S^x = \pm 1/2$, and for the spin 1 dyon $S^x = \pm 1, 0$.

The potential functions $\delta^2 A_0^a$ and $\delta^2 \overleftarrow{A}^a$ are amenable to straightforward interpretation. Given that

$$\lim_{r \rightarrow \infty} \frac{1}{g} \overrightarrow{\mathcal{D}}\Phi^a = \frac{1}{r^2} \frac{e_r^a}{n_m}, \quad (71)$$

then $\delta^2 A_{a0}^a$ and $\delta^2 A_{ai}^a$ in the limit of large r approach the classical electric and magnetic dipole potentials. The factor of 2 preceding each dipole moment is the gyromagnetic (“gyroelectric”) ratio (Kastor and Na have previously obtained the gyroelectric ratio in their analysis of magnetic monopoles within the $N = 2$ super Yang-Mills theory [8]).

It is apparent that the electric dipole field derived from the potential $\delta^2 A_0^a$ is equal but opposite to the field derived from the potential $\delta^2 A_4^a$. Not as obvious is the fact that the magnetic dipole field derived from the potential $\delta^2 \overleftarrow{A}^a$ is also equal but opposite to that derived from the potential $\delta^2 A_5^a$. This relationship follows directly from the fact that $\mathcal{D}^i \mathcal{D}_i \Phi^a = 0$. A similar situation occurs in the Maxwell theory in which the magnetic field derived from the vector, dipole potential is, except for a minus sign, identical in form to the electric field derived from the scalar, dipole potential.

4. Dyons, Type IIB, and Type I $SO(32)$ Superstring Theory

The purpose of this section is to generalize the results of Section 3 to the superstring theory. As we show, the solutions obtained in Section 3 correspond, in superstring theory, to $D3$ -branes, which are embedded in an $M5$ -brane compactified on a type IIB torus [16].

First, the arena for discussing the dyon solutions of Section 3 is the $M5$ -brane. The $M5$ -brane is a $5 + 1$ hypersurface propagating in $D = 1 + 10$ dimensions [17]. The underlying theory is based on a single copy of the $D = 11$ Majorana fermions which in $D = 10$ superstring theory reduces to the two Majorana-Weyl fermions. The defining characteristic of these fermions, ϵ , is that they satisfy a constraint equation, i.e., κ -symmetry:

$$\Gamma^{(6)} \epsilon = \epsilon. \quad (72)$$

This is precisely the constraint placed on the spinors, equation (14), defining the dyon solutions in Section 3. Consequently, from the perspective of $D = 11$, the dyon solutions obtained previously live, in fact, on an $M5$ -brane.

The application of supersymmetry to string theory is fraught with significant, nontrivial technical issues. First, in the case of superstring theory, the bosonic part of the action based on the Lagrangian density (equation (1)) is replaced by the Dp-brane action which is given by the non-Abelian Dirac-Born-Infeld plus Wess-Zumino-like actions (note: the antisymmetric tensor $B_{AB} = 0$, where the only nonvanishing R-R potential is $C_{(1)}$, which is a constant background).

$$S = S_{DBI} + S_{WZ}, \quad (73)$$

where

$$S_{DBI} = -\tau_p \int_{\mathcal{M}_{p+1}} \text{STr} \left\{ e^{-\Phi} \sqrt{-\det(g + 2\pi\alpha' F)} \right\}, \quad (74)$$

$$S_{WZ} = \mu_p \int_{\mathcal{M}_{p+1}} P[C_{(1)}] \wedge \text{STr} \left\{ e^{2\pi\alpha' F} \right\}. \quad (75)$$

Here, τ_p is the physical tension of the Dp-brane, μ_p is its R-R charge, and $g_{\alpha\beta} = P[G_{MN}]$ is the pullback of the background metric G_{MN} . STr indicates a symmetric trace for terms involving products of the generators of the gauge group (see reference [1] and references therein). In equation (74), it is known that after expanding the square root as a power series in F_{AB} , computation of the symmetric trace yields ambiguous results in terms of order F^6 [18, 19].

The fermionic action based on the Lagrangian density (equation (1)) is replaced by the fermionic Dp-brane action:

$$S_F = \frac{\tau_p}{2} \int_{\mathcal{M}_{p+1}} e^{-\Phi} \sqrt{-\det(g + 2\pi\alpha' F)} \bar{\theta} (1 - \Gamma_{Dp}) \left[(\tilde{M}^{-1})^{\alpha\beta} \Gamma_{\beta} \mathcal{D}_{\alpha} - \Delta \right] \theta, \quad (76)$$

where Δ vanishes since spacetime background is flat for the cases we are considering. Here,

$$M_{\alpha\beta} = g_{\alpha\beta} + F_{\alpha\beta}, \quad (77)$$

and $\Gamma_{\alpha} = P[\Gamma_M]$. For type IIB $D(2n+1)$ -branes,

$$\Gamma_{D(2n+1)} = \sum_{q+r=n+1} \frac{A_{2n+1}(q, r)}{B(q, r)}, \quad (78)$$

and for type IIA $D(2n)$ -branes,

$$\Gamma_{D(2n)} = \sum_{q+r=n+1} \frac{A_{2n}(q, r)}{B(q, r)}, \quad (79)$$

where the $\Gamma_{D(2n)}$ for the type IIA theory and the $\Gamma_{D(2n+1)}$ for the IIB theory differ by a factor of -1 in references [17, 20]. The reason is derived from the fact that $\Gamma^{(10)}$, denoted $\Gamma_{(10)}$ in [20], is defined with indices raised, whereas in [17], $\Gamma_{(10)}$ is defined with indices lowered. We adopt the same convention for $\Gamma_{(10)}$, as [20].

$$A_{2n+1}(q, r) = (-1)^{r+1} (i\sigma_2) (\sigma_3)^r \epsilon^{\alpha_1 \dots \alpha_{2q} \beta_1 \dots \beta_{2r}} F_{\alpha_1 \alpha_2} \dots F_{\alpha_{2q-1} \alpha_{2q}} \Gamma_{\beta_1 \dots \beta_{2r}}, \quad (80)$$

$$A_{2n}(q, r) = (-1)^{r+1} \Gamma^{(10)} \epsilon^{\alpha_1 \dots \alpha_{2q} \beta_1 \dots \beta_{2r}} F_{\alpha_1 \alpha_2} \dots F_{\alpha_{2q-1} \alpha_{2q}} \Gamma_{\beta_1 \dots \beta_{2r}}, \quad (81)$$

$$B(q, r) = q!(2r)!2^q \sqrt{-\det(g + 2\pi\alpha' F)}. \quad (82)$$

Since our interest is the type I $SO(32)$, our focus will be the type IIB theory to which the type I $SO(32)$ is related. For the type IIB theory, θ is a 64 component double spinor:

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \quad (83)$$

Each θ_i , ($i = 1, 2$) is the 32 component Majorana-Weyl spinor of positive chirality, i.e., $\Gamma_{(10)}\theta_i = +1\theta_i$. In equation (80), the Pauli matrices act on the spinorial index i in θ_i . For the Abelian gauge theory, the fermionic action is invariant under κ -symmetry which acts on fermions:

$$\delta\bar{\theta} = \bar{\kappa} \left(1 + \Gamma_{D(2n+1)} \right), \quad (84)$$

The action S_F , equation (76), corresponding to the fermionic sector of the theory, strictly speaking, only applies to abelian gauge theories. The extension to non-Abelian gauge theories is plagued with problems similar to those occurring in the bosonic action. Specifically, expansion of the square root in terms of the gauge fields yields products of generators of the algebra whose symmetric trace is known to result in inconsistencies at order F^2 [18]. At first, we ignore these problems and assume that the action S_F applies to the non-Abelian theory, in which case \mathcal{D} corresponds to the gauge covariant derivative of the applicable non-Abelian gauge theory.

We now show that the BPS solutions given in Section 3 are exact solutions of type I $SO(32)$ superstring theory. Since the type I $SO(32)$ theory is derived from the type IIB theory, we, initially, focus on the type IIB theory. In [1], we have shown that the BPS solutions presented in Section 3 are also solutions of the equations of motion derived from the non-Abelian DBI action, equation (73), and are therefore solutions of the type IIB theory with the fermionic degrees of freedom equal to zero. In general, these bosonic solutions are not supersymmetric. In [17], Simón has shown that whether such a set of bosonic solutions preserves supersymmetry is equivalent to determining if there exist supersymmetry transformations ϵ :

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad (85)$$

which preserve the bosonic nature of these solutions, i.e., θ remains zero, and furthermore, the bosonic solutions remain unchanged to first order. To satisfy the condition that $\theta = 0$, the combined κ - and supersymmetry transformations must vanish, i.e.,

$$s\theta = \delta_\kappa\theta + \epsilon = 0. \quad (86)$$

Here, the κ -symmetry transformation is

$$\delta_\kappa\theta = \left(1 + \Gamma_{D(2n+1)} \right) \kappa, \quad (87)$$

Simón has shown that this condition is satisfied when

$$\Gamma_{D(2n+1)}\epsilon = \epsilon. \quad (88)$$

Simón has solved equation (88) for a supersymmetric $D3$ -brane configuration, i.e., $n = 1$, with the Abelian gauge field residing on the brane. We now show how the solutions obtained by Simón can be straightforwardly extended to the BPS solutions with non-Abelian gauge fields, given in Section 3.

For $n = 1$, equation (78) becomes

$$\Gamma_{D(3)} = \text{STr} \frac{1}{4! \sqrt{-\det(g + 2\pi\alpha' F)}} e^{\alpha_0 \dots \alpha_3} \cdot \left(\Gamma_{\alpha_0 \dots \alpha_3} i\sigma_2 + 6F_{\alpha_0 \alpha_1}^a T^a \Gamma_{\alpha_2 \alpha_3} \sigma_1 + 3F_{\alpha_0 \alpha_1}^a T^a F_{\alpha_2 \alpha_3}^b T^b i\sigma_2 \right). \quad (89)$$

Substituting equation (89) into equation (88) and rearranging terms, we obtain

$$\begin{aligned} \text{STr} \sqrt{-\det(g + F)} I_2 \epsilon = & \text{STr} \left[1 + \Gamma^i \Gamma_0 \mathcal{D}_i \Phi^a T^a (\cos \psi \sigma_3 + \sin \psi \sigma_1) \right. \\ & - \Gamma^i \Gamma_0 \sigma_3 E_i^a T^a + \Gamma^i \Gamma^j E_i^a T^a \mathcal{D}_j \Phi^b T^b \\ & \cdot (\cos \psi \sigma_3 + \sin \psi \sigma_1) - \Gamma^i \Gamma_0 B_i^a T^a \sigma_1 \\ & + B^{ai} T^a \mathcal{D}_i \Phi^b T^b \\ & \left. \cdot (\cos \psi \sigma_3 + \sin \psi \sigma_1) B^{ia} T^a E_i^b T^b i\sigma_2 \right] \epsilon, \end{aligned} \quad (90)$$

where I_2 is the identity matrix in two dimensions. In transitioning from equation (89) to equation (90), we have imposed the projection constraints

$$-\Gamma^{(4)}\sigma_2\epsilon = \epsilon, \quad (91)$$

$$\Gamma_0 \Gamma_\phi \epsilon = \epsilon. \quad (92)$$

The matrix Γ_ϕ is defined as

$$\Gamma_\phi \equiv \Gamma_4 \cos \psi + \Gamma_5 \sin \psi. \quad (93)$$

Substituting equations (41a), (41b), and equation (51) into the square root term in equation (90), we obtain (see the appendix for details)

$$\text{Tr} \left\{ \sqrt{\left[1 + \sin^2 \psi \left(Z_r^2 (T^r)^2 + Z_\theta^2 (T^\theta)^2 + Z_\phi^2 (T^\phi)^2 \right) \right]^2} \right\} I_2 \epsilon, \quad (94)$$

where

$$\begin{aligned} Z_r &= \mathcal{D}_r \Phi^r, \\ Z_\theta &= \frac{\mathcal{D}_\theta \Phi^\theta}{r}, \\ Z_\phi &= \frac{\mathcal{D}_\phi \Phi^\phi}{r \sin \theta}. \end{aligned} \quad (95)$$

Making the same substitutions into the terms to the right of the equal sign in equation (90), we obtain

$$Tr \left\{ \left[1 + \sin^2 \psi \left(Z_r^2 (T^r)^2 + Z_\theta^2 (T^\theta)^2 + Z_\phi^2 (T^\phi)^2 \right) \right] \right\} \sigma_1^2 \epsilon. \quad (96)$$

The reduction of the right-hand side of equation (90) proceeds, for the most part, as in [17] without requiring that the symmetric trace condition, with one exception. The second term in the second line of equation (90) requires invoking the symmetric trace condition to vanish. In obtaining both equation (94) and equation (96), we have used the fact

$$\begin{aligned} B^{ia} T^a &= B_i^a T^a = \sin \psi Z_i T^a \delta_i^a, \\ E^{ia} T^a &= E_i^a T^a = \cos \psi Z_i T^a \delta_i^a. \end{aligned} \quad (97)$$

Using Z_i , defined in equation (95), rather than $\mathcal{D}\Phi_i^a \delta_i^a$ in equation (94), is equivalent, geometrically, to transforming from the orthogonal basis vectors of spherical polar coordinates $(\partial_r, \partial_\theta, \partial_\phi)$ to the orthonormal basis vectors, $(\partial_r, \widehat{\theta}, \widehat{\phi})$, i.e., $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = dr^2 + \widehat{\theta}^2 + \widehat{\phi}^2$. The reason for this replacement is to facilitate a comparison of results, presented here, with those presented in [17], where the metric tensor is given in an orthonormal basis.

We, now, solve the constraint equations, equation (91) and equation (92) for ϵ , obtaining (because we have two independent supercharges, $(\mathbf{a} = +, -)$, there are two solutions for ϵ . We have omitted labeling ϵ with an additional subscript \mathbf{a} so that the notation is less cluttered)

$$\epsilon = \begin{pmatrix} \zeta_{\mathbf{a},+1} \\ -i\Gamma^{(4)} \zeta_{\mathbf{a},+1} \end{pmatrix}. \quad (98)$$

Up to a phase, which we take to be zero, we find that $\zeta_{\mathbf{a},-1}$ (equation (16)) is given by

$$\zeta_{\mathbf{a},-1} = \epsilon_2 = -i\Gamma^{(4)} \epsilon_1 = -i\Gamma^{(4)} \zeta_{\mathbf{a},+1}. \quad (99)$$

The relationship between the type IIB theory, $(N = 2, D = 10)$ supersymmetry, discussed here, and relevant type I theory $SO(32)$, $(N = 1, D = 10)$ supersymmetry, can be gleaned from equation (66e) and equation (99) (it is worth noting that there is a supersymmetric version of the non-Abelian Dirac-Born-Infeld action (equation (73)). Its construction is based on a generalization of the principles used here to extend the supersymmetric results of Section 3 to

superstring theory. See, for example, the work of Bergshoeff et al. [21]). We define

$$\epsilon_{\text{Type I}} = \cos \psi \epsilon_2 - \sin \psi \epsilon_1, \quad (100)$$

so that

$$\delta \lambda_{\mathbf{a},-1}^a = 2\mathcal{D}\Phi \Gamma^0 \epsilon_{\text{Type I}}. \quad (101)$$

In summary, we have shown that the dyon solutions obtained in Section 3 satisfy the κ -symmetry constraint, equation (88), and are therefore, also, solutions of type IIB (type I $SO(32)$) superstring theory.

5. T -Duality, Gauge/Gravity Duality, and Wormholes

In this section, we apply T -duality transformations to the superstring solutions derived in Section 4 and study the duality between the supersymmetric string theoretic solutions obtained therein and their gravitational analogue. Specifically, we apply the T -duality transformations to spatial dimensions x^4 and x^5 of the $M5$ -brane, transforming the gauge potential functions, $A_4^a T^a$ and $A_5^a T^a$, into embedding coordinates, $2\pi\alpha' A_4^a T^a$ and $2\pi\alpha' A_5^a T^a$. In order that such transformation be interpreted, straightforwardly, the potentials should not depend on the coordinates x^4 or x^5 and furthermore should also commute. The metric obtained on the two resulting $D3$ -branes is derived by pulling back the metric induced by the embedding coordinates [22], i.e., the metric, $g_{\mu\nu}$, is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{M,N=4}^5 \eta_{MN} \text{STr} \left(\mathcal{D}_\mu A_M^a T^a \mathcal{D}_\nu A_N^b T^b \right). \quad (102)$$

After including the back reaction in the T -duality transformations, we find that the potential functions $A_4^a T^a$ and $A_5^a T^a$, equations (46a) and (46b), in general, do not commute complicating their interpretation as embedding coordinates. Since the noncommutativity is present only for solutions with nonvanishing spin, we organize this section into two subsections: the first dealing with the case of vanishing spin and second dealing with the more complicated case of nonvanishing spin, which includes both spin 1/2 and spin 1 solutions.

5.1. Case 1: Spin 0 Solutions. Before applying T -duality transformations, we perform a coordinate transformation in the x^4, x^5 plane which induces a gauge transformation on A_4^a and A_5^a , thereby eliminating A_4^a . Since, for the spin 0 case (see equation (50)):

$$A_4^a T^a = \Phi^a T^a \cos \psi, \quad (103a)$$

$$A_5^a T^a = \Phi^a T^a \sin \psi. \quad (103b)$$

By rotating the x^4, x^5 coordinate axes through an angle

$(\pi/2 - \psi)$, we transform A_4^a and A_5^a :

$$\begin{aligned} A_4^a T^a &\longrightarrow \sin \psi A_4^a T^a - \cos \psi A_5^a T^a = 0, \\ A_5^a T^a &\longrightarrow \cos \psi A_4^a T^a + \sin \psi A_5^a T^a = \Phi^a T^a = vQ(r) T^r. \end{aligned} \quad (104)$$

We note that this transformation leaves unchanged the components of the Minkowski metric η_{MN} , ($M, N = 4, 5$). After diagonalizing the matrix T^r , we apply a T -duality transformation to the x^5 -coordinate axis. As a consequence, we obtain two $D3$ -branes embedded in a subspace of the $M5$ -brane, where the embedding coordinates of the two $D3$ -branes are $x^5 = \pm L_{D3} Q(r)$ ($L_{D3} = 2\pi\alpha' v = 2\pi\alpha' M_{\text{gluon}}/g_{D3}$). In addition, the value of the electric or magnetic charge associated with one $D3$ -brane is opposite in sign of the corresponding charge on the other $D3$ -brane (see Figure 1).

The geometrical interpretation of L_{D3} is straightforward. It is one half of the separation between the $D3$ -branes in the asymptotic region of space, i.e., $r \rightarrow \infty$ (alternatively, we can transform $A_4^a T^a \rightarrow A_0^a T^a - A_4^a T^a = 0$, which, also, results in the transformation of the Minkowski metric, coincidentally, identical in form to equation (102). Furthermore, the quantity L_{D3} now depends only on the magnetic charge, q_m , and not on the electric charge, q_e). This is a consequence of the fact that $\lim_{r \rightarrow \infty} Q(r) = 1$. As noted previously, L_{dyon} is the characteristic size of the dyon (see equation (54)).

Using equations (102) and (104), we can calculate the metric tensor

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -dt dt + \frac{4\pi^2 \tilde{v}^2 \alpha'^2 g_{D3}^2 \sin^2 \psi ((d/du)q(u))^2 + \alpha'}{\sin^2 \psi g_{D3}^2 \tilde{v}^2} du du \\ &+ \frac{(4\pi^2 \tilde{v}^2 \alpha' g_{D3}^2 q(u)^2 \tilde{w}^2(u) \sin^2 \psi + u^2) \alpha'}{\sin^2 \psi g_{D3}^2 \tilde{v}^2} d\theta d\theta \\ &+ \frac{(4\pi^2 \tilde{v}^2 \alpha' g_{D3}^2 q(u)^2 \tilde{w}^2(u) \sin^2 \psi + u^2) \alpha'}{\sin^2 \psi g_{D3}^2 \tilde{v}^2} \sin^2 \theta d\phi d\phi, \end{aligned} \quad (105)$$

where $\tilde{w}(u) = -1 + w(u)$ and $\tilde{v} = v\sqrt{\alpha'}$, and the functions $q(u)$ and $w(u)$ are defined in equations (52a) and (52b). It is straightforward to calculate the scalar curvature. Details of performing this calculation can be found elsewhere [1]. We omit presenting the scalar curvature here, since it comprises a large number of terms and is not amenable to obvious interpretation. Nonetheless, we can show that the scalar curvature

$$\lim_{r \rightarrow \infty} R \longrightarrow \begin{cases} -\frac{24\pi^2 \sin^4(\psi) g_{D3}^2 \tilde{v}^6 L_{\text{dyon}}^6}{\alpha' r^6}, & \text{if } \psi \neq 0, \\ 0, & \text{if } \psi = 0, \end{cases} \quad (106)$$

so that the geometry of each $D3$ -brane is asymptotically flat. Furthermore, we can also show that the scalar curvature, R , is finite everywhere. In particular, for small values of r , the

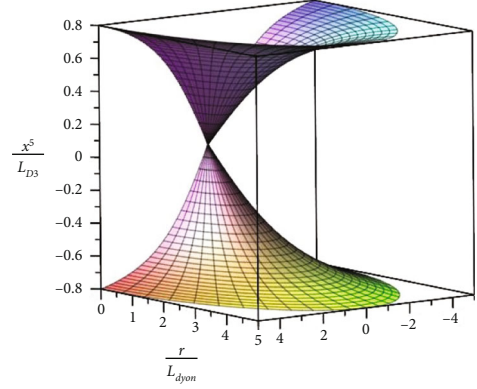


FIGURE 1: Wormhole. Shown is the embedding diagram of the two $D3$ -branes with azimuthal angle suppressed. The radial coordinate, r , has been replaced with the dimensionless coordinate r/L_{dyon} , and the embedding coordinate x^5 has been replaced with the dimensionless coordinate x^5/L_{D3} .

scalar curvature is given by

$$\lim_{r \rightarrow 0} R \longrightarrow \frac{216\pi^2 \sin^4(\psi) g_{D3}^2 \tilde{v}^6}{(4\pi^2 \sin^2(\psi) \tilde{v}^4 + 9)^2 \alpha'} + \left(\frac{r}{L_{\text{dyon}}} \right)^2 \frac{8\tilde{v}^6 g_{D3}^2 \pi^2 \sin^4(\psi) (100\pi^2 \sin^2(\psi)^2 \tilde{v}^4 - 1089)}{5(4\pi^2 \sin^2(\psi)^2 \tilde{v}^4 + 9)^3 \alpha'}. \quad (107)$$

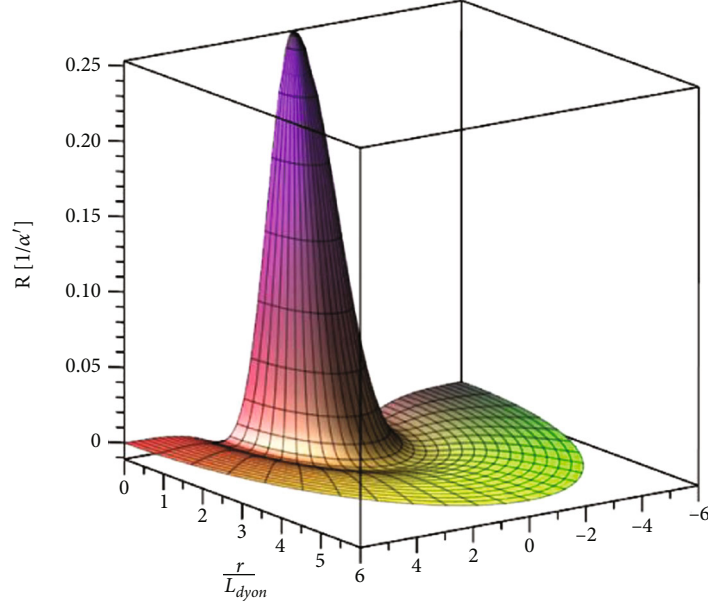
We are constraining the Yang-Mills coupling constant on the $D3$ -branes such that $0 \leq g_{D3}^2 \leq 4\pi$. Solutions when $g_{D3}^2 > 4\pi$ are obtained using weak/strong duality; i.e., the dual theory is obtained by interchanging electric and magnetic charge and letting $g_{D3}^2 \rightarrow (4\pi)/g_{D3}^2$.

In Figure 2, we compare plots of the scalar curvature of a magnetic monopole, with one unit of magnetic charge, to a dyon with one unit of magnetic and one unit of electric charge. The mass of the gluon $M_{\text{gluon}} = M_P/2$ (the Planck mass, $M_P \approx 1/\sqrt{\alpha'}$), and the Yang-Mills coupling constant $g_{D3} = 1$. Note that the maximum scalar curvature of the dyon is less than that of the monopole. In addition, it can be shown that when $\psi \rightarrow 0$ i.e., $q_e \rightarrow \infty$, the two $D3$ -branes merge into a single $D3$ -brane whose scalar curvature $R \rightarrow 0$. Furthermore, it can also be shown, independent of the value of ψ , that as $r \rightarrow \infty$, the metric $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ and that the geometry of each $D3$ -brane is asymptotically flat.

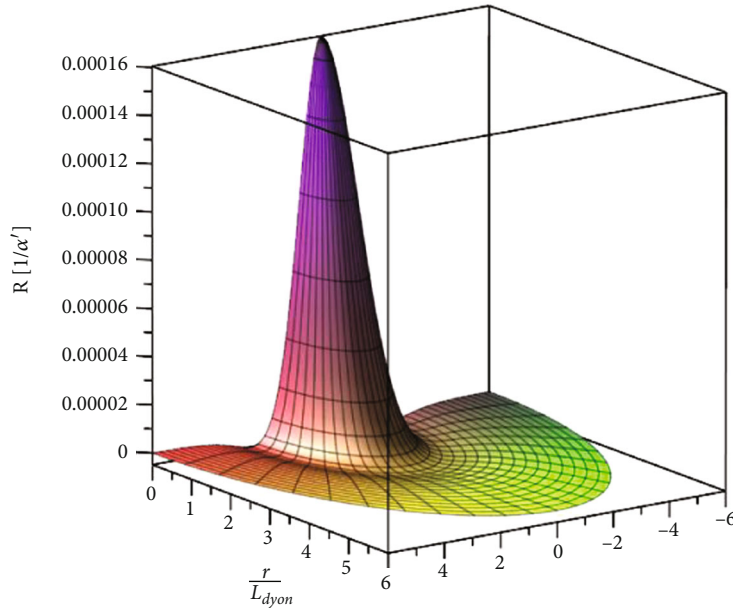
5.2. Case 2: Spin 1/2 and Spin 1 Solutions. For the case of nonvanishing spin, the potential functions $A_4^a T^a$ and $A_5^a T^a$ do not commute, except in the spin 1 case when the x component of the spin, S^x , vanishes. We consider, first, the case when $S^x \neq 0$.

For the case of nonvanishing spin,

$$A_4^a T^a = \Phi^a T^a \cos \psi + 2\vec{\mu}_m \cdot \frac{1}{g} \vec{\mathcal{D}}\Phi^a T^a, \quad (108a)$$



(a) Magnetic monopole: the magnetic charge of the monopole is $q_m = 4\pi$, the electric charge $q_e = 0$, i.e., $n_e = 0$, and the θ -term=0. Its mass is $4\pi M_{\text{gluon}}$



(b) Dyon: the magnetic charge of the dyon is $q_m = 4\pi$, the electric charge $q_e = 1/2$, i.e., $n_e = 0$, and the θ -term= $-\pi$. Its mass is $\sqrt{1/4 + 16\pi^2} M_{\text{gluon}}$

FIGURE 2: Scalar curvature: depicted in (a) and (b) are the scalar curvatures of a magnetic monopole and dyon, each without spin. In each figure, the scalar curvature, R , in units of $1/\alpha'$, is plotted as a function of the dimensionless, spherical coordinates r/L_{dyon} , and θ . The mass of the gluon $M_{\text{gluon}} = (M_p/2)$. The Yang-Mills coupling constant $g_{D3} = 1$.

$$A_5^a T^a = \Phi^a T^a \sin \psi + 2\vec{\mu}_e \cdot \frac{1}{g} \vec{\mathcal{D}}\Phi^a T^a. \quad (108b)$$

See equation (50), equation (66c), and equation (66d). The noncommutativity of $A_4^a T^a$ and $A_5^a T^a$ is derived from the fact that $\Phi^a T^a$ and $\vec{e}_x \cdot \vec{\mathcal{D}}\Phi^a T^a$ do not commute. We resolve the problem of noncommutativity by performing

the following coordinate transformation:

$$x^4 \longrightarrow x^4 - \cos 2\psi x^0 - \sin 2\psi x^5. \quad (109)$$

This induces the following gauge transformation:

$$A_4^a T^a \longrightarrow A_4^a T^a - \cos 2\psi A_0^a T^a - \sin 2\psi A_5^a T^a = 0, \quad (110)$$

thereby eliminating the noncommutativity of $A_4^a T^a$ and $A_5^a T^a$. In addition, the metric tensor η_{MN} is transformed to

$$(\eta_{MN}) \longrightarrow \begin{pmatrix} -\sin(2\psi)^2 & \cos(2\psi) & \sin(2\psi)\cos(2\psi) \\ \cos(2\psi) & 1 & \sin(2\psi) \\ \sin(2\psi)\cos(2\psi) & \sin(2\psi) & -\cos(2\psi)^2 + 2 \end{pmatrix}, \quad (111)$$

for components $M, N = 0, 4, 5$. The remaining η_{MN} are unchanged.

When $S^x \neq 0$, we limit our consideration to solutions where the electric charge, $q_e = 0$, i.e., magnetic monopole solutions. Our reason for this limitation is that some calculations, including the curvature tensor, are calculatingly challenging and furthermore comprise such a large number of terms that they are not straightforward to interpret. For the magnetic monopole solutions, $\psi = \pi/2$ so that the metric simplifies to

$$(\eta_{MN}) \longrightarrow \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (112)$$

Using equation (102), equation (112), and equation (108b), we can calculate the metric tensor, $g_{\mu\nu}$. For these solutions,

$$g_{00} = \frac{A}{B}. \quad (113)$$

Here,

$$A = 4 \left(-12\pi^2 \tilde{v}^2 \alpha' g_{D_3}^2 \tilde{w}(u)^2 q(u)^2 + u^2 \right) \times (S^x)^2 q(u)^4 \alpha' 2\pi^2 \tilde{w}(u)^2 g_{D_3}^6 \sin^2 \theta, \quad (114a)$$

$$B = u^2 \left(4\pi^2 \tilde{v}^2 \alpha' g_{D_3}^2 \tilde{w}(u)^2 q(u)^2 + u^2 \right). \quad (114b)$$

The spatial components of the metric, g_{ij} , are the same as those of the spin 0 case, equation (105), with an important difference in the ϕ component of the metric. Because of the nonvanishing component of the spin S^x , reference frames retrogress about the x -axis so that the ϕ coordinate is transformed:

$$\phi \longrightarrow \phi - \Omega t. \quad (115)$$

The angular speed, Ω , is

$$\Omega = - \frac{8\pi^2 \alpha' q(u)^3 \tilde{w}^2(u) \{ g_{D_3}^5 \tilde{v}^2 S^x \}}{4\pi^2 \alpha' q(u)^2 \tilde{w}^2(u) g_{D_3}^2 \tilde{v}^2 + u^2}. \quad (116)$$

Relative to spatial infinity, inertial frames are dragged

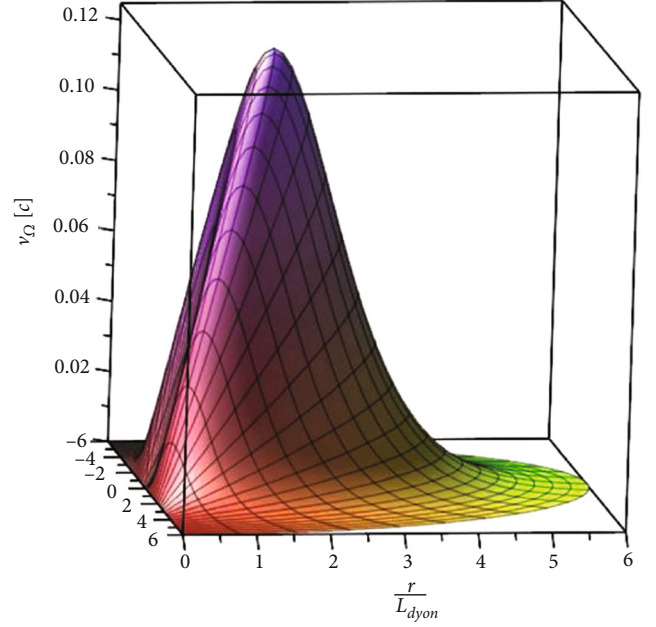


FIGURE 3: Frame dragging. Shown in the plot is the speed, with which reference frames are dragged around the wormhole relative to a stationary reference frame in a region of space far from the wormhole, i.e., $r \rightarrow \infty$. The speed, v_{Ω} , is given as a multiple of the speed of light.

with speed

$$v_{\Omega} = L_{\text{dyon}} u \sin \theta |\Omega|. \quad (117)$$

We note that these solutions are, strictly speaking, only accurate to $\mathcal{O}(\alpha')$ or, equivalently, accurate for values of the gluon mass, M_{gluon} , less than the Planck mass, M_P . In fact, we can show that whenever, $M_{\text{gluon}} \ll M_P$, the speed with which inertial frames are dragged is less than the speed of light, thus avoiding the possibility of closed time-like curves. In Figure 3, we show a plot of the spatial dependence of v_{Ω} for $M_{\text{gluon}} = (1/2)M_P$.

The scalar curvature in the case of nonvanishing spin is markedly different from that of vanishing spin. Regarding its general features, we can show that the scalar curvature, $R \propto 1/\rho^2$, is the cylindrical polar coordinate, $\rho \rightarrow 0$. Here, $\rho \equiv \sqrt{y^2 + z^2}$. In particular, as $r \rightarrow 0$, i.e., $\rho, x \rightarrow 0$,

$$\lim_{\rho, x \rightarrow 0} R \longrightarrow - \left(\frac{L_{\text{dyon}}}{\rho} \right)^2 \frac{18 g_{D_3}^2 \tilde{v}^2}{(4\pi^2 \tilde{v}^4 + 9) \alpha'}, \quad (118)$$

where $\tilde{v} = v\sqrt{\alpha'}$. Furthermore, we can also show that

$$\lim_{x \rightarrow \infty} R \longrightarrow - \frac{2(\rho^2 + L_{\text{dyon}}^2) g_{D_3}^2 \tilde{v}^2}{\rho^2 \alpha'}. \quad (119)$$

In Figure 4(a), we show a plot of the scalar curvature when $M_{\text{gluon}} = (1/2)M_P$ and $g_{D_3} = 1$. In Figure 4(b), we show

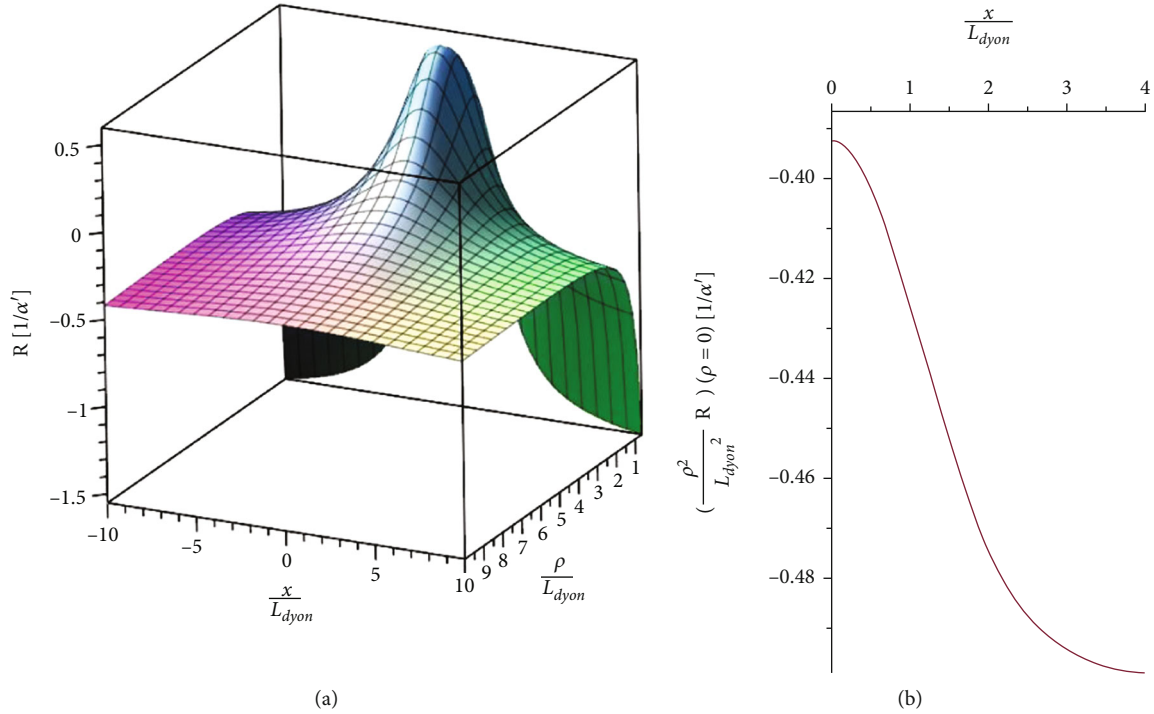


FIGURE 4: Rotating magnetic monopole: in (a), the scalar curvature, R , in units of $1/\alpha'$, is plotted as a function of the dimensionless, cylindrical coordinates ρ/L_{dyon} and x/L_{dyon} . In (b), the scalar curvature, which has been rescaled to remove its divergent behavior, is plotted as a function of x/L_{dyon} at $\rho=0$. The charge of the monopole is $q_m = 4\pi$, i.e., the electric charge $q_e = 0$ and the θ -term = 0. Its mass is $4\pi M_{\text{gluon}}$. The Yang-Mills coupling constant $g_{D_3} = 1$.

a plot of $\rho^2 R$, evaluated at $\rho = 0$. The purpose of scaling by ρ^2 is to remove the divergent part of the scalar curvature.

Surprisingly, the scalar curvature is independent of S^x . The reason is that the R_{00} component of the Ricci tensor is the only component which depends on S^x . Specifically, $R_{00} \propto (S^x)^2$. In addition, the g_{00} component of the metric tensor is the only component which depends on S^x , i.e., $g_{00} \propto (S^x)^2$. Thus, after contracting the metric tensor with the Ricci tensor, the scalar curvature is independent of S^x . As a result, after contracting the metric tensor with the Ricci tensor, the scalar curvature is independent of S^x . This is consequential for the spin one monopole solutions when $S^x = 0$. For spin one, when $S^x = 0$, equations (108a) and (108b) reduce to equations (103a) and (103b), which would seem to indicate that the scalar curvature is the same as for the spin 0 case. Alternatively, as well as preferably, we can obtain the case $S^x = 0$ as the limit $S^x \rightarrow 0$ for the case of nonvanishing S^x . Taking the limit, $S^x \rightarrow 0$, we find that $g_{00} \rightarrow 0$, $R_{00} \rightarrow 0$, while all other components of the metric tensor, the Ricci tensor, and scalar curvature remain unchanged. This analysis demonstrates that the case of a spin one, magnetic monopole with $S^x = 0$, is inherently different from the that of a magnetic monopole with vanishing spin.

It is interesting to contrast the geometries of vanishing spin with nonvanishing spin solutions in the asymptotic region of space. For the spin zero monopole, as $r \rightarrow \infty$, the metric approaches the Minkowski metric, and the scalar curvature $R \rightarrow 0$. For the nonvanishing spin monopole, the

metric also approaches the Minkowski metric, transformed using “light cone” coordinates; however in contrast, as $\rho \rightarrow \infty$, the scalar curvature

$$R \rightarrow -\frac{2g_{D_3}^2 \tilde{v}^2}{\alpha'}. \quad (120)$$

6. Conclusions

In this study, we have investigated the superstring analogue of the ‘t Hooft/Polyakov monopole. We have conducted this study in several steps. First, because superstring theory naturally resides in ten dimensions, we have reviewed the dimensional reduction of $D = 10$, $N = 1$ supersymmetry in a way that specifically applies to this study. The theory underlying the ‘t Hooft/Polyakov monopole is based on a real-valued scalar boson, which undergoes spontaneous symmetry breaking. In this study, we assume this boson to be complex-valued so that the monopole (dyon) possesses both magnetic and electric charge. We, next, recast the scalar dyon theory as a supersymmetric gauge theory in six dimensions. The complex scalar field is replaced by two real fields which correspond to components of the gauge field in the two extra dimensions. Applying supersymmetry transformations to the gauge fields, we obtain a theory comprising of dyons with spin zero, one half, or one. In addition to possessing both magnetic and electric charge, the dyon possesses both electric and magnetic dipole moments. We show that

both the gyromagnetic and gyroelectric ratios of each are exactly two, as would be expected [23] (the gyroelectric ratio has been reported, previously, by Kastor and Na). Next, we reinterpret the supersymmetric dyon solutions as solutions in $N=2$ type IIB superstring theory. For $N=2$ type IIB superstring theory, the number of on-shell bosonic and fermionic degrees of freedom, in general, is unequal, the number of fermionic degrees is sixteen, and the number of bosonic degrees is eight. Supersymmetry requires that the degrees of freedom of each be equal. To prove that the supersymmetric dyon solutions are also supersymmetric solutions in $N=2$ type IIB string theory, we show that the solutions satisfy κ -symmetry constraint equations, which, when satisfied, remove half of the fermionic degrees of freedom. We then recast the solutions in the type IIB theory as solutions in the type I $SO(32)$ theory. We, next, perform a T -duality transformation on the two components of the gauge field in the two extra dimensions. The T -duality transformation is complicated by the fact that the two gauge fields do not commute, a complication we resolve by eliminating one of the two components of the gauge field by a judiciously chosen coordinate/gauge transformation. The transformed solution is a rotating wormhole joining two $D3$ -branes. The electric or magnetic charge of the dyon associated with each $D3$ -brane is opposite in sign to that of the other $D3$ -brane. Finally, we analyze the geometry of the $D3$ -branes for two cases: one corresponding to a dyon with vanishing spin and the other corresponding to a magnetic monopole with nonvanishing spin. For the case of vanishing spin, we calculate the metric tensor and scalar curvature. We find that the scalar curvature is finite, everywhere (we have obtained comparable results in a previous study [1]). In particular, the scalar curvature vanishes, asymptotically far from the throat of the wormhole. For the case of nonvanishing spin, we similarly calculate the metric tensor and find that the spin of the magnetic monopole causes frame dragging, equation (117). We then calculate the scalar curvature. Unlike the case of vanishing spin, the scalar curvature diverges along the spin quantization axis. Specifically, it diverges as $1/\rho^2$, ρ being the radial, cylindrical coordinate. Also, in contrast to the case of vanishing spin, we find that as $\rho \rightarrow \infty$, the scalar curvature, on the boundary, approaches a constant negative value, equation (120).

In summary, we note that the wormhole solutions, obtained in this study, provide an example of a gauge, gravity duality. Furthermore, because they correspond to BPS states and are based on supersymmetry where quantum corrections are expected to be well controlled, we expect such quantum corrections not to modify these solutions in a significant way. Consequently, the underlying theoretical principles may provide some insight in formulating a theory of quantum gravity.

Appendix

In the appendix, we present a heuristic derivation of equation (94). First, we can show by direct calculation that if T^r , T^θ , and T^ϕ commute, then the left side of equation (90) evaluates to

$$Tr \left\{ \sqrt{1 + \sin^2 \psi \left(Z_r^2 (T^r)^2 + Z_\theta^2 (T^\theta)^2 + Z_\phi^2 (T^\phi)^2 \right)} \right\} I_2 \epsilon. \quad (A.1)$$

On the other hand, if T^r , T^θ , and T^ϕ do not commute, there are additional terms. For example, one such term is proportional to $\sin^6 \psi$:

$$\begin{aligned} & -Z_r^2 Z_\theta^2 Z_\phi^2 \left(T^{\phi 2} T^\theta T^r T^r T^\theta - T^\theta T^\phi T^r T^\theta T^\phi T^r - T^\phi T^\theta T^\theta T^r T^r T^\phi \right. \\ & \left. + T^{\theta 2} T^r T^\phi T^\phi T^r + T^{r 2} T^\theta T^\phi T^\phi T^\theta - T^{r 2} T^{\theta 2} T^{\phi 2} \right) \sin^6 \psi. \end{aligned} \quad (A.2)$$

After expanding the square root, we obtain products of such terms. In applying the symmetric trace condition to expressions like equation (A.2), we first symmetrize each of the terms containing T^r , T^θ , and T^ϕ with respect to their superscript index. This effectively makes such terms commute so that the terms vanish. Consequently, the expression for the square root reduces to equation (94).

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

- [1] E. A. Olszewski, "Dyons, superstrings, and wormholes: exact solutions of the non-Abelian Dirac-Born-Infeld action," *Advances in High Energy Physics*, vol. 2015, Article ID 960345, 14 pages, 2015.
- [2] K. Jusufi, A. Övgün, A. Banerjee, and I. Sakalli, "Gravitational lensing by wormholes supported by electromagnetic, scalar, and quantum effects," *The European Physical Journal Plus*, vol. 134, no. 9, 2019.
- [3] S. H. Mazharimousavi, M. Halilsoy, and Z. Amirabi, "Higher-dimensional thin-shell wormholes in Einstein-Yang-Mills-Gauss-Bonnet gravity," *Classical and Quantum Gravity*, vol. 28, no. 2, article 025004, 2010.
- [4] S. Habib Mazharimousavi, M. Halilsoy, and Z. Amirabi, "d-Dimensional nonasymptotically flat thin-shell wormholes in Einstein-Yang-Mills-dilaton gravity," *Physics Letters A*, vol. 375, no. 3, pp. 231-236, 2011.
- [5] A. Övgün, K. Jusufi, and I. Sakalli, "Exact traversable wormhole solution in bumblebee gravity," *Physical Review D*, vol. 99, no. 2, 2019.
- [6] I. Sakalli and A. Ovgun, "Tunnelling of vector particles from Lorentzian wormholes in 3+1 dimensions," *The European Physical Journal Plus*, vol. 130, no. 6, 2015.
- [7] J. A. Harvey, "Magnetic monopoles, duality, and supersymmetry," <https://arxiv.org/abs/hep-th/9603086>.
- [8] D. Kastor and E. Na, "Electric dipole moment of a BPS monopole," *Physics Review D*, vol. 60, no. 2, 1999.

- [9] J. Polchinski, *String Theory Volume II*, Cambridge University Press, New York, NY, USA, 1998.
- [10] S. P. Martin, “A supersymmetry primer,” <https://arxiv.org/abs/hep-ph/9709356>.
- [11] E. Witten and D. Olive, “Supersymmetry algebras that include topological charges,” *Physics Letters B*, vol. 78, no. 1, pp. 97–101, 1978.
- [12] G. t Hooft, “Magnetic monopoles in unified gauge theories,” *Nuclear Physics B*, vol. 79, pp. 276–284, 1974.
- [13] A. M. Polyakov, “Particle spectrum in quantum field theory,” *JETP Letters*, vol. 20, no. 6, pp. 194–195, 1974.
- [14] E. A. Olszewski, “Dyons and magnetic monopoles revisited,” *Particle Physics Insights*, vol. 5, pp. 1–12, 2012.
- [15] E. Witten, “Dyons of charge $e\theta/2\pi$,” *Physics Letters B*, vol. 86, pp. 283–287, 1979.
- [16] G. Shiber, “The d3-brane, self-duality, the super-Hamiltonian action and the M5-brane,” 2018, <https://www.georgeshiber.com/the-d3-braneself-duality-the-super-hamiltonian-action-and-them5-brane>.
- [17] J. Simón, “Brane effective actions, kappa-symmetry and applications,” *Living Reviews in Relativity*, vol. 15, no. 1, 2012.
- [18] E. A. Bergshoeff, M. de Roo, and A. Sevrin, “Non-Abelian Born-Infeld and kappa-symmetry,” *Journal of Mathematical Physics*, vol. 42, no. 7, pp. 2872–2888, 2001.
- [19] A. A. Tseytlin, “On the non-Abelian generalization of Born-Infeld action in string theory,” *Nuclear Physics B*, vol. 501, no. 1, pp. 41–52, 1997.
- [20] L. Martucci, J. Rosseel, D. B. den Bleeken, and A. Van Proeyen, “Dirac actions for D-branes on backgrounds with fluxes,” *Classical and Quantum Gravity*, vol. 22, no. 13, pp. 2745–2763, 2005.
- [21] E. Bergshoeff, M. Roo, and A. Sevrin, “On the supersymmetric non-Abelian Born-Infeld action,” *Nuclear Physics B - Proceedings Supplements*, vol. 49, pp. 50–55, 2000.
- [22] C. V. Johnson, *D-Branes*, Cambridge University Press, New York, NY, USA, 2003.
- [23] I. Giannakis and J. T. Liu, “N=2 supersymmetry and dipole moments,” <https://arxiv.org/abs/hep-th/9711173>.