Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 8868043, 7 pages
https://doi.org/10.1155/2020/8868043

Research Article

An Extended b-Metric-Type Space and Related Fixed Point Theorems with an Application to Nonlinear Integral Equations

Kushal Roy,[1] Sayantan Panja,[1] Mantu Saha,[1] and Vahid Parvaneh[2]

[1]*Department of Mathematics, The University of Burdwan, Purba Bardhaman, 713104 West Bengal, India*
[2]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*

Correspondence should be addressed to Vahid Parvaneh; zam.dalahoo@gmail.com

Received 1 September 2020; Revised 7 October 2020; Accepted 20 October 2020; Published 17 November 2020

Academic Editor: Su Yan

In this paper, the concept of sequential p-metric spaces has been introduced as a generalization of usual metric spaces, b-metric spaces and specially of p-metric spaces. Several topological properties of such spaces have been discussed here. In view of this notion, we prove fixed point theorems for some classes of contractive mappings over such spaces. Supporting examples have been given in order to examine the validity of the underlying space and in respect to our proven fixed point theorems.

1. Introduction and Preliminaries

In the last few decades, several generalizations of usual metric structure have been made by the researchers working in the area of fixed point theory. Different topological structured spaces play vital roles for obtaining fixed point theorems using several contractive or expansive or non-expansive type mappings. There are many interesting two-variable metric-type spaces such as b-metric space [1, 2], rectangular metric space [3], extended b-metric space [4], p-metric space [5], JS-metric space [6], modular metric space [7], multiplicative metric space [8], bipolar metric space [9, 10], cone metric space [11], and C^*-algebra-valued metric space [12]. Also, for considering and analyzing more generalizations of the concept of metric spaces, one can consider the following works dealing with (double) controlled metric spaces and generalized b-metric spaces [13–16]. Due to the presence of such interesting spaces and various types of applications of fixed point theorems therein, fixed point theory gains attention in the mathematical community specially to the new researchers working on functional analysis. In the context of various metric-type spaces which are the combination of the above-mentioned spaces, several authors have proved different types of fixed point theorems therein (see [17, 18]). Now, we give definitions of some generalized spaces which are relevant to our research work.

Definition 1. b-metric space (see [1, 2]). Let Λ be a nonempty set and s be a real number satisfying $s \geq 1$. A function $\rho_b : \Lambda \times \Lambda \longrightarrow \mathbb{R}^+$ is a b -metric on Λ if

(1) $\rho_b(\iota, \kappa) = 0$ if and only if $\iota = \kappa$,

(2) $\rho_b(\iota, \kappa) = \rho_b(\kappa, \iota)$ for all $\iota, \kappa \in \Lambda$,

(3) $\rho_b(\iota, z) \leq s[\rho_b(\iota, \kappa) + \rho_b(\kappa, z)]$ for all $\iota, \kappa, z \in \Lambda$.

The space (Λ, ρ_b) is called a b-metric space.

Let Λ be a nonempty set and $\rho_g : \Lambda \times \Lambda \longrightarrow [0,\infty]$ be a mapping. For any $\iota \in \Lambda$, let us define the set

$$C\left(\rho_g, \Lambda, \iota\right) = \left\{\{\iota_n\} \subset \Lambda : \lim_{n\to\infty} \rho_g(\iota_n, \iota) = 0\right\}. \quad (1)$$

Definition 2. JS-metric space (see [6]). Let $\rho_g : \Lambda \times \Lambda \longrightarrow [0,\infty]$ be a mapping which satisfies

(1) $\rho_g(\iota, \kappa) = 0$ implies $\iota = \kappa$,

(2) for every $\iota, \kappa \in \Lambda$, we have $\rho_g(\iota, \kappa) = \rho_g(\kappa, \iota)$,

(3) if $(\iota, \kappa) \in \Lambda \times \Lambda$ and $\{\iota_n\} \in C(\rho_g, \Lambda, \iota)$, then $\rho_g(\iota, \kappa) \leq p \limsup_{n\to\infty} \rho_g(\iota_n, \kappa)$, for some $p > 0$.

The pair (Λ, ρ_g) is called a generalized metric space, usually known as a JS-metric space.

Definition 3. p-metric space (see [5]). Let Λ be a nonempty set. A function $\rho_p : \Lambda \times \Lambda \longrightarrow [0,\infty)$ is said to be p -metric if there exists a strictly increasing continuous function $\Omega : [0,\infty) \longrightarrow [0,\infty)$ with $\Omega^{-1}(t) \le t \le \Omega(t)$ for all $t \ge 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$ such that for all $\iota, \kappa, z \in \Lambda$

(1) $\rho_p(\iota, \kappa) = 0$ if and only if $\iota = \kappa$,

(2) $\rho_p(\iota, \kappa) = \rho_p(\kappa, \iota)$,

(3) $\rho_p(\iota, z) \le \Omega(\rho_p(\iota, \kappa) + \rho_p(\kappa, z))$.

The pair (Λ, ρ_p) is called a p-metric space.

For various examples of the above spaces, one can see the research papers in the reference section of this manuscript regarding such notions and also the references cited in these papers.

Now, we are ready to prove our main results. We define a new metric-type structure, which is the main concept of this paper.

2. Introduction to Sequential p-Metric Space

In this section, we introduce a new type of extended b-metric spaces. To develop such a notion, first, we define $S(\sigma, \Lambda, a) \coloneqq \{\{a_n\} \subset \Lambda : \lim_{n\to\infty} \sigma(a_n, a) = 0\}$, where $\sigma : \Lambda \times \Lambda \longrightarrow [0,\infty]$ is a given mapping.

Definition 4. Let Λ be a nonempty set. A mapping $\sigma : \Lambda \times \Lambda \longrightarrow [0,\infty]$ is said to be *a* sequential p -metric if for all $a, b \in \Lambda$

(a) $\sigma(a, b) = 0$ implies $a = b$,

(b) $\sigma(a, b) = \sigma(b, a)$,

(c) $\sigma(a, b) \le \Omega(\limsup_{n\to\infty} \sigma(a_n, b))$, where $\{a_n\} \in S(\sigma, \Lambda, a)$ and $\Omega : [0,\infty] \longrightarrow [0,\infty]$ is a strictly increasing continuous function with $\Omega^{-1}(t) \le t \le \Omega(t)$ for all $0 \le t < \infty$ with $\Omega^{-1}(s) = s = \Omega(s)$ for $s \in \{0,\infty\}$.

The triplet $(\Lambda, \sigma, \Omega)$ is called a sequential p-metric space. We express a sequential p-metric space simply as (Λ, σ).

Example 1. Let $\Lambda = N$ and the metric $\sigma : \Lambda^2 \longrightarrow [0,\infty)$ be defined by

$$\begin{cases} \sigma(1,1) = 0, \\ \sigma(n,n) = e - 1, \text{ for } n \ge 2, \\ \sigma(1,n) = \sigma(n,1) = e^{1/(n+1)} - 1, \text{ for } n \ge 2, \\ \sigma(n,m) = \sigma(m,n) = e^{mn} - 1, \text{ for all } n, m \ge 2 \text{ with } n \ne m. \end{cases} \quad (2)$$

Then, clearly, $\sigma(m, n) = 0$ implies $m = n$ and $\sigma(m, n) = \sigma(n, m)$ for all $m, n \in \Lambda$. Now, we show that σ satisfies condition (iii) of Definition 4.

For $n \ge 2$, $S(\sigma, \Lambda, n) = \varnothing$. Let $\{n_k\} \in S(\sigma, \Lambda, 1)$. If all but finitely many terms of $\{n_k\}$ are 1 then we are done. So, suppose that $\{n_k\}$ only have finitely many 1's. Without loss of generality, we can exclude such 1's and then we get $\limsup_{k\to\infty} \sigma(m, n_k) = \lim_{k\to\infty} [e^{mn_k} - 1] = \infty$. Therefore, $\sigma(1, m) \le \limsup_{k\to\infty} \sigma(m, n_k)$ for all $m \ge 2$.

Hence, σ is a sequential p-metric on Λ for $\Omega_1(t) = t$ for all $t \ge 0$ and $\Omega_2(t) = e^t - 1$ for all $t \ge 0$.

Proposition 5. *If (Λ, σ) is a JS -metric space (see Definition 2) then σ is also a sequential p -metric on Λ.*

Proof. If (Λ, σ) is a JS-metric space then σ clearly satisfies the first two conditions of Definition 4. We just show that σ also satisfies the third condition of Definition 4.

Since σ is a JS-metric then for all $a, b \in \Lambda$ and for any sequence $\{a_n\} \in S(\sigma, \Lambda, a)$, we have $\sigma(a, b) \le \kappa \limsup_{n\to\infty} \sigma(a_n, b)$, where $\kappa > 0$.

Then, if we choose $\Omega(t) = st$ for all $t \in [0,\infty]$ with $s \ge \max\{1, \kappa\}$, then we have $\sigma(a, b) \le \Omega(\limsup_{n\to\infty} \sigma(a_n, b))$ for all $a, b \in \Lambda$ and $\{a_n\} \in S(\sigma, \Lambda, a)$. Therefore, (Λ, σ) is also a sequential p-metric space.

Remark 6. . (i) We know that any metric space, $b-$ metric space [1, 2], dislocated metric space, and modular metric space with the Fatou property [7] are JS -metric spaces; therefore, these spaces are also sequential p -metric spaces. (ii) Any extended b-metric space or p-metric space is clearly a sequential p-metric space.

Proposition 7. *Let (Λ, σ)be aJS -metric space with coefficient $\kappa \ge 1$. Let $\sigma'(a, b) \coloneqq \Gamma(\sigma(a, b))$, where Γ is a strictly increasing continuous function with $t \le \Gamma(t)$ for all $t \ge 0$ and $\Gamma(0) = 0$. Then, σ' is a sequential p -metric for $\Omega(t) = \Gamma_\kappa(t) = \Gamma(\kappa t)$ for all $t \ge 0$.*

Proof. Here, we show that σ' satisfies all the conditions of Definition 4.

(a) $\sigma'(a, b) = 0$ gives $\Gamma(\sigma(a, b)) = 0$. Then $\sigma(a, b) = \Gamma^{-1}(0) = 0$ implies $a = b$

(b) $\sigma'(a, b) = \sigma'(b, a)$ holds trivially

(c) For all $a, b \in \Lambda$ we have, $\sigma'(a, b) = \Gamma(\sigma(a, b)) \le \Gamma(\kappa \limsup_{n\to\infty} \sigma(a_n, b))$ where $\{a_n\} \in S(\sigma, \Lambda, a) = S(\sigma', \Lambda, a)$

Now, $\sigma(a_n, b) \le \Gamma(\sigma(a_n, b)) = \sigma'(a_n, b)$ for all $n \in N$. Then, $\limsup_{n\to\infty} \sigma(a_n, b) \le \limsup_{n\to\infty} \sigma'(a_n, b)$.

Therefore, $\sigma'(a,b) \leq \Gamma(\kappa \limsup_{n\to\infty} \sigma'(a_n,b)) = \Gamma_\kappa(\limsup_{n\to\infty} \sigma'(a_n,b))$. This proves our proposition.

Definition 8. Let (Λ,σ) be a sequential p -metric space. Also let $\{a_n\}$ be a sequence in Λ and $a \in \Lambda$.

(i) $\{a_n\}$ is said to be convergent and converges to a if $\{a_n\} \in S(\sigma,\Lambda,a)$,

(ii) $\{a_n\}$ is said to be Cauchy if $\lim_{n,m\to\infty} \sigma(a_n,a_m) = 0$,

(iii) Λ is called complete if any Cauchy sequence in Λ is convergent.

Definition 9. Let (Λ,σ) and (Δ,σ^*) be two sequential p -metric spaces. A mapping $T : \Lambda \longrightarrow \Delta$ is called continuous at a point $a \in \Lambda$ if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $\iota \in \Lambda, \sigma^*(T\iota, Ta) < \varepsilon$ whenever $\sigma(\iota,a) < \delta_\varepsilon . T$ is said to be continuous on Λ if T is continuous at each point of Λ.

Proposition 10. *In a sequential p -metric space (Λ,σ) if a sequence $\{a_n\}$ is convergent then it converges to a unique element in Λ.*

Proof. Suppose $a, b \in \Lambda$ be such that $a_n \longrightarrow a$ and $a_n \longrightarrow b$ as $n \longrightarrow \infty$. Then, we have, $\sigma(a,b) \leq \Omega(\limsup_{n\to\infty} \sigma(a_n,b))$ implying that $\sigma(a,b) \leq \Omega(0) = 0$, i.e., $a = b$.

Proposition 11. *Let (Λ,σ) be a sequential p -metric space and $\{a_n\} \subset \Lambda$ converges to some $a \in \Lambda$ then $\sigma(a,a) = 0$.*

Proof. Since $\{a_n\}$ converges to $a \in \Lambda$, so $\lim_{n\to\infty} \sigma(a_n,a) = 0$. Therefore, we have $\sigma(a,a) \leq \Omega(\limsup_{n\to\infty} \sigma(a_n,a)) = \Omega(0) = 0$ which implies $\sigma(a,a) = 0$.

Proposition 12. *Let $\{a_n\}$ be a Cauchy sequence in a sequential p -metric space (Λ,σ,Ω) such that Ω^{-1} is continuous. If $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ which converges to $a \in \Lambda$, then $\{a_n\}$ also converges to $a \in \Lambda$.*

Proof. From condition (c) of Definition 4, we have $\sigma(a_n,a) \leq \Omega(\limsup_{k\to\infty} \sigma(a_n,a_{n_k}))$ which implies that $\Omega^{-1}(\sigma(a_n,a)) \leq \limsup_{k\to\infty} \sigma(a_n,a_{n_k})$ for all $n \in N$.

Due to the Cauchyness of $\{a_n\}$, it follows that $\lim_{n,k\to\infty} \sigma(a_n,a_{n_k}) = 0$and thus $\Omega^{-1}(\sigma(a_n,a)) \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that $\sigma(a_n,a) \longrightarrow 0$ as $n \longrightarrow \infty$, since Ω^{-1} is continuous. Hence, $\{a_n\}$ converges to $a \in \Lambda$.

Proposition 13. *In a sequential p -metric space (Λ,σ), if a self-mapping T is continuous at $a \in \Lambda$ then $\{Ta_n\} \in S(\sigma,\Lambda, Ta)$ for any sequence $\{a_n\} \in S(\sigma,\Lambda,a)$.*

Proof. Let $\varepsilon > 0$ be given. Since T is continuous at a, then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\sigma(c,a) < \delta_\varepsilon$ implies $\sigma(Tc,Ta) < \varepsilon$.

As $\{a_n\}$ converges to a, so for $\delta_\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sigma(a_n,a) < \delta_\varepsilon$ for all $n \geq N$. Therefore, for any $n \geq N$, $\sigma(Ta_n,Ta) < \varepsilon$ and thus $Ta_n \longrightarrow Ta$ as $n \longrightarrow \infty$.

Some observations regarding sequential p-metric spaces are as follows:

(1) In a metric space, a convergent sequence is always Cauchy, but it is not true in a sequential p-metric space. In Example 1, the sequence $\{n\}_{n\geq 2}$ converges to 1, but $\sigma(n,m) = e^{nm} - 1 \not\longrightarrow 0$ whenever $n, m \longrightarrow \infty$

(2) In a metric space, if $\{a_n\}$ and $\{b_n\}$ are two sequences converging to a and b, respectively, then $\sigma(a_n,b_n) \longrightarrow \sigma(a,b)$ as $n \longrightarrow \infty$. In particular, if $a = b$ then $\sigma(a_n,b_n) \longrightarrow 0$ as $n \longrightarrow \infty$. But this does not always hold in a sequential p-metric space. In Example 1, let us consider two sequences $\{2n\}_{n\geq 1}$ and $\{2n+1\}_{n\geq 1}$ in Λ. Then, both of these two sequences converge to $1 \in \Lambda$, but $\sigma(2n, 2n+1) \not\longrightarrow 0$ as $n \longrightarrow \infty$

(3) A p-metric is always a sequential p-metric space, but the converse is not true in general. The metric σ defined in Example 1 is not a p-metric for any Ω. Otherwise, $\sigma(n,m) \leq \Omega(\sigma(n,1) + \sigma(1,m))$ for all $n, m \geq 2$ with $n \neq m$, which at once implies that $e^{nm} - 1 \leq \Omega(e^{1/(n+1)} + e^{1/(m+1)} - 2)$ for all $n, m \geq 2$ with $n \neq m$, arrives at a contradiction

3. Cantor's Intersection-Like Theorem on Sequential p-Metric Spaces

Let (Λ,σ) be a sequential p-metric space with supporting function Ω. Define

$$\begin{aligned} \mathcal{B}(a,\eta) &:= \{b \in \Lambda : \sigma(a,b) < \sigma(a,a) + \eta\}, \\ \mathcal{B}[a,\eta] &:= \{b \in \Lambda : \sigma(a,b) \leq \sigma(a,a) + \eta\}, \end{aligned} \quad (3)$$

for all $a \in \Lambda$ and $\eta > 0$.

Remark 14. One can easily check that the collection $\tau_\sigma := \{\emptyset\} \cup \{\mathcal{U}(\neq \emptyset) \subset \Lambda$: for any $a \in U$, there exists $\eta > 0$ such that $\mathcal{B}(a,\eta) \subset \mathcal{U}\}$ forms a topology on Λ.

Definition 15. A set $\mathcal{F}$ is said to be closed if there exists an open set $\mathcal{U} \subset \Lambda$ such that $\mathcal{F} = \mathcal{U}^c$, where $\mathcal{U}^c$ denotes the complement of $\mathcal{U}$ in Λ.

Proposition 16. *Let (Λ,σ) be a sequential p -metric space and $\mathcal{F} \subset \Lambda$ be closed. Let $\{a_n\} \subset \mathcal{F}$ such that $a_n \longrightarrow a$ as $n \longrightarrow \infty$. Then, $a \in \mathcal{F}$.*

Proof. Suppose that $a \notin \mathcal{F}$. Then, $a \in \mathcal{F}^c$. Since $\mathcal{F}^c \in \tau_\sigma$, so there exists $\eta > 0$ such that $\mathcal{B}(a, \eta) \subset \mathcal{F}^c$. Again, since $\sigma(a_n, a) \longrightarrow 0$ as $n \longrightarrow \infty$, so for $\eta > 0$ there exists $N \in N$ such that $\sigma(a_n, a) < \sigma(a, a) + \eta$ for all $n \geq N$. That is, $a_n \in \mathcal{B}(a, \eta) \subset \mathcal{F}^c$ for all $n \geq N$, which is a contradiction. Hence, $a \in \mathcal{F}$.

Proposition 17. *Let (Λ, σ) be a complete sequential p -metric space and $\mathcal{F} \subset \Lambda$ be closed. Then, $(\mathcal{F}, \sigma)$ is also complete.*

Proof. Let $\{a_n\}$ be a Cauchy sequence in $\mathcal{F} \subset \Lambda$. Then, $\{a_n\}$ is convergent in Λ, since Λ is complete. Let $\lim_n a_n = a \in \Lambda$. Then, by Proposition 16, it follows that $a \in \mathcal{F}$. Consequently, $\mathcal{F}$ is complete.

Definition 18. In a sequential p -metric space (Λ, σ), for $\mathcal{A} \subset \Lambda$, we define

$$\operatorname{diam}(\mathcal{A}) \coloneqq \sup \{\sigma(a, b) \colon a, b \in \mathcal{A}\}. \tag{4}$$

Theorem 19. *Let (Λ, σ) be a complete sequential p -metric space and $\{\mathcal{F}_n\}$ be a decreasing sequence of nonempty closed subsets of Λ such that $diam(\mathcal{F}_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Then, the intersection $\cap_{n=1}^{\infty} \mathcal{F}_n$ contains exactly one point.*

Proof. Let $a_n \in \mathcal{F}_n$ be chosen as arbitrary for all $n \in \mathbb{N}$. Since $\{\mathcal{F}_n\}$ is decreasing, we have $\{a_n, a_{n+1}, \cdots\} \subset \mathcal{F}_n$ for all $n \in \mathbb{N}$.

Now, for any $n, m \in \mathbb{N}$ with $n, m \geq k$, we have $\sigma(a_n, a_m) \leq \operatorname{diam}(\mathcal{F}_k)$, $k \geq 1$. Let $\varepsilon > 0$ be given. Then, there exists some $l \in \mathbb{N}$ such that $\operatorname{diam}(\mathcal{F}_l) < \varepsilon$, since $\operatorname{diam}(\mathcal{F}_n) \longrightarrow 0$ as $n \longrightarrow \infty$. From this, it follows that $\sigma(a_n, a_m) < \varepsilon$ whenever $n, m \geq l$. Therefore, $\{a_n\}$ is Cauchy in Λ. By the completeness of Λ, there exists some $a \in \Lambda$ such that $\{a_n\}$ converges to a. Since $\{a_n, a_{n+1}, \cdots\} \subset \mathcal{F}_n$ and $\mathcal{F}_n$ is closed for each $n \in \mathbb{N}$, using Proposition 16, we have $a \in \cap_{n=1}^{\infty} \mathcal{F}_n$.

Next, we prove the uniqueness of the point a. Let $c \in \cap_{n=1}^{\infty} \mathcal{F}_n$ be another point, then $\sigma(a, c) > 0$. As $\operatorname{diam}(\mathcal{F}_n) \longrightarrow 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\operatorname{diam}(\mathcal{F}_n) < \sigma(a, c) \leq \operatorname{diam}(\mathcal{F}_n), \tag{5}$$

for all $n \geq N_0$, a contradiction. Hence, $\cap_{n=1}^{\infty} \mathcal{F}_n = \{a\}$ and this completes the proof of our theorem.

4. Some Fixed Point Theorems

Theorem 20. *Let (Λ, σ) be a complete sequential p -metric space and $Y : \Lambda \longrightarrow \Lambda$ be a mapping so that*

(i) $\sigma(Ya, Yb) \leq \alpha \sigma(a, b)$ for all $a, b \in \Lambda$ and for some $\alpha \in (0, 1)$

(ii) there exists $a_0 \in \Lambda$ such that $\delta(\sigma, Y, a_0) \coloneqq \sup \{\sigma(Y^i a_0, Y^j a_0) \colon i, j = 1, 2, \cdots\} < \infty$

Then, Y has at least one fixed point in Λ. Moreover, if a and b are two fixed points of Y in Λ with $\sigma(a, b) < \infty$ then $a = b$.

Proof. Let us define $\delta(\sigma, Y^{p+1}, a_0) \coloneqq \sup \{\sigma(Y^{p+i} a_0, Y^{p+j} a_0) \colon i, j = 1, 2, \cdots\}$, for all $p \geq 1$. Clearly, $\delta(\sigma, Y^{p+1}, a_0) \leq \delta(\sigma, Y, a_0) < \infty$ for all $p \geq 1$.

Then, for all $p \geq 1$ and for all $i, j = 1, 2, \cdots$,

$$\sigma\left(Y^{p+i} a_0, Y^{p+j} a_0\right) \leq \alpha \sigma\left(Y^{p-1+i} a_0, Y^{p-1+j} a_0\right) \leq \alpha \delta(\sigma, Y^p, a_0), \tag{6}$$

implies, for all $p \geq 1$,

$$\begin{aligned} \delta\left(\sigma, Y^{p+1}, a_0\right) &= \sup_{i,j \geq 1} \sigma\left(Y^{p+i} a_0, Y^{p+j} a_0\right) \\ &\leq \alpha \delta(\sigma, Y^p, a_0) \\ &\leq \alpha^2 \delta\left(\sigma, Y^{p-1}, a_0\right) \vdots \leq \alpha^p \delta(\sigma, Y, a_0). \end{aligned} \tag{7}$$

Denote $a_i = Y a_{i-1} = Y^i a_0$ for all $i \in \mathbb{N}$. For $1 \leq n < m$, we have

$$\begin{aligned} \sigma(a_n, a_m) = \sigma(Y^n a_0, Y^m a_0) &= \sigma\left(Y^{n-1+1} a_0, Y^{n-1+(m-n+1)} a_0\right) \\ &\leq \delta(\sigma, Y^n, a_0) \leq \alpha^{n-1} \delta(\sigma, Y, a_0) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{8}$$

Therefore, $\{a_n\}$ is a Cauchy sequence in Λ. Due to the completeness of Λ, $\{a_n\}$ is convergent and let $\lim_n a_n = a \in \Lambda$.

Now, $\sigma(Ya, Ya_n) \leq \alpha \quad \sigma(a, a_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $a_{n+1} \longrightarrow Ya$ as $n \longrightarrow \infty$. Hence, by Proposition 10, it follows that $Ya = a$, i.e., $a \in \Lambda$ is a fixed point of Y.

Now, if a and b are two fixed points of Y in Λ with $\sigma(a, b) < \infty$, then we have $\sigma(a, b) = \sigma(Ya, Yb) \leq \alpha \sigma(a, b)$ which gives $\sigma(a, b) = 0$ implies $a = b$.

Theorem 21. *Let (Λ, σ) be a complete sequential p-metric space and $Y : \Lambda \longrightarrow \Lambda$ such that*

(i) $\sigma(Ya, Yb) \leq \gamma[\sigma(a, Ya) + \sigma(b, Yb)]$ for all $a, b \in \Lambda$ and for some $\gamma \in (0, 1/2)$,

(ii) there exists $a_0 \in \Lambda$ such that $\delta(\sigma, Y, a_0) \coloneqq \sup \{\sigma(Y^i a_0, Y^j a_0) \colon i, j = 1, 2, \cdots\} < \infty$

Then, the Picard iterating sequence $\{a_n\}$, $a_n = Y^n a_0$ for all $n \in \mathbb{N}$, converges to some $a \in \Lambda$. If $\sigma(a, Ya) < \infty$ and $\gamma t < \Omega^{-1}(t)$ for all $t > 0$, then $a \in \Lambda$ is a fixed point of Y. Moreover, if b is a fixed point of Y in Λ such that $\sigma(a, b) < \infty$ and $\sigma(b, b) < \infty$ then $a = b$.

Proof. For all $p \geq 1$ and for all $i, j = 1, 2, \cdots$,

$$\begin{aligned} \sigma\left(Y^{p+i} a_0, Y^{p+j} a_0\right) &\leq \gamma\left[\sigma\left(Y^{p-1+i} a_0, Y^{p+i} a_0\right) + \sigma\left(Y^{p-1+j} a_0, Y^{p+j} a_0\right)\right] \\ &\leq 2\gamma \delta(\sigma, Y^p, a_0). \end{aligned} \tag{9}$$

This implies that $\delta(\sigma, Y^{p+1}, a_0) = \sup_{i,j \geq 1} \sigma(Y^{p+i} a_0, Y^{p+j} a_0) \leq 2\gamma \delta(\sigma, Y^p, a_0)$ for all $p \geq 1$. Then, proceeding in a similar way as in Theorem 20, it can be easily shown that $\{a_n\}$ is a

Cauchy sequence in Λ, and by the completeness of Λ, there exists some $a \in \Lambda$ such that $\lim_n a_n = a$.

Now, $\sigma(a_{n+1}, Ya) = \sigma(Ya_n, Ya) \le \gamma[\sigma(a_n, a_{n+1}) + \sigma(a, Ya)]$ for all $n \ge 0$, which implies that $\limsup_{n\to\infty} \sigma(a_{n+1}, Ya) \le \gamma \sigma(a, Ya) < \infty$. Then,

$$\sigma(a, Ya) \le \Omega\left(\limsup_{n\to\infty} \sigma(a_{n+1}, Ya)\right) \le \Omega(\gamma\sigma(a, Ya)). \quad (10)$$

If $\sigma(a, Ya) > 0$ then $\Omega^{-1}(\sigma(a, Ya)) \le \gamma\sigma(a, Ya) < \Omega^{-1}(\sigma(a, Ya))$, a contradiction. Hence $Ya = a$, i.e., $a \in \Lambda$ is a fixed point of Y.

Now, if b is a fixed point of Y in Λ with $\sigma(a, b) < \infty$ and $\sigma(b, b) < \infty$, then we have $\sigma(a, b) = \sigma(Ya, Yb) \le \gamma[\sigma(a, Ya) + \sigma(b, Yb)] = 0$, as $\sigma(b, b) = 0$, implying that $a = b$.

Theorem 22. *Let (Λ, σ) be a complete sequential p -metric space and $Y : \Lambda \longrightarrow \Lambda$ be a mapping satisfying the following conditions:*

(i) $\sigma(Ya, Yb) \le \beta[\sigma(a, Yb) + \sigma(b, Ya)]$ for all $a, b \in \Lambda$ and for some $\beta \in (0, 1/2)$

(ii) there exists $a_0 \in \Lambda$ such that $\delta(\sigma, Y, a_0) := \sup\{\sigma(Y^i a_0, Y^j a_0) : i, j = 1, 2, \cdots\} < \infty$

Then, the Picard iterating sequence $\{a_n\}$, $a_n = Y^n a_0$ for all $n \ge 1$, converges to some $a \in \Lambda$. If $\limsup_{n\to\infty} \sigma(a_n, Ya) < \infty$ then $a \in \Lambda$ is a fixed point of Y. Also, if b is a fixed point of Y in Λ such that $\sigma(a, b) < \infty$ then $a = b$.

Proof. by similar argument as in Theorem 20, $\{a_n\}$ is a Cauchy sequence in Λ, and by completeness of Λ, it converges to an element say $a \in \Lambda$.

Now, for all $n \in \mathbb{N} \cup \{0\}$, $\sigma(a_{n+1}, Ya) = \sigma(Ya_n, Ya) \le \beta[\sigma(a_n, Ya) + \sigma(a_{n+1}, a)]$, which implies that $\limsup_{n\to\infty} \sigma(a_{n+1}, Ya) \le \beta \limsup_{n\to\infty} \sigma(a_n, Ya)$ and hence $\limsup_{n\to\infty} \sigma(a_n, Ya) = 0$. Therefore, $\sigma(a, Ya) \le \Omega(\limsup_{n\to\infty} \sigma(a_n, Ya)) = \Omega(0) = 0$, and consequently, $Ya = a$.

If b is a fixed point of Y in Λ with $\sigma(a, b) < \infty$, then we have $\sigma(a, b) = \sigma(Ya, Yb) \le \beta[\sigma(a, Yb) + \sigma(b, Ya)] = 2\beta\sigma(a, b)$ which implies $\sigma(a, b) = 0$ that is $a = b$.

Example 2. Consider $\Lambda = [0, 1]$ and $\sigma(a, b) = (|a| + |b|) + \ln(1 + |a| + |b|)$ for all $a, b \in \Lambda$. Then, σ forms a sequential p -metric on Λ with the function $\Omega(t) = t + \ln(1 + t)$ for all $t \ge 0$.

(i) Define $Y : \Lambda \longrightarrow \Lambda$ by $Ya = a/3$ for all $a \in \Lambda$. Then, Y satisfies all the conditions of Theorem 20 for $\alpha = 1/2$ and clearly Y has a unique fixed point $0 \in \Lambda$

(ii) Define, $Ya = \begin{cases} a/11 & \text{if } 0 \le a < 1/2 \\ a/12 & \text{if } 1/2 \le a \le 1 \end{cases}$. Then, Y satisfies all the conditions of Theorem 21 for $\gamma = 2/5$ and clearly Y has a unique fixed point $0 \in \Lambda$

5. An Application to Nonlinear Integral Equations

In this section, we discuss about the existence of solutions for nonlinear integral equations as an application of Theorem 20.

Let $\Lambda = C[a, b]$ be the set of all real valued continuous functions on $[a, b]$ and $\sigma : \Lambda^2 \longrightarrow [0,\infty)$ be defined by

$$\sigma(u, v) = \sinh\left(\max_{a\le t\le b} |u(t) - v(t)|^p\right) \text{for all } u, v \in \Lambda, \quad p \ge 1. \quad (11)$$

Then, (Λ, σ) is a complete sequential p-metric space with $\Omega(t) = \sinh(2^{p-1}t)$ for all $t \ge 0$. Now, let us consider the integral equation

$$u(t) = h(t) + \int_a^b F(t, s)\mathcal{K}(t, s, u(s))ds, \quad (12)$$

where $h : [a, b] \longrightarrow \mathbb{R}$, $F : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ and $\mathcal{K} : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

Theorem 23. *Assume that the following hypotheses are satisfied:*

(i) for all $t, s \in [a, b]$, we have

$$|\mathcal{K}(t, s, u(s)) - \mathcal{K}(t, s, v(s))|^p \le A\|u - v\|^p, p \ge 1, A > 0, \quad (13)$$

where $\|u - v\| = \max_{a\le t\le b} |u(t) - v(t)|$, $u, v \in \Lambda$,

(ii) $\max_{a\le t\le b} \int_a^b |F(t, s)|^p ds \le M$, where $M > 0$ is such that $AM(b - a) < 1$

Then, integral equation (12) has a unique solution $u \in \Lambda$.

Proof. Let us define $Y : \Lambda \longrightarrow \Lambda$ by

$$Y(u)(t) = h(t) + \int_a^b F(t, s)\mathcal{K}(t, s, u(s))ds, \text{u} \in \Lambda, \text{t, s} \in [\text{a, b}]. \quad (14)$$

Then, by conditions (i) and (ii), for all $\varphi, \psi \in \Lambda$, we get

$$\begin{aligned}
&|Y(\varphi)(t) - Y(\psi)(t)|^p \\
&= \left| \int_a^b F(t,s)\{\mathscr{K}(t,s,\varphi(s)) - \mathscr{K}(t,s,\psi(s))\} ds \right|^p \\
&\le \left(\int_a^b |F(t,s)|^q ds \right)^{p/q} \left(\int_a^b |\mathscr{K}(t,s,\varphi(s)) - \mathscr{K}(t,s,\psi(s))|^p ds \right) \\
&\le \mathrm{AM}(b-a) \|\varphi - \psi\|^p \\
&\le \mu \|\varphi - \psi\|^p \text{ for all } t \in [a,b],
\end{aligned} \tag{15}$$

where $\mu = \mathrm{AM}(b-a) < 1$.

Therefore, $\| Y(\varphi) - Y(\psi)\|^p \le \mu \|\varphi - \psi\|^p$. Thus,

$$\begin{aligned}
\sigma(Y(\varphi), Y(\psi)) &= \sinh \left(|Y(\varphi) - Y(\psi)||^p \right) \\
&\le \sinh \left(\mu \mid |\varphi - \psi||^p \right) \\
&\le \mu \sinh \left(|\varphi - \psi||^p \right) \\
&= \mu \quad \sigma(\varphi, \psi),
\end{aligned} \tag{16}$$

for $\mu \in (0,1)$ and for all $\varphi, \psi \in \Lambda$. Hence, the conditions of Theorem 20 are satisfied, and therefore, Y has a unique fixed point in Λ, provided $\delta(\sigma, Y, u_0) < \infty$ for some $u_0 \in \Lambda$, i.e., nonlinear integral equation (12) has a unique solution in $C[a,b]$.

Now, we give a numerical example in support of Theorem 23.

Example 3. Let us consider the complete sequential p -metric space (Λ, σ) defined in Theorem 23 for $a = 0$, $b = 1$, $p = 2$, and the nonlinear integral equation given by

$$u(t) = e^t + \int_0^1 \sqrt{ts} \left(t + s + \sqrt{\frac{2}{3}} \quad u(s) \right) ds, t \in [0,1]. \tag{17}$$

Then, for all $u, v \in \Lambda$, we have

$$\begin{aligned}
&|\mathscr{K}(t,s,u(s)) - \mathscr{K}(t,s,v(s))|^2 \le \frac{2}{3} ||u - v||^2, \text{for all } t, s \in [0,1], \\
&\max_{0 \le t \le 1} \int_0^1 |F(t,s)|^2 ds = \frac{1}{2}.
\end{aligned} \tag{18}$$

Therefore, Y satisfies the contractive condition of Theorem 23 for $\mu = 1/3$, and also, we see that for $u_0 \in \Lambda$ defined by $u_0(t) = 0$ for all $t \in [a,b]$, $\sigma(Y^i u_0, Y^j u_0) \le \sinh [36(e + (16/15))^2] < \infty$ for all $i, j \ge 1$. Hence, all the conditions of Theorem 20 are satisfied and therefore (17) has a unique solution in Λ.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgments

The first and second authors acknowledge the financial support awarded by the Council of Scientific and Industrial Research, New Delhi, India, through research fellowship for carrying out research work leading to the preparation of this manuscript.

References

[1] I. A. Bakhtin, "The contraction mapping principle in quasi-metric spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[2] S. Czerwik, "Contraction mappings in b – metric spaces," *Acta mathematica et informatica universitatis ostraviensis*, vol. 1, pp. 5–11, 1993.

[3] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Universitatis Debreceniensis*, vol. 57, pp. 31–37, 2000.

[4] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of *b* -metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, 2017.

[5] V. Parvaneh and S. J. H. Ghoncheh, "Fixed points of $(\psi, \phi)_\Omega$-contractive mappings in ordered p-metric spaces," *Global Analysis and Discrete Mathematics*, vol. 4, no. 1, pp. 15–29, 2020.

[6] M. Jleli and B. Samet, "A generalized metric space and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 61, 2015.

[7] W. M. Kozlowski, "Moduler Function Spaces," in *Monographs and Textbooks in Pure and Applied Mathematics*, vol. 122, Dekker, New York, 1988.

[8] A. E. Bashirov, E. M. Kurpınar, and A. Özyapıcı, "Multiplicative calculus and its applications," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 36–48, 2008.

[9] A. Mutlu and U. Gürdal, "Bipolar metric spaces and some fixed point theorems," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 9, pp. 5362–5373, 2016.

[10] A. Mutlu, K. Özkan, and U. Gürdal, "Coupled fixed point theorems on bipolar metric spaces," *European journal of pure and applied Mathematics*, vol. 10, no. 4, pp. 655–667, 2017.

[11] L. G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[12] Z. Ma, L. Jiang, and H. Sun, "C∗-algebra-valued metric spaces and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2014, no. 1, Article ID 206, 2014.

[13] E. Karapinar, S. Czerwik, and H. Aydi, "(α, ψ)-Meir-Keeler contraction mappings in Generalized *b*-Metric spaces," *Journal of Function spaces*, vol. 2018, Article ID 3264620, 4 pages, 2018.

[14] T. Abdeljawad, N. Mlaiki, H. Aydi, and N. Souayah, "Double controlled metric type spaces and some fixed point results," *Mathematics*, vol. 6, no. 12, 2018.

[15] V. Parvaneh and Z. Kadelburg, "Extended partial b-metric spaces and some fixed point results," *Filomat*, vol. 32, no. 8, pp. 2837–2850, 2018.

[16] N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad, "Controlled metric type spaces and the related contraction principle," *Mathematics*, vol. 6, no. 10, 2018.

[17] D. Dey, K. Roy, and M. Saha, "On generalized contraction principles over S-metric spaces with application to homotopy," *Journal of New Theory*, vol. 31, pp. 95–103, 2020.

[18] K. Roy, I. Beg, and M. Saha, "Sequentially compact S^{JS}-metric spaces," *Communications in Optimization Theory*, vol. 2020, article 4, pp. 1–7, 2020.