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Research Article

Vector Solutions for Linearly Coupled Choquard Type Equations with Lower Critical Exponents

Huiling Wu

College of Mathematics and Data Science, Minjiang University, Fuzhou, Fujian 350108, China

Correspondence should be addressed to Huiling Wu; huilingwu@mju.edu.cn

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The existence, nonexistence, and multiplicity of vector solutions of the linearly coupled Choquard type equations $\begin{cases} -\Delta u + V_1(x)u = (I_\alpha * |u|^{(N+\alpha)/N})|u|^{\alpha/N-1}u + \lambda v, x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = (I_\alpha * |v|^{(N+\alpha)/N})|v|^{\alpha/N-1}v + \lambda u, x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases}$ are proved, where $\alpha \in (0, N)$, $N \geq 3$, $V_1(x)V_2(x) \in L^\infty(\mathbb{R}^N)$ are positive functions, and I_α denotes the Riesz potential.

1. Introduction

We deal with the linearly coupled Choquard type equations:

$$\begin{cases} -\Delta u + V_1(x)u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|u|^{\frac{\alpha}{N}-1}u + \lambda v, x \in \mathbb{R}^N, \\ -\Delta u + V_2(x)v = \left(I_\alpha * |v|^{\frac{N+\alpha}{N}}\right)|v|^{\frac{\alpha}{N}-1}v + \lambda u, x \in \mathbb{R}^N, \\ u, v \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1}$$

where $N \geq 3$, $\alpha \in (0, N)$, and $V_1, V_2 \in L^\infty(\mathbb{R}^N)$ are positive functions, $(N + \alpha)/N$ is the lower critical exponent with respect to a Hardy-Littlewood-Sobolev inequality (see ([1], Theorem 3.1) or ([2], Theorem 4.3)), and I_α denotes the Riesz potential defined on $\mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma((N-\alpha)/2)}{2^\alpha \pi^{N/2}\Gamma(\alpha/2)|x|^{N-\alpha}}. \tag{2}$$

The single equation

$$-\Delta u + V(x)u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u, \; u \in H^1\left(\mathbb{R}^N\right), \tag{3}$$

appears in various physical contexts (see [3–6]). Mathematically, equations of this type have received considerable attention due to the appearance of the nonlocal term $(I_\alpha * |u|^p)|u|^{p-2}u$, which makes the problem challenging and interesting. The readers can refer to [4, 7–18] and references therein for research on related problems.

Recently, Chen and Liu [19] established the existence and asymptotic behavior of the vector ground state of the linearly coupled system:

$$\begin{cases} -\Delta u + u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u + \lambda v, \; x \in \mathbb{R}^N, \\ -\Delta v + v = (I_\alpha * |v|^q)|v|^{q-2}v + \lambda u, \; x \in \mathbb{R}^N, \\ u, v \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{4}$$

where $0 < \lambda < 1$, $(N + \alpha)/N < p, q < (N + \alpha)/(N - 2)$. Xu, Ma and Xing [20] extended the results in [19] to (4) in the case that $(I_\alpha * |u|^p)|u|^{p-2}u$ and $(I_\alpha * |u|^q)|u|^{q-2}u$ are replaced with general subcritical nonlinearities $(I_\alpha * F(u))F'(u)$ and $(I_\alpha * G(u))G'(u)$, respectively. Yang et al. [21] obtained the existence of the vector ground state of (4) in the following three cases:

$$p = \frac{N+\alpha}{N}, \; \frac{N+\alpha}{N} < q < \frac{N+\alpha}{N-2},$$

$$p = \frac{N+\alpha}{N-2},\ \frac{N+\alpha}{N} < q < \frac{N+\alpha}{N-2},$$
$$p = \frac{N+\alpha}{N},\ q = \frac{N+\alpha}{N-2}. \tag{5}$$

They also proved that (4) has no nontrivial solutions if $p = q = (N+\alpha)/N$ or $p = q = (N+\alpha)/(N-2)$.

As we know, when $\alpha \longrightarrow 0$, the local system

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + \lambda v,\ x \in \mathbb{R}^N, \\ -\Delta v + v = |v|^{q-2}v + \lambda u,\ x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{6}$$

which has application in a large number of physical problems such as in nonlinear optics, can be regarded as a limiting system of (4). Systems of this type have received great attention in recent years (see [22–28] for instance). However, linearly coupled systems with nonlocal nonlinearities have been less studied.

In this paper, we are interested in the existence, nonexistence, and multiplicity of solutions of system (1) with positive nonconstant potentials. We assume that

(H1) $V_i(x) \geq C > 0,\ V_i(x) \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\longrightarrow\infty} V_i(x) = 1, i = 1, 2$

(H2) $\liminf_{|x|\longrightarrow\infty}(1 - V_i(x))|x|^2 \geq (N^2(N-2))/(4(N+1)), i = 1, 2$

(H3) $0 < |\lambda| < \inf_{x\in\mathbb{R}^N}\sqrt{V_1(x)V_2(x)}$

For simplicity, the integral $\int_{\mathbb{R}^N} \cdot\, dx$ is denoted by $\int \cdot$. According to (H1), the norm in $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ can be defined by

$$\|(u, v)\| := \sqrt{\|u\|_1^2 + \|v\|_2^2}, \tag{7}$$

where

$$\|u\|_1^2 := \int \left(|\nabla u|^2 + V_1(x)u^2\right),$$
$$\|u\|_2^2 := \int \left(|\nabla u|^2 + V_2(x)u^2\right). \tag{8}$$

Then, a solution of system (1) can be found as a critical point of the energy functional $E : H \mapsto \mathbb{R}$ defined by

$$E_\lambda(u, v) := \frac{1}{2}\|(u, v)\|^2 - \int \lambda uv - \frac{N}{2(N+\alpha)}\int \left(\left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{(N+\alpha)/N} + \left(I_\alpha * |v|^{(N+\alpha)/N}\right)|v|^{(N+\alpha)/N}\right). \tag{9}$$

Set

$$\mathcal{N} := \left\{(u, v) \in H \setminus \{(0, 0)\} \mid \left\langle E'_\lambda(u, v), (u, v)\right\rangle = 0\right\},$$
$$c_\lambda := \inf_{\mathcal{N}} E_\lambda(u, v). \tag{10}$$

We first show that c_λ is attained.

Theorem 1. *Assume that (H1), (H2), and (H3) hold. Then, there exists a vector ground state (u_λ, v_λ) of system (1). Additionally, if $\{\lambda_n\} \subset (0, \inf_{x\in\mathbb{R}^N}\sqrt{V_1(x)V_2(x)})$ is a sequence satisfying $\lambda_n \longrightarrow 0^+$ as $n \longrightarrow +\infty$, then up to a subsequence, either $(u_{\lambda_n}, v_{\lambda_n}) \longrightarrow (\widehat{u}, 0)$ or $(u_{\lambda_n}, v_{\lambda_n}) \longrightarrow (0, \widehat{v})$ in H as $n \longrightarrow \infty$, where $\widehat{u}$ is a ground state of*

$$-\Delta u + V_1(x)u = \left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{\alpha/N-1}u,\ u \in H^1(\mathbb{R}^N), \tag{11}$$

and $\widehat{v}$ is a ground state of

$$-\Delta u + V_2(x)u = \left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{\alpha/N-1}u,\ u \in H^1(\mathbb{R}^N). \tag{12}$$

Remark 2. We call a solution $(u, v) \in H$ of system (1) a nontrivial solution if $(u, v) \neq (0, 0)$ and a vector solution if $u \neq 0$ and $v \neq 0$. A nontrivial solution (u, v) satisfying $E_\lambda(u, v) \leq E_\lambda(h, k)$ for any nontrivial solutions $(h, k) \in H$ of system (1) is called a ground state.

Remark 3. Under assumptions (H1) and (H2), the existence of ground states of equations (11) and (12) has been proved by Moroz and Van Schaftingen ([17], Theorem 3 and Theorem 6).

To prove Theorem 1, it is crucial to give an estimate of the upper bound of the least energy c_λ due to the lack of compactness. In our case, the estimate is quite involved, since we are dealing with a coupled system, which is more complex than a single equation. The method we follow can be sketched as follows. We first study the minimizing problem

$$S_0 = \inf_{(u,v)\in L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)\setminus\{(0,0)\}} \frac{\int \left(|u|^2 + |v|^2 - 2\lambda uv\right)}{\left(\int \left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{(N+\alpha)/N} + \int \left(I_\alpha * |v|^{(N+\alpha)/N}\right)|v|^{(N+\alpha)/N}\right)^{N/(N+\alpha)}}, \tag{13}$$

which can be considered an extension of the classical problem

$$S_1 = \inf_{u\in L^2(\mathbb{R}^N)\setminus\{0\}} \frac{\int |u|^2}{\left(\int \left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{(N+\alpha)/N}\right)^{N/(N+\alpha)}}. \tag{14}$$

By the results that S_1 is attained if and only if

$$u(x) = U_b(x) := A\left(\frac{b}{b^2+|x-a|^2}\right)^{N/2}, \tag{15}$$

where $A>0$ is a fixed constant, $a\in\mathbb{R}^N$, and $b\in(0,\infty)$ (see ([1], Theorem 3.1) or ([2], Theorem 4.3)), and studying the minimum point of a function $h(\tau)$ defined on $[0,+\infty)$ by

$$h(\tau) = \frac{1+\tau^2-2\,|\,\lambda\,|\,\tau}{\left(1+\tau^{(2(N+\alpha))/N}\right)^{N/(N+\alpha)}}, \tag{16}$$

we show that S_0 is attained at $(U_b, \tau_{\min}U_b)$ if $0<\lambda<1$ and at $(U_b,-\tau_{\min}U_b)$ if $-1<\lambda<0$ (see Theorem 7 in Section 2), which combined with the existence of ground states for equations (11) and (12) enables us to obtain the precise upper bound of c_λ.

Our second goal is to show the existence of a higher energy vector solution of (1).

Theorem 4. *Assume that (H1) and (H2) hold. Then, for some $\lambda^*\in(0,\inf_{x\in\mathbb{R}^N}\sqrt{V_1(x)V_2(x)})$, there exists a vector solution $(\bar{u}_\lambda,\bar{v}_\lambda)$ of system (1) if $0<\lambda<\lambda^*$. Additionally, if $\{\lambda_n\}\subset(0,\lambda^*)$ is a sequence satisfying $\lambda_n\longrightarrow 0^+$ as $n\longrightarrow+\infty$, then up to a subsequence, $(\bar{u}_{\lambda_n},\bar{v}_{\lambda_n})\longrightarrow(\bar{u},\bar{v})$ in H, where $\bar{u}$ is a ground state of (11) and $\bar{v}$ is a ground state of (12).*

Remark 5. For $\lambda>0$ sufficiently small, it is trivial to see that the solutions obtained in Theorem 1 and Theorem 4 are different, which implies that there exists at least two vector solutions of system (1) if $\lambda>0$ is small enough.

Finally, we prove the nonexistence of the nontrivial solution of system (1) by establishing the Pohozaev type identity.

Theorem 6. *Assume that (H3) holds. If $V_i(x)\in W^{1,1}_{loc}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $i=1,2$ and*

$$\sup_{x\in\mathbb{R}^N}|x|^2\nabla V_i(x)\cdot x < \frac{(N-2)^2}{2},\ i=1,2, \tag{17}$$

then, system (1) has no nontrivial solutions in H.

This paper is structured as follows. Some preliminary results are provided in Section 2. The proofs of Theorems 1 and 4 are presented in Section 3 and Section 4, respectively. In Section 5, we show the nonexistence of nontrivial solutions.

2. Preliminary Results

In this section, we show the sharp constant S_0 defined in (13) is attained and give an estimate of the upper bound of c_λ.

Theorem 7. *If $0<|\lambda|<1$, then S_0 is attained. Moreover, $(U_b, \tau_{\min}U_b)$ (or $(U_b,-\tau_{\min}U_b)$) is a solution of (13) for $0<\lambda<1$ (or $-1<\lambda<0$), where $\tau_{\min}>0$ is a minimum point of $h(\tau)$ defined on $[0,+\infty)$ by*

$$h(\tau) = \frac{1+\tau^2-2\,|\,\lambda\,|\,\tau}{\left(1+\tau^{(2(N+\alpha))/N}\right)^{N/(N+\alpha)}}. \tag{18}$$

Proof. First, we show that there exists $\tau_{\min}>0$ such that

$$h(\tau_{\min}) = \min_{\tau\geq 0} h(\tau). \tag{19}$$

Calculating directly, we have

$$h'(\tau) = \frac{\tau-|\lambda|+|\lambda\,|\,\tau^{(2(N+\alpha))/N}-\tau^{(N+2\alpha)/N}}{2\left(1+\tau^{(2(N+\alpha))/N}\right)^{(2N+\alpha)/(N+\alpha)}}. \tag{20}$$

Set $f(\tau)=\tau-|\lambda|+|\,\lambda\,|\,\tau^{(2(N+\alpha))/N}-\tau^{(N+2\alpha)/N}$. It can be easily seen that $f(\tau)\longrightarrow-|\,\lambda\,|$ as $\tau\longrightarrow 0$, and $f(\tau)\longrightarrow+\infty$ as $\tau\longrightarrow+\infty$. Then, there is $\tau_{\min}>0$ such that $f(\tau_{\min})=0$, and $h(\tau_{\min})=\min_{\tau\geq0}h(\tau)$.

In the next step, we prove

$$S_0 = h(\tau_{\min})S_1, \tag{21}$$

where S_1 is defined in (14). We employ the idea in ([29], Theorem 5) to prove (21). For the case $\lambda>0$, taking $(u,v)=(U_b,\tau_{\min}U_b)$ gives

$$S_0 \leq \frac{\left(1+\tau_{\min}^2-2\lambda\tau_{\min}\right)\int|U_b|^2}{\left((1+\tau_{\min})^{(2(N+\alpha))/N}\int\left(I_\alpha*|U_b|^{(N+\alpha)/N}\right)|U_b|^{(N+\alpha)/N}\right)^{N/(N+\alpha)}} = h(\tau_{\min})S_1. \tag{22}$$

Let $(u_n,v_n)\in L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$ be a minimizing sequence for S_0. Set $z_n=t_nu_n$, where $t_n=\left(\int|v_n|^2/\int|u_n|^2\right)^{1/2}$. Then,

$$\int|z_n|^2 = t_n^2\int|u_n|^2 = \int|v_n|^2, \tag{23}$$

$$\int z_nv_n = t_n\int u_nv_n \leq \int|v_n|^2 = \int|z_n|^2. \tag{24}$$

Collecting (23) and (24) leads to

$$S_0 + o(1) = \frac{\int \left(|u_n|^2 + |v_n|^2 - 2\lambda u_n v_n\right)}{\left(\int \left(I_\alpha * |u_n|^{(N+\alpha)/N}\right)|u_n|^{(N+\alpha)/N} + \int \left(I_\alpha * |v_n|^{(N+\alpha)/N}\right)|v_n|^{(N+\alpha)/N}\right)^{N/(N+\alpha)}}$$
$$\geq \frac{t_n^{-2}\int |z_n|^2 + \int |v|_n^2 - 2\lambda t_n^{-1}\int |v_n|^2}{\left(t_n^{-(2(N+\alpha))/N}\int \left(I_\alpha * |z_n|^{(N+\alpha)/N}\right)|z_n|^{(N+\alpha)/N} + \int \left(I_\alpha * |v_n|^{(N+\alpha)/N}\right)|v_n|^{(N+\alpha)/N}\right)^{N/(N+\alpha)}} \geq h\left(t_n^{-1}\right)S_1 \geq h(\tau_{\min})S_1. \tag{25}$$

Then, (21) follows from (22) and (25). For the case $\lambda < 0$, the conclusions follow by replacing $(U_b, \tau_{\min}U_b)$ with $(U_b, -\tau_{\min}U_b)$ and repeating the proof previously.

Lemma 8. *Assume that (H1) and (H3) holds, then for any $(u, v) \in H \setminus \{(0,0)\}$, there exists $t_0 > 0$ such that $t_0(u, v) \in \mathcal{N}$ and*

$$E_\lambda(t_0 u, t_0 v) = \max_{t\geq 0} E_\lambda(tu, tv). \tag{26}$$

Proof. This result is standard and the proof can be found in ([30], Lemma 12). We omit it.

For equations (11) and (12), we set

$$J_i(u) = \frac{1}{2}\|u\|_i^2 - \frac{N}{2(N+\alpha)}\int \left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{(N+\alpha)/N}, \tag{27}$$

and $B_i = \inf_{\mathcal{N}_i} J_i(u)$, where

$$\mathcal{N}_i = \left\{u \in H^1(\mathbb{R}^N)/\{0\} \mid \left\langle J_i'(u), u\right\rangle = 0\right\}, i = 1, 2. \tag{28}$$

Then, according to ([17], Theorem 3 and Theorem 6), we have

$$B_i < \frac{\alpha}{2(N+\alpha)}S_1^{(N+\alpha)/\alpha},\ i = 1, 2, \tag{29}$$

and B_i is achieved, where S_1 is defined in (14). By Theorem 7 and ([17], Theorem 3 and Theorem 6), we are able to get the following estimate.

Lemma 9. *Assume that (H1), (H2), and (H3) hold. Then,*

$$0 < c_\lambda < \min\left\{B_1, B_2, \frac{\alpha}{2(N+\alpha)}S_0^{(N+\alpha)/\alpha}\right\}. \tag{30}$$

Proof. We first show the positivity of c_λ. By (H3), we have

$$C\|(u,v)\|^2 \leq \|(u,v)\|^2 - 2\int \lambda uv = \int \left(\left(I_\alpha * |u|^{(N+\alpha)/N}\right)|u|^{(N+\alpha)/N} + \left(I_\alpha * |v|^{(N+\alpha)/N}\right)|v|^{(N+\alpha)/N}\right) \leq \left(S_1^{-1}\int |u|^2\right)^{(N+\alpha)/N} + \left(S_1^{-1}\int |v|^2\right)^{(N+\alpha)/N} \leq S_1^{-(N+\alpha)/N}\|(u,v)\|^{(2(N+\alpha))/N}, \tag{31}$$

for some $C > 0$, which suggests that there exists $M_1 > 0$ such that $\|(u, v)\| > M_1$. Thus, we obtain

$$c_\lambda = \inf_{\mathcal{N}} E_\lambda(u, v) = \inf_{\mathcal{N}} \frac{\alpha}{2(N+\alpha)}\left(\|(u,v)\|^2 - 2\int \lambda uv\right) \geq \frac{\alpha}{2(N+\alpha)}CM_1^2 > 0. \tag{32}$$

Second, we show

$$c_\lambda < \frac{\alpha}{2(\alpha+N)}S_0^{(N+\alpha)/\alpha}. \tag{33}$$

From the assumptions (H1)–(H3), we see that $0 < |\lambda| < 1$, and so Theorem 7 holds. For the case $\lambda > 0$, by Lemma 8, there is $t > 0$ such that $t(U_b, \tau_{\min}U_b) \in \mathcal{N}$; then, we have

$$\begin{aligned} c_\lambda &\leq E_\lambda(tU_b, t\tau_{\min}U_b) \\ &= \frac{t^2}{2}\int \left(\left(1+\tau_{\min}^2\right)|\nabla U_b|^2 + \left(V_1(x) + \tau_{\min}^2 V_2(x) - 2\tau_{\min}\lambda\right)|U_b|^2\right) \\ &\quad - \frac{Nt^{2(N+\alpha)/N}}{2(N+\alpha)}\left(1+\tau_{\min}^{2(N+\alpha)/N}\right)\int \left(I_\alpha * |U_b|^{(N+\alpha)/N}\right)|U_b|^{(N+\alpha)/N} \\ &= \frac{t^2}{2}\left(1+\tau_{\min}^2 - 2\tau_{\min}\lambda\right)\int |U_b|^2 - \frac{Nt^{2(N+\alpha)/N}}{2(N+\alpha)}\left(1+\tau_{\min}^{2(N+\alpha)/N}\right) \\ &\quad \cdot \int \left(I_\alpha * |U_b|^{(N+\alpha)/N}\right)|U_b|^{(N+\alpha)/N} + \frac{t^2}{2}\int \left(\left(1+\tau_{\min}^2\right)|\nabla U_b|^2\right. \\ &\quad \left. + \left((V_1(x)-1) + \tau_{\min}^2(V_2(x)-1)\right)|U_b|^2\right) \leq \frac{\alpha}{2(\alpha+N)}S_0^{(N+\alpha)/\alpha} \\ &\quad + \frac{t^2}{2}\int \left(\left(1+\tau_{\min}^2\right)|\nabla U_b|^2 + \left((V_1(x)-1)\right.\right. \\ &\quad \left.\left. + \tau_{\min}^2(V_2(x)-1)\right)|U_b|^2\right). \end{aligned} \tag{34}$$

The last inequality in (34) follows from Theorem 7 and direct calculation. Denote

$$L_i(u) = \frac{t^2}{2}\int \left(\left|\nabla u\right|^2 + (V_i(x)-1)\left|u\right|^2\right), i = 1, 2. \tag{35}$$

To prove (33), it is enough to show

$$L_i(U_b) < 0,\ i = 1, 2, \tag{36}$$

for some $b > 0$. Since

$$\int \frac{|x|^2}{\left(1+|x|^2\right)^{N+2}} = \frac{N-2}{4(N+1)} \int \frac{1}{x^2(1+x^2)^N}, \tag{37}$$

we have

$$\int |\nabla U_b|^2 = \frac{N^2(N-2)}{4(N+1)} \int \frac{|U_b|^2}{|x|^2}. \tag{38}$$

Then, by a transformation $x = a + bz$, we get

$$L_i(u_b) = \int \left(\frac{N^2(N-2)}{4(N+1)|z|^2} - b^2(1 - V_i(a+bz)) \right) \frac{C^2}{\left(1+|z|^2\right)^N} dz. \tag{39}$$

Taking the assumption (H2) into consideration, we see that that (36) holds. Then, (33) follows from (34).

Now, it remains to show

$$c_\lambda < \min\{B_1, B_2\}. \tag{40}$$

Denote ground states of (11) and (12) by U and V, respectively. Since $(U, 0) \in \mathcal{N}$ and $(0, V) \in \mathcal{N}$, we have $c_\lambda \leq \min\{B_1, B_2\}$. If $c_\lambda = \min\{B_1, B_2\}$, then we see that at least one of $(U, 0)$ and $(0, V)$ is a solution of system (1), which is impossible since $\lambda \neq 0$, so (40) holds.

3. Proof of Theorem 11

Lemma 10. *Assume that (H1), (H2), and (H3) hold. Then, there exists a vector ground state of system (1).*

Proof. According to Ekeland's variational principle, there exists $\{(u_n, v_n)\} \subset \mathcal{N}$ such that

$$E_\lambda(u_n, v_n) \longrightarrow c_\lambda, E_\lambda{}'(u_n, v_n)\big|_{\mathcal{N}} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{41}$$

For simplicity, we denote $I_\lambda(u, v) \coloneqq \langle E_\lambda{}'(u, v), (u, v)\rangle$. First, we prove $E_\lambda{}'(u_n, v_n) \longrightarrow 0$. Indeed,

$$o(1) = E_\lambda{}'(u_n, v_n)\big|_{\mathcal{N}} = E_\lambda{}'(u_n, v_n) - \sigma_n I_\lambda{}'(u_n, v_n) \tag{42}$$

for some σ_n and sufficiently large n. Particularly,

$$\begin{aligned} o(1) &= \left\langle E_\lambda{}'(u_n, v_n), (u_n, v_n) \right\rangle \\ &\quad - \sigma_n \left\langle I_\lambda{}'(u_n, v_n), (u_n, v_n) \right\rangle \\ &= -\sigma_n \left\langle I_\lambda{}'(u_n, v_n), (u_n, v_n) \right\rangle. \end{aligned} \tag{43}$$

From the proof of Lemma 9, we observe that there exists $M_1, M_2 > 0$ such that $M_1 \leq \|(u_n, v_n)\| \leq M_2$. Then, we have

$$\begin{aligned} \left\langle I_\lambda'(u_n, v_n), (u_n, v_n) \right\rangle &= 2\left(\|(u_n, v_n)\|^2 - 2\int \lambda u_n v_n \right) \\ &\quad - \frac{2(N+\alpha)}{N} \int \left(\left(I_\alpha * |u_n|^{(N+\alpha)/N} \right) |u_n|^{(N+\alpha)/N} \right. \\ &\quad \left. + \left(I_\alpha * |v_n|^{(N+\alpha)/N} \right) |v_n|^{(N+\alpha)/N} \right) \\ &\quad - \frac{2\alpha}{N} \left(\|(u_n, v_n)\|^2 - 2\int \lambda u_n v_n \right) \\ &\leq -C\|(u_n, v_n)\|^2 \leq -CM_1 < 0. \end{aligned} \tag{44}$$

Taking (43) into consideration, we obtain that $\sigma_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, from (42), we get $E_\lambda{}'(u_n, v_n) \longrightarrow 0$.

We may assume that

$$\begin{gathered} (u_n, v_n) \rightharpoonup (u, v) \text{ in } H, \\ (u_n, v_n) \to (u, v) \text{ in } L^r_{\text{loc}}\left(\mathbb{R}^N\right) \times L^r_{\text{loc}}\left(\mathbb{R}^N\right) (2 \leq r < 2^*), \\ (u_n, v_n) \to (u, v) \text{ a.e in } \mathbb{R}^N. \end{gathered} \tag{45}$$

Then, $E'_\lambda(u, v) = 0$. To complete the proof, it is sufficient to prove that $(u, v) \neq (0, 0)$. Actually, if $(u, v) \neq (0, 0)$. By Fatou's lemma, we get

$$\begin{aligned} c_\lambda \leq E_\lambda(u, v) &= \frac{\alpha}{2(N+\alpha)} \int \left(\left(I_\alpha * |u|^{(N+\alpha)/N} \right) |u|^{(N+\alpha)/N} \right. \\ &\quad \left. + \left(I_\alpha * |v|^{(N+\alpha)/N} \right) |v|^{(N+\alpha)/N} \right) \\ &\leq \liminf_{n\to\infty} \frac{\alpha}{2(N+\alpha)} \int \left(\left(I_\alpha * |u_n|^{(N+\alpha)/N} \right) |u_n|^{(N+\alpha)/N} \right. \\ &\quad \left. + \left(I_\alpha * |v_n|^{(N+\alpha)/N} \right) |v_n|^{(N+\alpha)/N} \right) = c_\lambda. \end{aligned} \tag{46}$$

Furthermore, since $E_\lambda{}'(u, v) = 0$ and $\lambda \neq 0$, we see from (1) that $u \equiv 0$ and $v \equiv 0$, that is, (u, v) is a vector ground state of (1).

Suppose the assertion is false, that is, $(u, v) = (0, 0)$. On the one hand, we know from (H1) that

$$\begin{aligned} \int \left(V_1(x)u_n^2 + V_2(x)v_n^2 \right) &= \int_{B_r(0)} \left(V_1(x)u_n^2 + V_2(x)v_n^2 \right) \\ &\quad + \int_{\mathbb{R}^N \setminus B_r(0)} \left(V_1(x)u_n^2 + V_2(x)v_n^2 \right) \\ &= \int_{B_r(0)} \left(V_1(x)u_n^2 + V_2(x)v_n^2 \right) \\ &\quad + \int_{\mathbb{R}^N \setminus B_r(0)} \left(u_n^2 + v_n^2 \right) + o(1) \\ &= \int \left(u_n^2 + v_n^2 \right) + o(1), \end{aligned} \tag{47}$$

as $n \longrightarrow \infty$. Then, it follows

$$\begin{aligned}&\int\left(|\nabla u_n|^2+|\nabla v_n|^2+u_n^2+v_n^2-2\lambda u_n v_n\right)+o(1)\\&\quad=\|(u_n,v_n)\|^2-2\int\lambda u_n v_n\end{aligned}\tag{48}$$

as $n \longrightarrow \infty$. Using Theorem 7, we obtain

$$\begin{aligned}c_\lambda&=E_\lambda(u_n,v_n)-\frac{1}{2}\left\langle E_\lambda'(u_n,v_n),(u_n,v_n)\right\rangle+o(1)\\&=\frac{\alpha}{2(N+\alpha)}\int\left(\left(I_\alpha*|u_n|^{(N+\alpha)/N}\right)|u_n|^{(N+\alpha)/N}\right.\\&\quad\left.+\left(I_\alpha*|v_n|^{(N+\alpha)/N}\right)|v_n|^{(N+\alpha)/N}\right)\\&\quad+o(1)\le\frac{\alpha}{2(N+\alpha)}S_0^{-(N+\alpha)/N}\left(\int\left(u_n^2+v_n^2-2\lambda u_n v_n\right)\right)^{(N+\alpha)/N}\\&\quad+o(1)\le\frac{\alpha}{2(N+\alpha)}S_0^{-(N+\alpha)/N}\\&\quad\cdot\left(\int\left(|\nabla u_n|^2+|\nabla v_n|^2+u_n^2+v_n^2-2\lambda u_n v_n\right)\right)^{(N+\alpha)/N}\\&\quad+o(1)=\frac{\alpha}{2(N+\alpha)}S_0^{-(N+\alpha)/N}\left(\|u_n,v_n\|^2-2\int\lambda u_n v_n\right)^{(N+\alpha)/N}\\&\quad+o(1).\end{aligned}\tag{49}$$

On the other hand,

$$c_\lambda=\frac{\alpha}{2(N+\alpha)}\left(\|(u_n,v_n)\|^2-2\int\lambda u_n v_n\right)+o(1).\tag{50}$$

Collecting (49) and (50) yields

$$c_\lambda\ge\frac{\alpha}{2(N+\alpha)}S_0^{(N+\alpha)/\alpha},\tag{51}$$

which contradicts Lemma 9. Thus, $(u,v)\neq(0,0)$.

Proof of Theorem 11. By Lemma 10, we need only show the asymptotic behavior of the vector ground state when $\lambda \longrightarrow 0^+$. First, we claim that c_λ decreases strictly monotonically with respect to $\lambda\in(0,\inf_{x\in\mathbb{R}^N}\sqrt{V_1(x)V_2(x)})$. Indeed, fix λ_1, $\lambda_2\in(0,\inf_{x\in\mathbb{R}^N}\sqrt{V_1(x)V_2(x)})$ with $\lambda_1<\lambda_2$. Denoting a vector ground state of system (1) when $\lambda=\lambda_1$ by $(u_{\lambda_1},v_{\lambda_1})$ and letting $t>0$ be the constant such that $(tu_{\lambda_1},tv_{\lambda_1})\in\mathcal{N}_{\lambda=\lambda_2}$, we obtain

$$\begin{aligned}&\left\|\left(u_{\lambda_1},v_{\lambda_1}\right)\right\|^2-2\int\lambda_1 u_{\lambda_1}v_{\lambda_1}\\&\quad=\int\left(\left(I_\alpha*\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right.\\&\qquad\left.+\left(I_\alpha*\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right),\\&\left\|\left(u_{\lambda_1},v_{\lambda_1}\right)\right\|^2-2\int\lambda_2 u_{\lambda_1}v_{\lambda_1}\\&\quad=t^{2\alpha/N}\int\left(\left(I_\alpha*\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right.\\&\qquad\left.+\left(I_\alpha*\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right).\end{aligned}\tag{52}$$

Then, by $\lambda_1<\lambda_2$, we deduce that $t<1$, which gives

$$\begin{aligned}c_{\lambda_2}&\le E_{\lambda_2}\left(tu_{\lambda_1},tv_{\lambda_1}\right)\\&=\frac{\alpha}{2(N+\alpha)}t^{2(N+\alpha)/N}\int\left(\left(I_\alpha*\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right.\\&\quad\left.+\left(I_\alpha*\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right)\\&<\frac{\alpha}{2(N+\alpha)}\int\left(\left(I_\alpha*\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|u_{\lambda_1}\right|^{(N+\alpha)/N}\right.\\&\quad\left.+\left(I_\alpha*\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right)\left|v_{\lambda_1}\right|^{(N+\alpha)/N}\right)=c_{\lambda_1}.\end{aligned}\tag{53}$$

The claim is proved.

Now, choose $\{\lambda_n\}\subset(0,\inf_{x\in\mathbb{R}^N}\sqrt{V_1(x)V_2(x)})$ satisfying $\lambda_n\longrightarrow 0^+$ as $n\longrightarrow\infty$ and denote a vector ground state of system (1) when $\lambda=\lambda_n$ by $(u_{\lambda_n},v_{\lambda_n})$. Then $\forall(h_1,h_2)\in H$, we have

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left\langle {E_0}'\left(u_{\lambda_n},v_{\lambda_n}\right),(h_1,h_2)\right\rangle\\&\quad=\lim_{n\longrightarrow\infty}\left\langle {E_{\lambda_n}}'\left(u_{\lambda_n},v_{\lambda_n}\right),(h_1,h_2)\right\rangle=0,\\&\lim_{n\longrightarrow\infty}E_0\left(u_{\lambda_n},v_{\lambda_n}\right)=\lim_{n\longrightarrow\infty}E_{\lambda_n}\left(u_{\lambda_n},v_{\lambda_n}\right).\end{aligned}\tag{54}$$

By $c_{\lambda_n}<\min\{B_1,B_2\}$ and (29), we obtain

$$\begin{aligned}\lim_{n\longrightarrow\infty}E_0\left(u_{\lambda_n},v_{\lambda_n}\right)&=\lim_{n\longrightarrow\infty}c_{\lambda_n}\le\min\{B_1,B_2\}\\&<\frac{\alpha}{2(N+\alpha)}S_1^{(N+\alpha)/\alpha}.\end{aligned}\tag{55}$$

Repeating an argument as in Lemma 10, we deduce that $(u_{\lambda_n},v_{\lambda_n})\longrightarrow(\widehat{u},0)$ or $(u_{\lambda_n},v_{\lambda_n})\longrightarrow(0,\widehat{v})$ in H, where $\widehat{u}$ and $\widehat{v}$ are ground states of (11) and (12), respectively.

4. Proof of Theorem 18

In this section, we study the existence of a higher energy vector solution of system (1) for $\lambda>0$ sufficiently small. We suppose that $B_1\le B_2$ without loss of generality. Let U, V be ground states of (11) and (12), respectively. Then, we may assume that U and V are positive since $|U|$ and $|V|$ are also ground states of (11) and (12), respectively. Now, we set

$$\mathscr{A}=\mathscr{A}_1\times\mathscr{A}_2,\tag{56}$$

where

$$\mathscr{A}_i = \left\{ u \in H^1\left(\mathbb{R}^N\right), J_i{}'(u) = 0, J_i(u) = B_i \right\}. \quad (57)$$

Then $(U, V) \in \mathscr{A}$. Moreover, by a similar argument as that in ([28], Lemma 12), we obtain the following.

Lemma 12. *Assume that (H1) and (H2) hold. Then, $\mathscr{A} \subset H$ is compact, and there exist $0 < a_1 < a_2$ such that*

$$a_1 \le \|u\|_1, \|v\|_2 \le a_2, \forall (u, v) \in \mathscr{A}. \quad (58)$$

Proof. The proof can be found in ([28], Lemma 12) and will be omitted here.

By the definition of U and V, we know that

$$B_1 = J_1(U) = \max_{t>0} J_1(tU), B_2 = J_2(V) = \max_{t>0} J_2(tV), \quad (59)$$

$$\begin{aligned} J_1(tU) &\le \frac{B_1}{4}, \\ \forall t &\in (0, t_1] \cup [t_2, +\infty), \\ J_2(sV) &\le \frac{B_2}{4}, \\ \forall s &\in (0, s_1] \cup [s_2, +\infty), \end{aligned} \quad (60)$$

for some t_1, t_2, s_1, and s_2 satisfying $0 < t_1 < 1 < t_2$ and $0 < s_1 < 1 < s_2$. Denote $A := [0, t_2] \times [0, s_2]$ and define $\widehat{\gamma} : A \mapsto H$ by

$$\widehat{\gamma}(t, s) = (\widehat{\gamma}_1(t), \widehat{\gamma}_2(s)) := (tU, sV). \quad (61)$$

Then, $\max_{(t,s)\in A} \|\widehat{\gamma}(t, s)\| \le a_0$ for some $a_0 > 0$. Define

$$\widehat{c}_\lambda := \inf_{\gamma \in \widehat{\Gamma}} \max_{(t,s)\in A} E_\lambda(\gamma(t, s)), m_\lambda := \max_{(t,s)\in A} E_\lambda(\widehat{\gamma}(t, s)), \quad (62)$$

where

$$\widehat{\Gamma} := \left\{ \gamma \in \mathbb{C}(A, H) \middle| \max_{(t,s)\in A} \|\gamma(t, s)\| \le 2a_2 + a_0, \ \gamma(t, s) = \widehat{\gamma}(t, s) \text{ for } (t, s) \in A \setminus \{(t_1, t_2) \times (s_1, s_2)\} \right\}, \quad (63)$$

and a_2 is defined in Lemma 12. Obviously, $\widehat{\gamma}(t, s) \in \widehat{\Gamma}$. Moreover, we havethe following.

Lemma 13. *Assume that (H1), (H2), and $0 < \lambda < \inf_{x\in\mathbb{R}^N} \sqrt{V_1(x)V_2(x)}$ hold. Then,*

$$\lim_{\lambda \longrightarrow 0^+} \widehat{c}_\lambda = \lim_{\lambda \longrightarrow 0^+} m_\lambda = \widehat{c}_0 = m_0 = B_1 + B_2. \quad (64)$$

Proof. By $\lambda > 0$, we see that $E_\lambda(\widehat{\gamma}(t, s)) \le E_0(\widehat{\gamma}(t, s))$ and

$$\begin{aligned} m_\lambda \le m_0 &= \max_{(t,s)\in A} E_0(\widehat{\gamma}(t, s)) \\ &= \max_{t\in[0,t_2]} J_1(\widehat{\gamma}_1(t)) + \max_{s\in[0,s_2]} J_2(\widehat{\gamma}_2(s)) \\ &= J_1(U) + J_2(V) = B_1 + B_2. \end{aligned} \quad (65)$$

Observing that $\widehat{c}_\lambda \le m_\lambda$ since $\widehat{\gamma} \in \widehat{\Gamma}$, we deduce

$$\limsup_{\lambda \longrightarrow 0^+} \widehat{c}_\lambda \le \liminf_{\lambda \longrightarrow 0^+} m_\lambda \le \limsup_{\lambda \longrightarrow 0^+} m_\lambda \le m_0, \ \widehat{c}_0 \le m_0. \quad (66)$$

Now, for $\gamma(t, s) = (\gamma_1(t), \gamma_2(s)) \in \widehat{\Gamma}$, define a function $f(\gamma)$ on $[t_1, t_2] \times [s_1, s_2]$ by

$$f(\gamma)(t, s) := (\phi_1(\gamma_1(t)) - \phi_2(\gamma_2(s)), \phi_1(\gamma_1(t)) + \phi_2(\gamma_2(s)) - 2), \quad (67)$$

where $\phi_1, \phi_2 : H \mapsto \mathbb{R}$ are given by

$$\begin{aligned} \phi_1(u) &:= \begin{cases} \dfrac{\int \left(I_\alpha * |u|^{(N+\alpha)/N}\right) |u|^{(N+\alpha)/N}}{\int \left(|\nabla u|^2 + V_1(x)u^2\right)}, & \text{if } u \ne 0, \\ 0, & \text{if } u = 0, \end{cases} \\ \phi_2(u) &:= \begin{cases} \dfrac{\int \left(I_\alpha * |u|^{(N+\alpha)/N}\right) |u|^{(N+\alpha)/N}}{\int \left(|\nabla u|^2 + V_2(x)u^2\right)}, & \text{if } u \ne 0, \\ 0, & \text{if } u = 0. \end{cases} \end{aligned} \quad (68)$$

Noting that ϕ_1, ϕ_2 are continuous and $f(\widehat{\gamma})(1, 1) = 0$, we deduce $\deg(f(\widehat{\gamma}), [t_1, t_2] \times [s_1, s_2], (0, 0)) = 1$. Moreover, we know from (60) that $f(\gamma)(t, s) = f(\widehat{\gamma})(t, s) \ne (0, 0)$ for any $(t, s) \in \partial([t_1, t_2] \times [s_1, s_2])$, which implies $\deg(f(\gamma), [t_1, t_2] \times [s_1, s_2], (0, 0))$ is well defined and

$$\begin{aligned} &\deg(f(\gamma), [t_1, t_2] \times [s_1, s_2], (0, 0)) \\ &\quad = \deg(f(\widehat{\gamma}), [t_1, t_2] \times [s_1, s_2], (0, 0)) = 1. \end{aligned} \quad (69)$$

Therefore, there exists $(t^*, s^*) \in [t_1, t_2] \times [s_1, s_2]$ satisfying $f(\gamma)(t^*, s^*) = (0, 0)$, that is, $\phi_1(\gamma_1(t^*)) = \phi_2(\gamma_2(s^*)) = 1$. Recalling the definition of ϕ_1, ϕ_2, we have $\gamma_1(t^*) \in \mathscr{N}_1$, $\gamma_2(s^*) \in \mathscr{N}_2$. Then, it follows

$$\begin{aligned} \max_{(t,s)\in A} E_0(\gamma(t, s)) &\ge E_0(\gamma(t^*, s^*)) = J_1(\gamma_1(t^*)) + J_2(\gamma_2(s^*)) \\ &\ge B_1 + B_2 = m_0. \end{aligned} \quad (70)$$

Thus, $\widehat{c}_0 \ge m_0$. Taking account of (66), we obtain $\widehat{c}_0 = m_0$. Now, it remains to prove

$$\liminf_{\lambda \longrightarrow 0^+} \widehat{c}_\lambda \ge m_0. \quad (71)$$

If (12) is not true, there exists a sequence $\lambda_n \longrightarrow 0^+$,

$\gamma_n(t,s) = (\gamma_{1,n}(t), \gamma_{2,n}(s)) \in \widehat{\Gamma}$ and $\varepsilon > 0$ satisfying

$$\max_{(t,s)\in A} E_{\lambda_n}(\gamma_n(t,s)) \le m_0 - 2\varepsilon. \tag{72}$$

For the ε given above, the definition of $\widehat{\Gamma}$ leads to

$$\max_{(t,s)\in A} \lambda_n \int |\gamma_{1,n}(t)\gamma_{2,n}(s)| \le C\lambda_n < \varepsilon,\ \forall\, n \ge N_0 \tag{73}$$

for some $N_0 > 0$ sufficiently large. Then, it follows

$$\max_{(t,s)\in A} E_0(\gamma_n(t,s)) \le \max_{(t,s)\in A} E_{\lambda_n}(\gamma_n(t,s)) + \varepsilon \le m_0 - \varepsilon,\ \forall n \ge N_0, \tag{74}$$

which contradicts (70), implying that (71) holds.

Set

$$\begin{aligned} \mathscr{A}^d &:= \{(u,v) \in H : \operatorname{dist}((u,v), \mathscr{A}) \le d\}, E_\lambda^c \\ &:= \{(u,v) \in H : E_\lambda(u,v) \le c\}. \end{aligned} \tag{75}$$

Then, we show the compactness of the PS sequence.

Lemma 14. *Assume that (H1) and (H2) hold. Denote* $d_0 := 1/2((2(N+\alpha)/\alpha)B_1)^{1/2}$ *and let* $d \in (0, d_0)$. *If* $\{\lambda_n\}$ *satisfies* $\lambda_n > 0$ *and* $\lambda_n \longrightarrow 0$ *as* $n \longrightarrow \infty$ *and* $\{(u_n, v_n)\} \subset \mathscr{A}^d$ *is a sequence with*

$$\lim_{n\to\infty} E_{\lambda_n}(u_n, v_n) \le \widehat{c}_0,\ \lim_{n\to\infty} E_{\lambda_n}'(u_n, v_n) = 0, \tag{76}$$

then, there exists $(u,v) \in \mathscr{A}$ *such that* $(u_n, v_n) \longrightarrow (u,v)$ *in* H.

Proof. Observing that $\{(u_n, v_n)\}$ is bounded by the choice of d and Lemma 12, we assume $(u_n, v_n) \rightharpoonup (u,v)$ in H. Then, by a similar argument as in ([28], Lemma 14), we obtain that

$$(u,v) \in \mathscr{A}^{2d}. \tag{77}$$

Moreover, using the definition of d again, we get $u \neq 0$, $v \neq 0$.

We now prove $(u_n, v_n) \longrightarrow (u,v) \in \mathscr{A}$. Actually, for $(h_1, h_2) \in H$, we have

$$\begin{aligned} \left\langle E_0'(u,v), (h_1,h_2) \right\rangle &= \lim_{n\longrightarrow\infty} \left\langle E_0'(u_n, v_n), (h_1,h_2) \right\rangle \\ &= \lim_{n\longrightarrow\infty} \left\langle E_{\lambda_n}'(u_n, v_n), (h_1,h_2) \right\rangle = 0, \end{aligned} \tag{78}$$

$$\lim_{n\longrightarrow\infty} E_0(u_n, v_n) = \lim_{n\longrightarrow\infty} E_{\lambda_n}(u_n, v_n) \le \widehat{c}_0. \tag{79}$$

Then, it holds

$$\begin{aligned} E_0(u,v) &= \frac{\alpha}{2(N+\alpha)} \int \left(\left(I_\alpha * |u|^{(N+\alpha)/N} \right) |u|^{(N+\alpha)/N} \right. \\ &\quad \left. + \left(I_\alpha * |v|^{(N+\alpha)/N} \right) |v|^{(N+\alpha)/N} \right) \\ &\le \liminf_{n\to\infty} \frac{\alpha}{2(N+\alpha)} \int \left(\left(I_\alpha * |u_n|^{(N+\alpha)/N} \right) |u_n|^{(N+\alpha)/N} \right. \\ &\quad \left. + \left(I_\alpha * |v_n|^{(N+\alpha)/N} \right) |v_n|^{(N+\alpha)/N} \right) \\ &= \liminf_{n\to\infty} E_0(u_n, v_n). \end{aligned} \tag{80}$$

Note that from (78), $(u,v) \in \mathscr{A}$. Then, combining Lemma 13 with (79) and (80), we have $E_0(u,v) = \widehat{c}_0$ and $(u_n, v_n) \longrightarrow (u,v) \in \mathscr{A}$ in H.

Next, we will construct a PS sequence using a perturbation approach.

Lemma 15. *Assume that (H1) and (H2) hold. Then, for a* $d \in (0, d_0/2)$, *where* d_0 *was defined in Lemma 14, there are* $\bar{\lambda} \in (0, \inf_{x\in\mathbb{R}^N} \sqrt{V_1(x)V_2(x)})$ *and* $a \in (0,1)$ *such that*

$$\|E_\lambda'(u,v)\| \ge a,\ \forall (u,v) \in E_\lambda^{m_\lambda} \cap \left(\mathscr{A}^d \setminus \mathscr{A}^{\frac{d}{2}} \right),\ \lambda \in (0, \bar{\lambda}). \tag{81}$$

Proof. We prove indirectly. Suppose that there exists $\{\lambda_n\}$ satisfying $\lim_{n\longrightarrow\infty} \lambda_n = 0$ and $\{(u_n, v_n)\} \subset E_{\lambda_n}^{m_{\lambda_n}} \cap (\mathscr{A}^d \setminus \mathscr{A}^{d/2})$ with $\|E_{\lambda_n}'(u_n, v_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then, we see immediately that $E_{\lambda_n}(u_n, v_n) \le \widehat{c}_0$ from Lemma 13, and $(u_n, v_n) \longrightarrow (u,v)$ in H for some $(u,v) \in \mathscr{A}$ by Lemma 14. Thus, $(u_n, v_n) \in \mathscr{A}^{d/2}$ for n sufficiently large, which is in contradiction with $\{(u_n, v_n)\} \subset E_{\lambda_n}^{m_{\lambda_n}} \cap (\mathscr{A}^d \setminus \mathscr{A}^{d/2})$, so the conclusion holds.

In the sequel, we assume that $d, a, \bar{\lambda}$ be fixed such that Lemma 15 holds.

Lemma 16. *Assume that (H1) and (H2) hold. Then, there exist* $\widehat{\lambda} \in (0, \bar{\lambda})$ *and* $\delta > 0$ *such that* $\forall \lambda \in (0, \widehat{\lambda})$

$$E_\lambda(\widehat{\gamma}(t,s)) \ge \widehat{c}_\lambda - \delta \text{ implies } \widehat{\gamma}(t,s) \in \mathscr{A}^{d/2}. \tag{82}$$

Proof. Arguing indirectly, we suppose that there exist $\lambda_n \longrightarrow 0$, $\delta_n \longrightarrow 0$ and $(t_n, s_n) \in A$ satisfying

$$E_{\lambda_n}(\widehat{\gamma}(t_n, s_n)) \ge \widehat{c}_{\lambda_n} - \delta_n,\ \widehat{\gamma}(t_n, s_n) \in \mathscr{A}^{d/2}. \tag{83}$$

We may suppose $(t_n, s_n) \longrightarrow (\hat{t}, \hat{s}) \in A$. Then, from Lemma 13 and (83), we deduce

$$E_0\left(\widehat{\gamma}\left(\hat{t}, \hat{s}\right)\right) \ge \lim_{n\longrightarrow\infty} \left(\widehat{c}_{\lambda_n} - \delta_n\right) = B_1 + B_2. \tag{84}$$

Recalling the definition of $\widehat{\gamma}(t,s)$, we have $(\widehat{t},\widehat{s})=(1,1)$, and so

$$\lim_{n\longrightarrow\infty}\|\widehat{\gamma}(t_n,s_n)-\widehat{\gamma}(1,1)\|=\lim_{n\longrightarrow\infty}\|\widehat{\gamma}(t_n,s_n)-(U,V)\|=0, \tag{85}$$

which is in contradiction with $\widehat{\gamma}(t_n,s_n)\in\mathscr{A}^{d/2}$.

For δ and $\widehat{\lambda}$ given in Lemma 16, we define

$$\delta_0:=\min\left\{\frac{\delta}{2},\frac{B_1}{4},\frac{1}{8}da^2\right\}. \tag{86}$$

Then,

$$|\widehat{c}_\lambda-m_\lambda|<\delta_0,|\widehat{c}_\lambda-(B_1+B_2)|<\delta_0,\ \forall\lambda\in(0,\lambda^*), \tag{87}$$

for some $\lambda^*\in(0,\widehat{\lambda}]$.

Lemma 17. *Assume that (H1) and (H2) hold. For fixed* $\lambda\in(0,\lambda^*)$, *there is* $\{(u_n,v_n)\}\subset\mathscr{A}^d\cap E_\lambda^{m_\lambda}$ *with*

$$E_\lambda{}'(u_n,v_n)\longrightarrow 0, as\ n\longrightarrow\infty. \tag{88}$$

Proof. For $\lambda\in(0,\lambda^*)$, suppose contradictorily that $\|E_\lambda{}'(u,v)\|\geq l(\lambda)$ for all $(u,v)\in\mathscr{A}^d\cap E_\lambda^{m_\lambda}$ and some $0<l(\lambda)<1$. Then, there is a pseudogradient vector field h_λ for E_λ on neighborhood S_λ of $\mathscr{A}^d\cap E_\lambda^{m_\lambda}$ such that

$$\begin{aligned}&\|h_\lambda(u,v)\|\leq 2\min\left\{1,\left\|E_\lambda{}'(u,v)\right\|\right\},\\&\left\langle E_\lambda{}'(u,v),h_\lambda(u,v)\right\rangle\geq\min\left\{1,\left\|E_\lambda{}'(u,v)\right\|\right\}\left\|E_\lambda{}'(u,v)\right\|.\end{aligned} \tag{89}$$

Define a function η_λ on H satisfying $0\leq\eta_\lambda\leq 1$, $\eta_\lambda\equiv 1$ on $\mathscr{A}^d\cap E_\lambda^{m_\lambda}$ and $\eta_\lambda\equiv 0$ on $H\setminus S_\lambda$, and a function ζ_λ on $\mathbb{R}$ with $0\leq\zeta_\lambda(t)\leq 1$, $\zeta_\lambda(t)\equiv 1$ if $|t-\widehat{c}_\lambda|\leq\delta/2$, and $\zeta_\lambda(t)\equiv 0$ if $|t-\widehat{c}_\lambda|\geq\delta$. Then both η_λ and ζ_λ are Lipschitz continuous. Set

$$g_\lambda(u,v)=\begin{cases}-\eta_\lambda(u,v)\zeta_\lambda(E_\lambda(u,v))h_\lambda(u,v),\ (u,v)\in S_\lambda,\\0,\ (u,v)\in H\setminus S_\lambda.\end{cases} \tag{90}$$

Then, the initial problem

$$\begin{cases}\dfrac{d}{d\theta}\psi_\lambda(u,v,\theta)=g_\lambda(\psi_\lambda(u,v,\theta)),\\\psi_\lambda(u,v,0)=(u,v),\end{cases} \tag{91}$$

has a global solution ψ_λ on $H\times[0,+\infty)$ with the properties:

(i) $\psi_\lambda(u,v,\theta)=(u,v)$ if $\theta=0$ or $(u,v)\in H\setminus S_\lambda$ or $|E_\lambda(u,v)-\widehat{c}_\lambda|\geq\delta$

(ii) $\|(d/d\theta)\psi_\lambda(u,v,\theta)\|\leq 2$

(iii) $(d/d\theta)E_\lambda(\psi_\lambda(u,v,\theta))=\langle E_\lambda{}'(\psi_\lambda(u,v,\theta)),g_\lambda(\psi_\lambda(u,v,\theta))\rangle\leq 0$

Now, the proof can be divided into two steps.

Step 1. We show that there exists $\theta_0\geq 0$ such that

$$\psi_\lambda(\widehat{\gamma}(t,s),\theta_0)\in E_\lambda^{\widehat{c}_\lambda-\delta_0} \tag{92}$$

for $(t,s)\in A$ and δ_0 defined in (86).

Arguing indirectly, we suppose

$$E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta))>\widehat{c}_\lambda-\delta_0,\ \forall\theta\geq 0, \tag{93}$$

for some $(t,s)\in A$. Noting $\delta_0<\delta$ and applying Lemma 16, we get $\widehat{\gamma}(t,s)\in\mathscr{A}^{d/2}$. By property (iii) and the fact that

$$E_\lambda(\widehat{\gamma}(t,s))\leq m_\lambda<\widehat{c}_\lambda+\delta_0, \tag{94}$$

we have

$$\widehat{c}_\lambda-\delta_0<E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta))\leq m_\lambda<\widehat{c}_\lambda+\delta_0,\forall\theta\geq 0. \tag{95}$$

Then, $\zeta_\lambda(E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta)))\equiv 1$. If $\psi_\lambda(\widehat{\gamma}(t,s),\theta))\in\mathscr{A}^d$ for all $\theta\geq 0$, then

$$\eta_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta)))\equiv 1,\|E_\lambda{}'(\psi_\lambda(\widehat{\gamma}(t,s),\theta))\|\geq l(\lambda),\forall\theta>0. \tag{96}$$

Thus,

$$E_\lambda\left(\psi_\lambda\left(\widehat{\gamma}(t,s),\frac{\delta}{l(\lambda)^2}\right)\right)\leq\widehat{c}_\lambda+\frac{\delta}{2}-\int_0^{\delta/l(\lambda)^2}l^2(\lambda)d\theta\leq\widehat{c}_\lambda-\frac{\delta}{2}, \tag{97}$$

which contradicts (93). Hence, $\psi_\lambda(\widehat{\gamma}(t,s),\theta_0)\in\mathscr{A}^d$ for some $\theta_0>0$. Observing that $\widehat{\gamma}(t,s)\in\mathscr{A}^{d/2}$, we deduce that for some θ_1,θ_2 with $0<\theta_1<\theta_2\leq\theta_0$, it holds $\psi_\lambda(\widehat{\gamma}(t,s),\theta_1)\in\partial\mathscr{A}^{d/2}$, $\psi_\lambda(\widehat{\gamma}(t,s),\theta_2)\in\partial\mathscr{A}^d$, and $\psi_\lambda(\widehat{\gamma}(t,s),\theta)\in\mathscr{A}^d\setminus\mathscr{A}^{d/2}$ for all $\theta\in(\theta_1,\theta_2)$. Then, according to Lemma 15, we get

$$\left\|E_\lambda'(\psi_\lambda(\widehat{\gamma}(t,s),\theta))\right\|\geq a,\forall\theta\in(\theta_1,\theta_2). \tag{98}$$

Moreover, we deduce from property (ii) that

$$\frac{d}{2}\leq\|\psi_\lambda(\widehat{\gamma}(t,s),\theta_1)-\psi_\lambda(\widehat{\gamma}(t,s),\theta_2)\|\leq 2|\theta_1-\theta_2|, \tag{99}$$

which yields $|\theta_1 - \theta_2| \geq d/4$. Then, we obtain

$$E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta_2)) \leq E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta_1)) + \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} E_\lambda(\psi_\lambda(u,v,\theta)) \leq \widehat{c}_\lambda + \delta_0 - a^2(\theta_2 - \theta_1) \leq \widehat{c}_\lambda + \delta_0 - \frac{1}{4} da^2, \leq \widehat{c}_\lambda - \delta_0, \tag{100}$$

which contradicts (93), and the proof of this step is complete.

By step 1, we define

$$G(t,s) \coloneqq \inf\{\theta \geq 0 : E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta)) \leq \widehat{c}_\lambda - \delta_0\} \tag{101}$$

and $\gamma(t,s) \coloneqq \psi_\lambda(\widehat{\gamma}(t,s), G(t,s))$. Obviously, $E_\lambda(\gamma(t,s)) \leq \widehat{c}_\lambda - \delta_0$ for all $(t,s) \in A$.

Step 2. We prove

$$\gamma(t,s) \in \widehat{\Gamma}. \tag{102}$$

Noting that

$$E_\lambda(\widehat{\gamma}(t,s)) \leq E_0(\widehat{\gamma}(t,s)) = J_1(\widehat{\gamma}_1(t)) + J_2(\widehat{\gamma}_2(s)) \leq \frac{B_1}{4} + B_2 \leq B_1 + B_2 - 3\delta_0 < \widehat{c}_\lambda - \delta_0, \tag{103}$$

for all $(t,s) \in \mathscr{A} \setminus (t_1,t_2) \times (s_1,s_2)$, we have $G(t,s) = 0$ and $\gamma(t,s) = \widehat{\gamma}(t,s)$.

It remains to show $\|\gamma(t,s)\| \leq 2a_2 + a_0$, $\forall (t,s) \in A$, and $G(t,s)$ is a continuous function of $(t,s) \in A$. For any $(t,s) \in A$, if $E_\lambda(\widehat{\gamma}(t,s)) \leq \widehat{c}_\lambda - \delta_0$, then $G(t,s) = 0$ and $\gamma(t,s) = \widehat{\gamma}(t,s)$, so $\|\gamma(t,s)\| = \|\widehat{\gamma}(t,s)\| \leq 2a_2 + a_0$. If $E_\lambda(\widehat{\gamma}(t,s)) > \widehat{c}_\lambda - \delta_0$, then $\widehat{\gamma}(t,s) \in \mathscr{A}^{d/2}$ and

$$\widehat{c}_\lambda - \delta_0 < E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta)) \leq m_\lambda < \widehat{c}_\lambda + \delta_0, \forall \theta \in [0, G(t,s)]. \tag{104}$$

So we get $\zeta_\lambda(E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta))) \equiv 1$ for $\theta \in [0, G(t,s)]$. Then, we can prove that

$$\gamma(t,s) = \psi_\lambda(\widehat{\gamma}(t,s), G(s,t))) \in \mathscr{A}^d. \tag{105}$$

Indeed, if not, by similar arguments as in step 1, we know that $E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s),\theta_2)) \leq \widehat{c}_\lambda - \delta_0$ for some θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < G(t,s)$, which is in contradiction with the definition of $G(t,s)$. Thus, $\gamma(s,t) \in \mathscr{A}^d$ and

$$\|\gamma(s,t) - (u,v)\| \leq d \leq \frac{a_0}{2}, \tag{106}$$

for some $(u,v) \in \mathscr{A}$. Hence, from Lemma 12,

$$\|\gamma(s,t)\| \leq \|(u,v)\| + \frac{a_0}{2} \leq 2a_2 + a_0. \tag{107}$$

Now, we prove that $G(t,s)$ is continuous. For fixed $(\bar{t},\bar{s}) \in A$, if $E_\lambda(\gamma(\bar{t},\bar{s})) < \widehat{c}_\lambda - \delta_0$, then $G(\bar{t},\bar{s}) = 0$ and $E_\lambda(\widehat{\gamma}(\bar{t},\bar{s})) < \widehat{c}_\lambda - \delta_0$. Since $\widehat{\gamma}$ is continuous, we have

$$E_\lambda(\widehat{\gamma}(t,s)) < \widehat{c}_\lambda - \delta_0, \forall (t,s) \in (\bar{t}-\tau, \bar{t}+\tau) \times (\bar{s}-\tau, \bar{s}+\tau) \cap A, \tag{108}$$

for some $\tau > 0$, which implies $G(t,s) = 0$, $\forall (t,s) \in (\bar{t}-\tau, \bar{t}+\tau) \times (\bar{s}-\tau, \bar{s}+\tau) \cap A$. Thus, the continuity of G at $(\bar{t},\bar{s})$ is proved. If $E_\lambda(\gamma(\bar{t},\bar{s})) = \widehat{c}_\lambda - \delta_0$, then we see from the proof previously that $\gamma(\bar{t},\bar{s}) = \psi_\lambda(\widehat{\gamma}(\bar{t},\bar{s}), G(\bar{t},\bar{s})) \in \mathscr{A}^d$, so

$$\left\|E_\lambda{}'(\psi_\lambda(\widehat{\gamma}(\bar{t},\bar{s}), G(\bar{t},\bar{s})))\right\| \geq l(\lambda) > 0, \tag{109}$$

and $E_\lambda(\psi_\lambda(\widehat{\gamma}(\bar{t},\bar{s}), G(\bar{t},\bar{s}) + \omega)) < \widehat{c}_\lambda - \delta_0$ for any $\omega > 0$. By the continuity of ψ_λ, we obtain $E_\lambda(\psi_\lambda(\widehat{\gamma}(t,s), G(\bar{t},\bar{s}) + \omega)) < \widehat{c}_\lambda - \delta_0$, $\forall (t,s) \in (\bar{t}-\tau, \bar{t}+\tau) \times (\bar{s}-\tau, \bar{s}+\tau) \cap A$ for some $\tau > 0$. Therefore, $G(t,s) \leq G(\bar{t},\bar{s}) + \omega$ and

$$0 \leq \limsup_{(t,s) \longrightarrow (\bar{t},\bar{s})} G(t,s) \leq G(\bar{t},\bar{s}). \tag{110}$$

If $G(\bar{t},\bar{s}) = 0$, then

$$\lim_{(t,s) \longrightarrow (\bar{t},\bar{s})} G(t,s) = G(\bar{t},\bar{s}). \tag{111}$$

If $G(\bar{t},\bar{s}) > 0$, then $E_\lambda(\psi_\lambda(\widehat{\gamma}(\bar{t},\bar{s}), G(\bar{t},\bar{s}) - \omega)) > \widehat{c}_\lambda + \delta_0$ for any ω with $0 < \omega < G(\bar{t},\bar{s})$. Then, since ψ_λ is continuous, we deduce

$$\liminf_{(t,s) \longrightarrow (\bar{t},\bar{s})} G(t,s) \geq G(\bar{t},\bar{s}). \tag{112}$$

Combining with (110), we obtain the continuity of $G(s,t)$ at $(\bar{t},\bar{s})$. Consequently, (102) holds.

By Step 1 and Step 2, we have showed that $\gamma(t,s) \in \widehat{\Gamma}$ and $\max_{(t,s)\in A} E_\lambda(\gamma(t,s)) \leq \widehat{c}_\lambda - \delta_0$, which is in contradiction with the definition of $\widehat{c}_\lambda$. Thus, the conclusion holds.

Proof of Theorem 18. Denote $d_0 \coloneqq 1/2((2(N+\alpha))/\alpha B_1)^{1/2}$. From Lemma 17, we obtain that there exists $\{(u_n^\lambda, v_n^\lambda)\} \subset \mathscr{A}^d$ such that

$$E_\lambda\left(u_n^\lambda, v_n^\lambda\right) \leq m_\lambda, E_\lambda'\left(u_n^\lambda, v_n^\lambda\right) \longrightarrow 0, \tag{113}$$

for fixed $\lambda \in (0, \lambda^*)$, where $\lambda^* \in (0, \inf_{x\in\mathbb{R}^N} \sqrt{V_1(x)V_2(x)})$. Then, by Lemma 14, $(u_n^\lambda, v_n^\lambda) \rightharpoonup (u_\lambda, v_\lambda)$ in H for some $(u_\lambda, v_\lambda) \in \mathscr{A}^{2d}$ and $E_\lambda{}'(u_\lambda, v_\lambda) = 0$. Moreover, recalling the definition of d, we have $u_\lambda \neq 0$, $v_\lambda \neq 0$, that is, (u_λ, v_λ) is a vector solution of system (1.1).

Now, choosing $\{\lambda_n\} \subset (0, \lambda^*)$ such that $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$, by a repeat of the proof in Lemma 14, we obtain $(u_{\lambda_n}, v_{\lambda_n}) \longrightarrow (\bar{u}, \bar{v}) \in \mathscr{A}$ in H, with $\bar{u}$ and $\bar{v}$ being ground states of (11) and (12), respectively, which completes the proof.

5. Proof of Theorem 20

Lemma 19. *Let $N \geq 3$, $V_i(x) \in W^{1,1}_{loc}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$, $i = 1, 2$, and $(u, v) \in H$ be a solution of system (1). If*

$$\sup_{x \in \mathbb{R}^N} |\nabla V_i(x) \cdot x| < \infty, i = 1, 2, \tag{114}$$

then (u, v) satisfies the Pohozaev identity

$$\begin{aligned} \frac{N-2}{2} \int \left(|\nabla u|^2 + |\nabla v|^2 \right) + \frac{N}{2} \int \Big(& V_1(x)|u|^2 \\ & + V_2(x)|v|^2 - 2\lambda uv \Big) + \frac{1}{2} \int \left(\nabla V_1(x) \cdot x |u|^2 + \nabla V_2(x) \cdot x |v|^2 \right) \\ & = \frac{N}{2} \int \Big(I_\alpha * |u|^{(N+\alpha)/N} |u|^{(N+\alpha)/N} \\ & + I_\alpha * |v|^{(N+\alpha)/N} |v|^{(N+\alpha)/N} \Big). \end{aligned} \tag{115}$$

Proof. The lemma can be proved by a similar argument as that in ([17], Proposition 11).

Proof of Theorem 20. Let (u, v) be a solution of system (1). By Lemma 19, we have

$$\int \left(|\nabla u|^2 + |\nabla v|^2 \right) = \frac{1}{2} \int \left(\nabla V_1(x) \cdot x |u|^2 + \nabla V_2(x) \cdot x |v|^2 \right). \tag{116}$$

Then, the conclusion follows from a classical Hardy inequality (see ([31], Theorem 6.4.10))

$$\frac{(N-2)^2}{4} \int \frac{|u|^2}{|x|^2} \leq \int |\nabla u|^2. \tag{117}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that she has no conflicts of interest in relation to this article.

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