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Research Article

Pullback Attractors for Nonautonomous Degenerate Kirchhoff Equations with Strong Damping

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In this paper, we obtain the existence of pullback attractors for nonautonomous Kirchhoff equations with strong damping, which covers the case of possible generation of the stiffness coefficient. For this purpose, a necessary method via "the measure of noncompactness" is established.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following Kirchhoff wave model with strong damping:

$$\begin{cases} u_{tt} - \Delta u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + f(u) = h(x,t), & \text{in } \Omega \times (\tau, \infty), \\ u|_{\partial\Omega} = 0, u(x,\tau) = u_\tau^0(x), u_t(x,\tau) = u_\tau^1(x), & x \in \Omega, \tau \in \mathbb{R}, \end{cases} \quad (1)$$

where $h(x,t)$ is a time-dependent external force term, u_τ^0 and u_τ^1 are initial data, and ϕ and f are nonlinear functions specified later.

To describe small vibrations of an elastic stretched string, Kirchhoff [1] introduced the equation

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + g, \quad (2)$$

where $u = u(x,t)$ is the lateral deflection, $0 < x < L$ the space coordinate, $t \geq 0$ the time, E the Young's modulus, ρ the mass density, h the cross-section area, L the length, p_0 the initial axial tension, and g the external force. It has been called the Kirchhoff equation since then. In general, we call the Kirchhoff equation nondegenerate if the stiffness ϕ satisfies the strict hyperbolicity condition $\phi(s) \geq c > 0$ and degenerate if $\phi(s) \geq 0$ on $\mathbb{R}^+$. Obviously, the degenerate stiffness coefficient $\phi(s)$ in (1) corresponds to the case that the initial axial tension equals zero.

From the mathematical point of view, global existence of the model like (2) has been proven in a multitude of special situations in $\Omega \subset \mathbb{R}^n$. We refer to [2–5] for the analytic data, [6–9] for the dispersive equations and small data, and [10–15] for the weak damped equations.

Introducing the strong damping term $-\Delta u_t$ provides an additional a priori estimate. Certainly, from the physical point of view, the dissipative plays an important spreading role for the energy gathered arising from the nonlinearity in a real process. Concerning Kirchhoff equations with strong dissipation, the first result on the well-posedness we are aware of was obtained by Nishihara [16]. He proved the global existence of the solution for the model $u_{tt} - \Delta u_t - m(\|\nabla u\|)\Delta u = 0$. In recent years, many mathematicians and physicists paid their attentions to this type of problem and obtained the well-posedness under different types of hypotheses, such as the absent source term [17] and the subcritical source term [18–23]. In general, the exponent $p^* = n + 2/(n-2)^+$ is called to be critical when someone studies the problem in $H_0^1(\Omega)) \times L^2(\Omega)$. Assuming the stiffness factor is nondegenerate $(\phi(s) \geq \phi_0 > 0)$, References [18–24] also proved the existence of the attractor. In the case of possible degeneration of the stiffness coefficient and the case of supercritical source term ($p^* < p < (n+4)/(n-4)^+$), the first result on the well-posedness we are aware of is given by Chueshov [25]. However, when he proved the existence of a global attractor for problem (1) in the natural energy space $(H_0^1(\Omega) \cap L^{p+1}(\Omega)) \times L^2(\Omega)$ endowed with a partially strong topology (in the sense, if $(u_0^n, u_1^n) \longrightarrow (u_0, u_1)$ with a partially strong topology, then $(u_0^n, u_1^n) \longrightarrow (u_0, u_1)$ strongly in $H_0^1 \times L^2$ and $u_0^n \rightharpoonup u_0$ weakly in L^{p+1}), he assumed that

$$\phi(s) > 0, \quad \forall s \geq 0, \phi \in C^1(\mathbb{R}^+). \tag{3}$$

Under this condition, one can conclude that $\phi(\|\nabla u(t)\|^2) \geq c_0 > 0$ if $\|\nabla u(t)\|$ is bounded for $t \in \mathbb{R}^+$. Recently, Ma et al. [26] proved the existence of the global attractor in the case of degeneration for the autonomous Kirchhoff system.

The pullback attractor is a basic concept to study the longtime dynamics of nonautonomous evolution equations (see [27–32] and references therein). It is worth mentioning that there are only a few recent results devoted to the pullback attractor for nonautonomous systems like (1). In 2013, Wang and Zhong [33] investigated the upper semicontinuity of pullback attractors for problem (1) with $\phi(s) = 1 + \varepsilon s$ ($\varepsilon > 0$) and $|f'(u)| \leq C(|u|^{2/(n-4)^+} + 1)(n \geq 3)$. Recently, Li and Yang [34] studied the robustness of pullback attractors with $\phi'(s) \geq 0, \phi(0) = \phi_0 > 0$. We notice that all these publications assume that the stiffness factor is nondegenerate, or more precisely, $\phi(0) > 0$ and ϕ is nondecreasing.

In this paper, we consider the problem (1) under the degenerate hyperbolicity condition $\phi(s) \geq 0$. We do not assume that ϕ is monotone and allow $\phi(0) = 0$, such as $\phi(s) = bs^\gamma$ (degenerate and monotone) or $\phi(s) = (1 + \sin^2 s)s^\gamma$ (degenerate and nonmonotone) with $\gamma \geq 1$. Based on the result in [25, 26], we prove the existence of pullback attractors in $H_0^1(\Omega)) \times L^2(\Omega)$ if ϕ is really degenerate. To overcome the difficulties caused by the degeneration, we first established a method (condition ($\mathscr{D}$-PC)) via "the measure of noncompactness" (some ideas come from [35, 36]) to prove that the process is pullback $\mathscr{D}$-asymptotically compact.

The paper is organized as follows. In Section 2, we introduce some preliminaries and establish a necessary abstract result (see Theorem 5). In Section 3, we discuss the existence of pullback attractors for the equation (1) (see Theorem 12).

2. Preliminaries

In this section, we will give some notations and results. As usual, we denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the norm and the inner product in $L^2(\Omega)$, respectively. Let $\mathscr{H} = H_0^1(\Omega) \times L^2(\Omega)$. We define the norms in $\mathscr{H}$ by $\|u_0, u_1\|_{\mathscr{H}}^2 = \|\nabla u_0\|^2 + \|u_1\|^2$.

Let X be a Banach space and $U(t, \tau)$ be a process acting on X. In the following, we recall some definitions and results related to the pullback attractors; more details can be found in [27, 29, 33].

Definition 1. A family of compact sets $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ is said to be a pullback attractor for process $U(\cdot, \cdot)$ if

(i) $\mathscr{A}$ is invariant, that is, $U(t, \tau)A(\tau) = A(t)$, for all $t \geq \tau$

(ii) $\mathscr{A}$ is pullback attracting, i.e., $d(U(t, t-\tau)B, A(t)) \longrightarrow 0$, as $\tau \longrightarrow +\infty$, for all bounded subset B of X, where $d(B, A)$ is the Hausdorff semidistance

Definition 2. A family of sets $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}}$ is said to be a pullback absorbing family for process $U(\cdot, \cdot)$, if for all $t \in \mathbb{R}$ and all bounded $B \subset X$, there exists $T = T(t, B) > 0$, such that $U(t, t-\tau)B \subset D(t)$, for all $\tau \geq T$. In addition, the family $\mathscr{D}$ is said to be pullback $\mathscr{D}$-absorbing, if for any $t \in \mathbb{R}$, there exists $T_t > 0$ such that $U(t, t-\tau)D(t-\tau) \subset D(t)$ for $\tau \geq T_t$.

Definition 3. A process $U(\cdot, \cdot)$ is said to be pullback $\mathscr{D}$-asymptotically compact in X, if for any $t \in \mathbb{R}$, any sequences $\tau_n \longrightarrow \infty$ and $x_n \in D(t-\tau_n)$; the sequence $\{U(t, t-\tau_n)x_n\}_{n\in\mathbb{N}}$ is relatively compact in X.

Lemma 4 (see [29]). *Let the family $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}}$ be pullback absorbing and $U(\cdot, \cdot)$ be continuous and pullback $\mathscr{D}$-asymptotically compact in X. Then, the family $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ defined by*

$$A(t) = \bigcap_{s\geq 0} \overline{\bigcup_{\tau\geq s} U(t, t\tau)D(t\tau)}, \tag{4}$$

is a pullback attractor for $U(\cdot, \cdot)$.

To verify the pullback $\mathscr{D}$-asymptotically compact property in X, it suffices to check the following condition.

2.1. $\mathscr{D}$-Pullback Condition ($\mathscr{D}$-PC). For any $\delta > 0$ and $t \in \mathbb{R}$, there exist $\tau_0 = \tau_0(t, \mathscr{D}, \delta) > 0$ and a finite dimensional space X_1 of X such that

$$\begin{aligned} &\{P(U(t, t-\tau_0)D(t-\tau_0))\} \text{ is bounded}, \\ &\quad \|(I-P)(U(t, t-\tau_0)x)\|_X < \delta, \quad \forall x \in D(t-\tau_0), \end{aligned} \tag{5}$$

where $P : X \longrightarrow X_1$ is a bounded projector.

Theorem 5. *Let the family $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}}$ be a pullback $\mathscr{D}$-absorbing family of the process $U(t, \tau)$. If the $\mathscr{D}$-pullback condition ($\mathscr{D}$ -PC) holds, then $U(\cdot, \cdot)$ is pullback $\mathscr{D}$-asymptotically compact in X.*

Proof. By Definition 3, the result will be proven if we can show that for any $t \in \mathbb{R}$, any sequences $\tau_n \longrightarrow \infty$ and $x_n \in D(t-\tau_n)$, $\{U(t, t-\tau_n)x_n\}_{n\in\mathbb{N}}$ is relatively compact in X.

For every $\delta > 0$, condition ($\mathscr{D}$-PC) implies that there exist $\tau_0 = \tau_0(t, \mathscr{D}, \delta) > 0$ and the finite dimensional space X_1, such that (5) holds. Then, we have

$$\begin{aligned} \gamma(U(t, t-\tau_0)D(t-\tau_0)) &\leq \gamma[P(U(t, t-\tau_0)D(t-\tau_0))] \\ &\quad + \gamma[((I-P)(U(t, t-\tau_0)D(t-\tau_0)))] \leq \gamma(N(0, \delta)) \leq 2\delta, \end{aligned} \tag{6}$$

where γ is the measure of noncompactness defined as

$$\gamma(B) = \inf \{\delta > 0 \mid B \text{ admits a finite cover by sets whose diameter} \le \delta\}. \tag{7}$$

On the other hand, the properties of $\mathscr{D}$ give that there exists $T_{t-\tau_0} > 0$, such that for $\tau \ge T_{t-\tau_0}$, $U(t-\tau_0, t-\tau_0-\tau) D(t-\tau_0-\tau) \subset D(t-\tau_0)$ and

$$\begin{aligned} &U(t, t-\tau_0-\tau)D(t-\tau_0-\tau) \\ &\quad = U(t, t-\tau_0)U(t-\tau_0, t-\tau_0-\tau)D(t-\tau_0-\tau) \\ &\quad \cdot \subset U(t, t-\tau_0)D(t-\tau_0). \end{aligned} \tag{8}$$

Then, we can find N_0, such that $\gamma(\bigcup_{n>N_0} U(t, t-\tau_n)x_n)) \le 2\delta$, which means that $\{U(t, t-\tau_n)x_n\}_{n\in\mathbb{N}}$ has a finite 4δ -net for any $\delta > 0$. The proof is complete. □

3. Existence of Pullback Attractors

In this section, we will prove the existence of the pullback attractor when $\phi(s)$ is really degenerate and $f(u)$ is subcritical. We assume that f, ϕ, and h satisfy the following conditions.

Assumption 6. The function $\phi \in C^1(\mathbb{R}^+)$, $\phi(s) \ge \min \{L_1 s^\alpha, L_2\}$ for $s \in \mathbb{R}^+$, and some constants $\alpha \ge 0$, $L_1, L_2 > 0$. Moreover, there exists $\delta_0 > 0$ such that

$$\liminf_{s\longrightarrow+\infty} (s\phi(s) - \delta_0 \Phi(s)) > -\infty, \tag{9}$$

where $\Phi(s) = \int_0^s \phi(t)\mathrm{d}t$.

Assumption 7. $f(u)$ is a C^1 function, $f(0) = 0$, $f'(s) \ge -c_1$, and $s \in \mathbb{R}$,

$$\mu_f = \lim_{|s|\longrightarrow+\infty} \inf \frac{f(s)}{s} > -\lambda_1 \phi_\infty \quad \text{with } \phi_\infty = \lim_{s\longrightarrow+\infty} \inf \phi(s), \tag{10}$$

and the following properties hold:

(i) if $n = 1$, then f is arbitrary

(ii) if $n = 2$, then

$$|f'(u)| \le C(1+|u|^{p-1}) \text{ with } 1 \le p < \infty, \tag{11}$$

(iii) if $n \ge 3$, then

$$|f'(u)| \le C(1+|u|^{p-1}) \text{ with } 1 \le p < p_* = \frac{n+2}{n-2}, \tag{12}$$

where c_1 and C are positive constants and λ_1 is the first eigenvalue of $-\Delta$.

Assumption 8. $h, \partial_t h \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ and

$$\int_{-\infty}^t \|h(\cdot, s)\|^2 \mathrm{d}s < +\infty, \quad \forall t \in \mathbb{R}. \tag{13}$$

Remark 9. (1) $\phi(s) = L_1 s^\alpha$ or $\widetilde{\phi}(s) = (1 + \sin^2 s)s^\alpha (\alpha \ge 1)$ satisfies Assumption 6. It indicates that we include into the consideration the case of possibly degenerate ϕ since $\phi(0) = 0$. Moreover, because $\phi_\infty = +\infty$ in this case, $\mu_f > -\lambda_1\phi_\infty$ becomes $\mu_f > -\infty$. If $\alpha = 0$, then $\phi(s)$ is a constant, and equation (1) is the nonlinear wave equation with strong damping.

(2) Assumptions 6 and 7 imply that there exist constants $c_0 > 0$, $\theta_1 > 0$ with $0 < \phi_1 < \phi_\infty$, $0 < \phi_1\lambda_1 - \theta_1 < <1$ such that

$$\Phi(s) \ge \phi_1 \cdot s - c_0\phi_1 \quad \forall s \in \mathbb{R}^+, \tag{14}$$

$$\begin{aligned} F(s) &\ge -\frac{\theta_1}{2}s^2 - c_2, f(s)s \ge -\theta_1 s^2 - c_2, f(s)s - F(s) \\ &\ge -\frac{c_1}{2}s^2, \quad \forall s \in \mathbb{R}, \end{aligned} \tag{15}$$

where $F(s) = \int_0^s f(t)\mathrm{d}t$.

The well-posedness of the problem

$$\begin{cases} \partial_{tt}u - \sigma(\|\nabla u\|^2)\Delta\partial_t u - \phi(\|\nabla u\|^2)\Delta u + f(u) = h(x), & \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, u(x,0) = u^0(x), u_t(x,0) = u^1(x), & x \in \Omega, \end{cases} \tag{16}$$

has been established by Chueshov [25] in the autonomous case. Noticing that the conditions of ϕ, f are more general than the above Assumptions 6–8, we can obtain the following Proposition 10 by a similar argument as in [25], except for the treatment of $h(x,t)$. The reader is referred to the Appendix for a detailed proof of these facts.

Proposition 10. *Let Assumptions 6–8 be in force. Then, for $\tau, T \in \mathbb{R} (\tau < T)$ and $(u^0_\tau, u^1_\tau) \in \mathscr{H}$, problem (1) has a unique weak solution u with $(u, u_t) \in C([\tau, T]; \mathscr{H})$ and*

(1) for every $t \in [\tau, T]$, there exists $C = C_{R,\tau,T} > 0$ such that

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \int_\tau^t \|\nabla u_t(s)\|^2 ds \le C, \tag{17}$$

$$\begin{aligned} &E(u(t), u_t(t)) + 2\int_s^t (\|\nabla u_t(r)\|^2 - (h, u_t))dr \\ &\quad = E(u(s), u_t(s)), \tau \le s < t, \end{aligned} \tag{18}$$

where $E(u_0, u_1) = \|u_1\|^2 + \Phi(\|\nabla u_0\|^2) + 2\int_\Omega F(u_0)dx$, $\|(u^0_\tau, u^1_\tau)\|_{\mathscr{H}} \le R$

(2) for every $t \in (\tau, T]$, there exists $K = K_{R,\tau,T} > 0$ such that

$$\begin{aligned} &\|u_{tt}(t)\|^2_{H^{-1}} + \|\nabla u_t(t)\|^2 \\ &\quad \le K\left(1 + \frac{1}{(t-\tau)^2}\right)\left(1 + \int_\tau^T (\|h(\cdot, s)\|^2 + \|h_t(\cdot, s)\|^2)ds\right). \end{aligned} \tag{19}$$

(3) the Lipschitz stability

$$\|(z(t), z_t(t)\|_{\mathscr{H}}^2 \le K\|(z(\tau), z_t(\tau)\|_{\mathscr{H}}^2, \tag{20}$$

holds for $z(t) = u^1(t) - u^2(t)$, *where* u^1, u^2 *are two weak solutions of problem (1) with initial data* $(u_{i,\tau}^0, u_{i,\tau}^1)$, $\|(u_{i,\tau}^0, u_{i,\tau}^1)\|_{\mathscr{H}} \le R, i = 1, 2$.

We define the solution operator $U(t, \tau)$: $\mathscr{H} \longrightarrow \mathscr{H}$ associated to problem (1) as

$$U(t, \tau)(u_\tau^0, u_\tau^1) = (u(t), u_t(t)), \quad \forall t \ge \tau, \tau \in \mathbb{R}, \tag{21}$$

where u is the weak solution of problem (1) corresponding to initial data $(u_\tau^0, u_\tau^1) \in \mathscr{H}$. Then, we know from Proposition 10 that $U(t, \tau)$: $\mathscr{H} \longrightarrow \mathscr{H}$ is a continuous evolution process. For convenience, we denote by $\xi_u(t) = (u(t), u_t(t))$ for any function $u(t)$. As $(u(\tau), u_t(\tau)) = (u_\tau^0, u_\tau^1)$, we also denote (u_τ^0, u_τ^1) by $\xi_u(\tau)$.

Lemma 11. *Let Assumptions 6–8 be valid. Then, the process* $U(\cdot, \cdot)$ *defined in (21) has a pullback* $\mathscr{D}$*-absorbing family* $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}}$. *Moreover,* $D(t)$ *is bounded in* $\mathscr{H}_1 = H_0^1(\Omega) \times H_0^1(\Omega)$ *for every* $t \in \mathbb{R}$.

Proof. As usual, the argument below can be justified by considering Galerkin approximations. Using the multiplier $u_t + \eta u$ in Equation (1), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} W^\eta(\xi_u(t)) + K(\xi_u(t)) = 0, \quad t \ge \tau, \tag{22}$$

where

$$\begin{aligned} W^\eta(\xi_u(t)) &= \|u_t\|^2 + \Phi(\|\nabla u\|^2) + 2(F(u), 1) \\ &\quad + \eta[\|\nabla u\|^2 + 2(u_t, u)] \ge (1-\eta)\|u_t\|^2 \\ &\quad + \phi_1 \cdot \|\nabla u\|^2 - c_0\phi_1 - \theta_1\|u\|^2 - 2c_2 \cdot mes\,\Omega \\ &\quad + \eta\|\nabla u\|^2 - \eta\|u\|^2 \ge \kappa\|\xi_u(t)\|_{\mathscr{H}}^2 - C_3, \end{aligned} \tag{23}$$

$$\begin{aligned} K(\xi_u(t)) &= 2\|\nabla u_t\|^2 - 2\eta\|u_t\|^2 + 2\eta[\phi(\|\nabla u\|^2)\|\nabla u\|^2 \\ &\quad + (f(u), u)] - 2(h, u_t + \eta u), \end{aligned} \tag{24}$$

for $\eta > 0$ which is small enough, $\kappa > 0$ is a positive constant, and κ, C_3 are independent of $\xi_u(t)$.

Since Assumption 6 implies that there exists $L_3 > 0$ such that

$$\delta_0\Phi(\|\nabla u\|^2) \le \phi(\|\nabla u\|^2)\|\nabla u\|^2 + L_3, \tag{25}$$

combining with (15), we have that

$$\begin{aligned} W^\eta(\xi_u(t)) &\le (1+\eta)\|u_t\|^2 + \Phi(\|\nabla u\|^2) + 2(f(u), u) \\ &\quad + c_1\|u\|^2 + \eta[\|\nabla u\|^2 + \|u\|^2 \le \kappa_1\|\xi_u(t)\|_{\mathscr{H}}^2 + \Phi(\|\nabla u\|^2) \\ &\quad + 2(f(u), u) \le \kappa_1\|\xi_u(t)\|_{\mathscr{H}}^2 + \Phi(\|\nabla u\|^2) + C_3\|u\|_{L^{p+1}}^{p+1} + C_4, \end{aligned} \tag{26}$$

$$\begin{aligned} K(\xi_u(t)) &\ge \|\nabla u_t\|^2 + (\lambda_1 - 2\eta)\|u_t\|^2 + (2\eta - \varepsilon)\phi(\|\nabla u\|^2)\|\nabla u\|^2 \\ &\quad + \varepsilon(\delta_0\Phi(\|\nabla u\|^2) - L_3) + 2\eta(f(u), u) - \delta\|u_t\|^2 - \delta\eta^2\|u\|^2 \\ &\quad - \frac{2}{\delta}\|h(\cdot, t)\|^2. \end{aligned} \tag{27}$$

Then, we can find $\eta > 0, \varepsilon > 0, \delta > 0$ small enough such that

$$\begin{aligned} K(\xi_u(t)) - \delta W^\eta(\xi_u(t)) &\ge \|\nabla u_t\|^2 + (\lambda_1 - 2\eta - \delta(1+\eta) - \delta)\|u_t\|^2 \\ &\quad + (2\eta - \varepsilon)\phi(\|\nabla u\|^2)\|\nabla u\|^2 + \varepsilon\delta_0\Phi(\|\nabla u\|^2) \\ &\quad - \varepsilon L_3 + 2(\eta - \delta)(f(u), u) - c_1\delta\|u\|^2 - \delta\Phi(\|\nabla u\|^2) \\ &\quad - \delta\eta\|\nabla u\|^2 - \delta(\eta^2 + \eta)\|u\|^2 - \frac{2}{\delta}\|h(\cdot, t)\|^2 \ge \|\nabla u_t\|^2 \\ &\quad + (2\eta - \varepsilon)(\phi_1 \cdot \|\nabla u\|^2 - c_0\phi_1) + (\varepsilon\delta_0 - \delta)\Phi(\|\nabla u\|^2) \\ &\quad - 2\eta\frac{\theta_1}{\lambda_1}\|\nabla u\|^2 - \delta\eta\|\nabla u\|^2 - \delta(c_1 + \eta^2 + \eta)\|u\|^2 - \frac{2}{\delta}\|h(\cdot, t)\|^2 \\ &\quad - C_4 \ge \|\nabla u_t\|^2 - C(1 + \|h(\cdot, t)\|^2). \end{aligned} \tag{28}$$

By (22) and (28), we get that

$$\frac{\mathrm{d}}{\mathrm{d}t} W^\eta(\xi_u(t)) + \delta W^\eta(\xi_u(t)) + \|\nabla u_t\|^2 \le C(1 + \|h(\cdot, t)\|^2). \tag{29}$$

According to the Gronwall inequality, we have

$$W^\eta(\xi_u(t)) \le W^\eta(\xi_u(\tau))\mathrm{e}^{-\delta(t-\tau)} + C\left(1 + \int_\tau^t \|h(\cdot, s)\|^2 \mathrm{d}s\right). \tag{30}$$

Then, (23), (26), and $H_0^1(\Omega)^\circ L^{p+1}(\Omega)$ yield that

$$\begin{aligned} \|\xi_u(t)\|_{\mathscr{H}}^2 &\le \frac{1}{\kappa}\left(W^\eta(\xi_u(\tau))\mathrm{e}^{-\delta(t-\tau)} + C\left(1 + \int_\tau^t \|h(\cdot, s)\|^2 \mathrm{d}s\right)\right) \\ &\le \left(\frac{\kappa_1}{\kappa}\|\xi_u(\tau)\|_{\mathscr{H}}^2 + \frac{1}{\kappa}\Phi(\|\nabla u\|^2) + \frac{C_3}{\kappa}\|u\|_{L^{p+1}}^{p+1} + \frac{C_4}{\kappa}\right)\mathrm{e}^{-\delta(t-\tau)} \\ &\quad + C\left(1 + \int_\tau^t \|h(\cdot, s)\|^2 \mathrm{d}s\right) \le C\Big(\|\xi_u(\tau)\|_{\mathscr{H}}^2 + \Phi(\|\nabla u\|^2) \\ &\quad + \|\nabla u\|^{p+1}\Big)\mathrm{e}^{-\delta(t-\tau)} + C\left(1 + \int_\tau^t \|h(\cdot, s)\|^2 \mathrm{d}s\right) \\ &\le C\left(\xi_u(\tau)_{\mathscr{H}}^2 + \Phi(\|\xi_u(\tau)\|_{\mathscr{H}}^2) + \left(\|\xi_u(\tau)\|_{\mathscr{H}}^2\right)^{(p+1)/2}\right)\mathrm{e}^{-\delta(t-\tau)} \\ &\quad + C\left(1 + \int_\tau^t \|h(\cdot, s)\|^2 \mathrm{d}s\right) \triangleq Q(\|\xi_u(\tau)\|_{\mathscr{H}}^2)\mathrm{e}^{-\delta(t-\tau)} \\ &\quad + C\left(1 + \int_\tau^t \|h(\cdot, s)\|^2 \mathrm{d}s\right), \end{aligned} \tag{31}$$

where $Q(x) = C(x + \Phi(x) + x^{(p+1)/2}) > 0$ is a monotone positive function on $\mathbb{R}^+$. Let

$$D_0(t) = \{\xi \in \mathscr{H} \mid \|\xi\|_{\mathscr{H}} \le R(t)\}, \text{ with } R^2(t) = 2C\left(1+\|h\|^2_{L^2(-\infty,t;L^2)}\right), \quad t \in \mathbb{R}. \tag{32}$$

Obviously, $\mathscr{D}_0 = \{D_0(t)\}_{t\in\mathbb{R}}$ is a pullback absorbing family of the process $U(t,\tau)$ in $\mathscr{H}$. Moreover, for every $t \in \mathbb{R}$, there exists a $T_t > 0$ such that

$$U(t, t-\tau)D_0(t-\tau) \subset D_0(t), \quad U(t-1, t-\tau)D_0(t-\tau) \subset D_0(t-1), \quad \text{for } \tau \ge T_t. \tag{33}$$

Let $D(t) = \overline{\bigcup_{\tau\ge T_t} U(t, t\tau)D_0(t\tau)}^{\mathscr{H}}$. By a standard procedure (see, e.g., Theorem 3.1 of [34]), we know that $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}}$ is a pullback absorbing family. Moreover, $D(t)$ is bounded in $\mathscr{H}_1$ for every $t \in \mathbb{R}$, and there exists a $T_t > 0$ such that $U(t, t-\tau)D(t-\tau) \subset D(t)$ for $\tau \ge T_t$. □

For simplicity, we assume that $\alpha > 0$ and $L_1 = L_2 = 1$ in the following.

Theorem 12. *Let Assumptions 6–8 be in force. Then, the process $U(\cdot,\cdot)$ possesses a pullback attractor $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ as shown in (4). Moreover, $A(t)$ is bounded in $\mathscr{H}_1$ for every $t \in \mathbb{R}$.*

Proof. According to Lemma 4, Theorem 5, Lemma 11, and the continuity of $U(t,\tau)$: $\mathscr{H} \longrightarrow \mathscr{H}$, it suffices to show that $U(t,\tau)$ satisfies the condition ($\mathscr{D}$-PC). Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis and $\{\lambda_j\}_{j=1}^\infty$ be the corresponding eigenvalues of $L^2(\Omega)$ which consists of eigenvectors of $-\Delta$, i.e., $-\Delta e_j = \lambda_j e_j, j \in \mathbb{N}$. Let $V_m \times W_m = \text{span}\{e_1,\cdots,e_m\} \times \text{span}\{e_1,\cdots,e_m\}$ in $\mathscr{H}$ and $P_m = (P_m^1, P_m^2)$: $\mathscr{H} \longrightarrow V_m \times W_m$ be an orthogonal projector. Denote $Q_m = I - P_m$, $u = P_m^1 u + Q_m^1 u = u^1 + u^2$, and $\xi_u^\tau(t) = (u(t), u_t(t)) = U(t,\tau)(u_\tau^0, u_\tau^1)$ with $(u_\tau^0, u_\tau^1) \in D(\tau)$, $t \ge \tau$.

Let $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ be given. Without loss of generality, we assume $\varepsilon < 1/4$.

For every $\tau \ge 1$ and every $(u^0_{t_0-\tau}, u^1_{t_0-\tau}) \in D(t_0-\tau)$, let

$$(u, u_t)(t) = \xi_u^{t_0-\tau}(t) = U(t, t_0-\tau)\left(u^0_{t_0-\tau}, u^1_{t_0-\tau}\right) \in U(t, t_0-\tau) \cdot D(t_0-\tau) \subset U(t, t_0-\tau)D_0(t_0-\tau). \tag{34}$$

Denote $Z(t) = (1/2)(\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2)$. It is easy to see that

$$\begin{aligned} Z(t_0-\tau+1) &\le \frac{1}{2}\left(1+\frac{1}{\lambda_1}\right)\left\|\xi_u^{t_0-\tau}(t_0-\tau+1)\right\|^2_{\mathscr{H}} \\ &\le \frac{1}{2}\left(1+\frac{1}{\lambda_1}\right)Q\left(\left\|\xi_u^{t_0-\tau}(t_0-\tau)\right\|^2_{\mathscr{H}} \cdot \mathrm{e}^{-\delta} + C\left(1+\int_{-\infty}^{t_0-\tau+1}\|h(\cdot,s)\|^2\mathrm{d}s\right)\right). \end{aligned} \tag{35}$$

Since $\xi_u^{t_0-\tau}(t_0-\tau) = (u^0_{t_0-\tau}, u^1_{t_0-\tau}) \in D(t_0-\tau) \subset D_0(t_0-\tau)$, we find

$$\left\|\xi_u^{t_0-\tau}(t_0-\tau)\right\|_{\mathscr{H}} \le R(t_0-\tau) \le R(t_0), \quad \forall \tau \ge 0. \tag{36}$$

Thus,

$$\begin{aligned} Z(t_0-\tau+1) &\le \frac{1}{2}\left(1+\frac{1}{\lambda_1}\right)\left(Q(R^2(t_0))\mathrm{e}^{-\delta} + C\left(1+\int_{-\infty}^{t_0}\|h(\cdot,s)\|^2\mathrm{d}s\right)\right) \\ &\triangleq C_5(t_0), \quad \forall \tau \ge 1, \end{aligned} \tag{37}$$

where $C_5(t_0)$ is independent of τ. Then, there exists $\tau_0 > 1$ such that

$$Z(t_0-(\tau_0-1))\mathrm{e}^{-2\varepsilon^{2\alpha}(\tau_0-1)} < \frac{\varepsilon^2}{2}. \tag{38}$$

On the other hand, for every $(u^0_{t_0-\tau_0}, u^1_{t_0-\tau_0}) \in D(t_0-\tau_0)$, using (16) and (18), we get that

$$\|\nabla u(t)\|^2 + \|\nabla u_t(t)\|^2 + \|u_{tt}(t)\|^2_{H^{-1}} \le K_0, \quad \text{for } t \in [t_0-\tau_0+1, t_0], \tag{39}$$

where $(u(t), u_t(t)) = U(t, t_0-\tau_0)(u^0_{t_0-\tau_0}, u^1_{t_0-\tau_0})$. Using $H_0^1(\Omega)^\circ L^q(\Omega)$ $(2 \le q \le p^* = 2n/(n-2))$, one can find $M \ge 1$, $L_0 < t_0$ (without loss of generality, we assume $L_0 < 0$), such that for every $t \in [t_0-\tau_0+1, t_0]$,

$$\|f(u(t))\|_{L^{(p+1)/p}} + \left(\int_t^{t+1}\|u_{tt}(s)\|^2_{H^{-1}}\mathrm{d}s\right)^{1/2} + \left(\int_{-\infty}^{t_0}\|h(\cdot,s)\|^2\mathrm{d}s\right)^{1/2} < M, \int_{-\infty}^{L_0}\|h(\cdot,s)\|^2\mathrm{d}s < \frac{\varepsilon^2}{4}. \tag{40}$$

By the Sobolev embedding theorem, we know that the embedding $H_0^1(\Omega)^\circ L^2(\Omega)$ is compact. Then, the boundedness of $\{u(t), u_t(t)\}_{t\in[t_0-\tau_0+1,t_0]}$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ implies that $\{u(t), u_t(t)\}_{t\in[t_0-\tau_0+1,t_0]}$ is compact in $L^2(\Omega) \times L^2(\Omega)$. Therefore, for $\varepsilon_1 = \varepsilon^{2+2\alpha}/4M(1+\sqrt{t_0-L_0})$, there exists $m_0 \in \mathbb{Z}_+$, such that for every $t \in [t_0-\tau_0+1, t_0]$,

$$\left\|\left(u^2(t), u_t^2(t)\right)\right\|_{L^2\times L^2} < \varepsilon_1, \tag{41}$$

$$\|u^2\|_{L^{p+1}} \le \|u^2\|^\theta \cdot \|u^2\|^{1-\theta}_{L^{p^*}} \le C\|u^2\|^\theta \cdot \|\nabla u^2\|^{1-\theta} \le C_6\|u^2\|^\theta < \frac{\varepsilon_1^2}{M}, \tag{42}$$

where $u = P^1_{m_0} u + (I - P^1_{m_0})u \triangleq u^1 + u^2$, $u_t = P^2_{m_0} u_t + (I - P^2_{m_0})u_t \triangleq u^1_t + u^2_t$, and $1/(p+1) = \theta/2 + (1-\theta)/p^*$. □

Now, we will consider two situations. Without loss of generality, we assume $0 < \varepsilon < 1/3$.

Case 1. For every $(u^0_{t_0-\tau_0}, u^1_{t_0-\tau_0}) \in D(t_0 - \tau_0)$, the inequality

$$\|\nabla u(t)\| > \varepsilon, \tag{43}$$

holds for any $t \in [t_0 - \tau_0 + 1, t_0]$, where $(u, u_t)(t) = \xi_u^{t_0-\tau_0}(t)$.

Multiplying (1) by u^2, we have that

$$\begin{aligned}&\frac{d}{dt}\left((u^2_t, u^2) + \frac{1}{2}\left\|\nabla u^2\right\|^2\right) + \phi\left(\|\nabla u(t)\|^2\right)\left\|\nabla u^2\right\|^2 \\ &\quad \leq \left\|u^2_t\right\|^2 + \left(f(u), u^2\right) + \left(h(\cdot, t), u^2\right).\end{aligned} \tag{44}$$

Let $Y(t) = (u^2_t, u^2) + (1/2)\|\nabla u^2\|^2$. Since $\phi(\|\nabla u\|^2) \geq \min\{\|\nabla u\|^{2\alpha}, 1\} \geq \min\{\varepsilon^{2\alpha}, 1\} = \varepsilon^{2\alpha}$ in this case, the above inequality implies that

$$\begin{aligned}\frac{d}{dt}Y(t) + 2\varepsilon^{2\alpha}Y(t) &\leq 2\varepsilon^{2\alpha}\left(u^2_t, u^2\right) + \left\|u^2_t\right\|^2 + \left(f(u), u^2\right) \\ &\quad + \left(h(\cdot, t), u^2\right) \leq 2\varepsilon^{2\alpha} \cdot \left\|u^2_t\right\| \cdot \left\|u^2\right\| + \left\|u^2_t\right\|^2 \\ &\quad + \|f(u)\|_{L^{(p+1)/p}} \cdot \left\|u^2\right\|_{L^{p+1}} + \|h(\cdot, t)\| \cdot \left\|u^2\right\| \triangleq W^\varepsilon(t).\end{aligned} \tag{45}$$

By Gronwall's inequality, we obtain that

$$Y(t) \leq Y(t - (\tau - 1))e^{-2\varepsilon^{2\alpha}(\tau-1)} + e^{-2\varepsilon^{2\alpha}t}\int_{t-(\tau-1)}^{t} e^{2\varepsilon^{2\alpha}s}W^\varepsilon(s)ds. \tag{46}$$

Since

$$Y(t_0 - \tau + 1) \leq Z(t_0 - \tau + 1) \leq C_5(t_0), \tag{47}$$

(37) yields that

$$Y(t_0 - (\tau_0 - 1))e^{-2\varepsilon^{2\alpha}(\tau_0-1)} < \frac{\varepsilon^2}{2}. \tag{48}$$

Combining (45), we have

$$\begin{aligned}Y(t_0) &\leq Y(t_0 - \tau_0 + 1)e^{-2\varepsilon^{2\alpha}(\tau_0-1)} + e^{-2\varepsilon^{2\alpha}t_0}\int_{t_0-\tau_0+1}^{t_0} e^{2\varepsilon^{2\alpha}s}W^\varepsilon(s)ds \\ &\leq \frac{\varepsilon^2}{2} + e^{-2\varepsilon^{2\alpha}t_0}\int_{t_0-\tau_0+1}^{t_0} e^{2\varepsilon^{2\alpha}s}2\varepsilon_1^2\left(\varepsilon^{2\alpha} + 1\right)ds \\ &\quad + e^{-2\varepsilon^{2\alpha}t_0}\int_{t_0-\tau_0+1}^{t_0} e^{2\varepsilon^{2\alpha}s} \cdot \|h(\cdot, s)\| \cdot \left\|u^2(s)\right\|ds \\ &\leq \frac{\varepsilon^2}{2} + \left(1 + \frac{1}{\varepsilon^{2\alpha}}\right)\varepsilon_1^2 + e^{-2\varepsilon^{2\alpha}t_0}\int_{t_0-\tau_0+1}^{t_0} e^{2\varepsilon^{2\alpha}s} \cdot \|h(\cdot, s)\| \cdot \left\|u^2(s)\right\|ds.\end{aligned} \tag{49}$$

If $L_0 \leq t_0 - \tau_0 + 1$, by the Hölder inequality, we have that

$$\begin{aligned}I_1 &= e^{-2\varepsilon^{2\alpha}t_0}\int_{t_0-\tau_0+1}^{t_0} e^{2\varepsilon^{2\alpha}s} \cdot \|h(\cdot, s)\| \cdot \left\|u^2(s)\right\|ds \\ &\leq e^{-2\varepsilon^{2\alpha}t_0} \cdot e^{2\varepsilon^{2\alpha}t_0}\varepsilon_1 \cdot \sqrt{t_0 - L_0} \cdot M < \frac{\varepsilon^2}{4}.\end{aligned} \tag{50}$$

On the other hand, if $L_0 > t_0 - \tau_0 + 1$, we get that

$$\begin{aligned}I_1 &\leq e^{-2\varepsilon^{2\alpha}t_0}\left[\int_{L_0}^{t_0} e^{2\varepsilon^{2\alpha}t_0} \cdot \|h(\cdot, s)\| \cdot \varepsilon_1 ds + \frac{1}{2}\int_{t_0-\tau_0+1}^{L_0}\right. \\ &\quad \left.\cdot\, e^{2\varepsilon^{2\alpha}s}\left(\left\|u^2(s)\right\|^2 + \|h(\cdot, s)\|^2\right)ds\right] \\ &< \varepsilon_1 \cdot \sqrt{t_0 - L_0} \cdot M + \frac{\varepsilon_1^2}{4\varepsilon^{2\alpha}} + \frac{\varepsilon^2}{8} < \frac{\varepsilon^2}{2}.\end{aligned} \tag{51}$$

The above inequalities guarantee that $Y(t_0) < 9\varepsilon^2/8$. And because

$$\begin{aligned}Y(t_0) &= \left(u^2_t(t_0), u^2(t_0)\right) + \frac{1}{2}\left\|\nabla u^2(t_0)\right\|^2 \geq \frac{1}{2}\left\|\nabla u^2(t_0)\right\|^2 \\ &\quad - \left\|u^2_t(t_0)\right\| \cdot \left\|u^2(t_0)\right\| \geq \frac{1}{2}\left\|\nabla u^2(t_0)\right\|^2 - \varepsilon_1^2,\end{aligned} \tag{52}$$

we get that

$$\left\|\nabla u^2(t_0)\right\|^2 \leq 2\left(Y(t_0) + \varepsilon_1^2\right) < 2\left(\frac{9\varepsilon^2}{8} + \frac{\varepsilon^2}{8}\right) < 4\varepsilon^2, \tag{53}$$

i.e., $\|\nabla u^2(t_0)\| < 2\varepsilon$.

Case 2. There exist $(u^0_{t_0-\tau_0}, u^1_{t_0-\tau_0}) \in D(t_0 - \tau_0)$ and $t_1 \in [t_0 - \tau_0 + 1, t_0]$ such that

$$\|\nabla u(t_1)\| \leq \varepsilon \text{ with } (u, u_t)(t) = \xi_u^{t_0-\tau}(t). \tag{54}$$

In this case, we claim that the following inequality is true, i.e., for every $t_1 \leq t \leq t_0$,

$$\left\|\nabla u^2(t)\right\| < 2\varepsilon, \quad \text{for } u^2 = Q^1_{m_0}u. \tag{55}$$

In fact, if this claim is not true, the continuity of $\|\nabla u^2(t)\|$ gives that

$$E = \left\{t \mid t \in [t_1, t_0], \left\|\nabla u^2(t)\right\| = 2\varepsilon\right\}, \tag{56}$$

is not an empty set. Let $t_3 = \inf E$. It is easy to prove that $\|\nabla u^2(t_3)\| = 2\varepsilon$. Moreover, by the definition of t_3, we have that

$$\left\|\nabla u^2(t)\right\| < 2\varepsilon, \quad \forall t \in [t_1, t_3). \tag{57}$$

According to the intermediate value theorem, we know that the set

$$E_1 = \left\{ t \mid t \in (t_1, t_3), \left\|\nabla u^2(t)\right\| = \frac{3}{2}\varepsilon \right\}, \tag{58}$$

is not empty. Denoting $t_2 = \sup E_1$, we can conclude from the definition of supremum that

$$\left\|\nabla u^2(t_2)\right\| = \frac{3}{2}\varepsilon. \tag{59}$$

Thus,

$$\begin{aligned}\frac{3}{2}\varepsilon &< \left\|\nabla u^2(t)\right\| < 3\varepsilon, \forall t \in (t_2, t_3], \left\|\nabla u^2(t_2)\right\| \\ &= \frac{3}{2}\varepsilon, \left\|\nabla u^2(t_3)\right\| = 2\varepsilon.\end{aligned} \tag{60}$$

Notice that $\|\nabla u\| \geq \|\nabla u^2\|$ and $\|\nabla u^2\| \leq 1$ for $t \in [t_2, t_3]$; we have that $\phi(\|\nabla u\|^2) \geq \|\nabla u^2\|^{2\alpha}$. Then, integrating (43) on (t_2, t_3), we have that

$$\begin{aligned}&\left(\left(u_t^2(t_3), u^2(t_3)\right) + \frac{1}{2}\left\|\nabla u^2(t_3)\right\|^2 \right) - \left(\left(u_t^2(t_2), u^2(t_2)\right) + \frac{1}{2}\left\|\nabla u^2(t_2)\right\|^2 \right) \\ &\quad + \int_{t_2}^{t_3} \phi\left(\|\nabla u(s)\|^2\right)\left\|\nabla u^2(s)\right\|^2 \mathrm{d}s \\ &\leq \int_{t_2}^{t_3} \left(\left\|u_t(s)^2\right\|^2 + \left(f(u(s)), u^2(s)\right) + \left(h(\cdot,s), u^2(s)\right) \right) \mathrm{d}s.\end{aligned} \tag{61}$$

It implies that

$$\begin{aligned}&\left\|\nabla u^2(t_3)\right\|^2 + 2\int_{t_2}^{t_3} \left\|\nabla u^2(s)\right\|^{2+2\alpha} \mathrm{d}s \leq \left\|\nabla u^2(t_2)\right\|^2 \\ &\quad - 2\left(u_t^2(t_3), u^2(t_3)\right) + 2\left(u_t^2(t_2), u^2(t_2)\right) \\ &\quad + 2\int_{t_2}^{t_3} \left(\left\|u_t(s)^2\right\|^2 + \left(f(u(s)), u^2(s)\right) + \left(h(\cdot,s), u^2(s)\right) \right) \mathrm{d}s.\end{aligned} \tag{62}$$

Combing (40), (41), and (59),we get

$$\begin{aligned}&\left\|\nabla u^2(t_3)\right\|^2 + 2\left(\frac{3}{2}\varepsilon\right)^{2\alpha+2} (t_3 - t_2) \leq \left\|\nabla u^2(t_3)\right\|^2 \\ &\quad + 2\int_{t_2}^{t_3} \left\|\nabla u^2(s)\right\|^{2\alpha+2} \mathrm{d}s \leq \left\|\nabla u^2(t_2)\right\|^2 + 2\left\|u_t^2(t_2)\right\| \\ &\quad \cdot \left\|u^2(t_2)\right\| + 2\left\|u_t^2(t_3)\right\| \cdot \left\|u^2(t_3)\right\| \\ &\quad + 2\int_{t_2}^{t_3} \left(\left\|u_t^2(s)\right\|^2 + \left(f(u), u^2\right) + \left(h(\cdot,s), u^2\right) \right) \mathrm{d}s \\ &\leq \left(\frac{3}{2}\varepsilon\right)^2 + 4\varepsilon_1^2 + 2\varepsilon_1^2(t_3 - t_2) + 2\int_{t_2}^{t_3} \|f(u)\|_{L^{(p+1)/p}} \left\|u^2\right\|_{L^{p+1}} \mathrm{d}s \\ &\quad + 2\varepsilon_1 \left(\int_{t_2}^{t_3} \|h(\cdot,s)\|^2 \mathrm{d}s \right)^{1/2} \sqrt{t_3 - t_2} \leq \left(\frac{3}{2}\varepsilon\right)^2 + 4\varepsilon_1^2 \\ &\quad + 2\varepsilon_1^2(t_3 - t_2) + 2\int_{t_2}^{t_3} M \cdot \frac{\varepsilon_1^2}{M} \mathrm{d}s + 2\varepsilon_1 M \sqrt{t_3 - t_2} \leq \left(\frac{3}{2}\varepsilon\right)^2 \\ &\quad + 4\varepsilon_1^2 + 4\varepsilon_1^2(t_3 - t_2) + \varepsilon_1 M (t_3 - t_2) + \varepsilon_1 M \leq \frac{9}{4}\varepsilon^2 + \frac{\varepsilon^{4+4\alpha}}{4} \\ &\quad + \frac{\varepsilon^{4+4\alpha}}{4}(t_3 - t_2) + \frac{\varepsilon^{2+2\alpha}}{4}(t_3 - t_2) + \frac{\varepsilon^{2+2\alpha}}{4} < \frac{11}{4}\varepsilon^2 + \frac{1}{2}\varepsilon^{2+2\alpha}(t_3 - t_2).\end{aligned} \tag{63}$$

Thus, $\left\|\nabla u^2(t_3)\right\|^2 < (11/4)\varepsilon^2$, which is in contradiction with (59), and condition ($\mathscr{D}$-PC) holds. This completes the proof.

Appendix

A. Proof of Proposition 10

We prove the well-posedness of Problem (1) using the same method as in [25].

Step 1. We start with the case when $u_\tau^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and assume that $\|(u_\tau^0, u_\tau^1)\|_{\mathscr{H}} \leq R$ with some $R > 0$. We seek for the approximate solutions of the form

$$u^N(t) = \sum_{k=1}^N g_k(t) e_k, \quad N = 1, 2, \cdots, \tag{A1}$$

satisfying the finite-dimensional projections of (1). Moreover, we have that

$$\begin{aligned}&\left\|\left(u^N(\tau), u_t^N(\tau)\right)\right\|_{\mathscr{H}} \leq C_R, \\ &\left\|\left(u^N(\tau) - u_\tau^0, u_t^N(\tau) - u_\tau^1\right)\right\|_{\mathscr{H}} \longrightarrow 0, \text{as } N \longrightarrow \infty.\end{aligned} \tag{A2}$$

We omit the superscript N in the sequel. Now, we use the multiplier $u_t(t)$ and get that

$$\frac{d}{dt}\left[\frac{1}{2}\left(\|u_t\|^2 + \Phi\left(\|\nabla u\|^2\right)\right) + F(u(t))\right] + \|\nabla u_t\|^2 - (h, u_t(t)) = 0. \tag{A3}$$

Similarly, multiplying (1) by u, we have that

$$\begin{aligned}\frac{d}{dt}\left[(u, u_t) + \frac{1}{2}\|\nabla u\|^2\right] &= \|u_t\|^2 - \phi\left(\|\nabla u\|^2\right)\|\nabla u\|^2 \\ &\quad - (f(u), u) + (h, u).\end{aligned} \tag{A4}$$

Let

$$\begin{aligned}E(u_0, u_1) &= \frac{1}{2}\left(\|u_1\|^2 + \Phi\left(\|\nabla u_0\|^2\right)\right) + F(u_0), \\ W^\eta(u_0, u_1) &= E(u_0, u_1) + \eta\left[(u_0, u_1) + \frac{1}{2}\|\nabla u_0\|^2\right].\end{aligned} \tag{A5}$$

From (A3) and (A4),

$$\begin{aligned}&\frac{d}{dt} W^\eta(u(t), u_t(t)) + \|\nabla u_t\|^2 - (h, u_t(t)) \\ &\quad = \eta\left(\|u_t\|^2 - \phi\left(\|\nabla u\|^2\right)\|\nabla u\|^2 - (f(u), u) + (h, u)\right).\end{aligned} \tag{A6}$$

Using (14), (24), and $|(h, u_t(t))| \leq \lambda_1/2\|u_t\|^2 + 1/2\lambda_1 \|h(\cdot,t)\|^2$, we find that

$$\frac{d}{dt}W^{\eta}(u(t),u_t(t))+\frac{1}{2}\|\nabla u_t\|^2\le \eta\|u_t\|^2-\eta\delta_0\Phi\left(\|\nabla u\|^2\right)+\eta\theta_1\|u\|^2+c_1\|h(\cdot,t)\|^2+c_2. \tag{A7}$$

where c_1, c_2 is independent of t. Obviously,

$$W^{\eta}(u_0,u_1)\le \|u_1\|^2+\frac{1}{2}\Phi\left(\|\nabla u_0\|^2\right)+\tilde{c}_0\|\nabla u_0\|^2. \tag{A8}$$

By (13) and (14), there exists $\eta_0 >, \delta_1 > 0$, for any $\eta \in (0, \eta_0)$,

$$W^{\eta}(u_0,u_1)\ge\left(\frac{1}{2}-\eta\right)\|u_1\|^2+\frac{1}{2}\phi_1\cdot\|\nabla u_0\|^2-\frac{\theta_1}{2}\|u_0\|^2-\eta\|u_0\|^2-\frac{\eta}{2}\|\nabla u_0\|^2-\tilde{c}_1\ge\frac{1}{4}\|u_1\|^2+\delta_1\|\nabla u_0\|^2-\tilde{c}_2. \tag{A9}$$

Combing (A8) and the above inequalities, we have that

$$\frac{d}{dt}W^{\eta}(u(t),u_t(t))+\frac{1}{2}\|\nabla u_t\|^2\le C_1W^{\eta}(u(t),u_t(t))+C_2\|h(\cdot,t)\|^2+C_3. \tag{A10}$$

Therefore, using Gronwall's inequality, we obtain

$$W^{\eta}(u(t),u_t(t))\le\tilde{C}_{R,T}+C_2e^{C_1T}\int_{-\infty}^{T}\|h(\cdot,s)\|^2\mathrm{d}s\triangleq C^1_{R,T},\quad \forall t\in[\tau,T], \tag{A11}$$

which means that

$$\|(u(t),u_t(t))\|_{\mathscr{H}}\le C_{R,T},\quad \forall t\in[\tau,T],\qquad \int_{\tau}^{T}\|\nabla u_t(t)\|^2dt\le C_{R,T}. \tag{A12}$$

Now, multiplying (1) by $-\Delta u$, we have

$$\frac{d}{dt}\left[-(u_t,\Delta u)+\frac{1}{2}\|\Delta u\|^2\right]+\phi\left(\|\nabla u(t)\|^2\right)\|\Delta u\|^2+\left(f'(u),|\nabla u|^2\right)\le\|\nabla u_t(t)\|^2+\frac{1}{2}\|h(\cdot,t)\|^2+\frac{1}{2}\|\Delta u\|^2. \tag{A13}$$

Since $H^1_0(\Omega)^{\circ}L^{p+1}(\Omega)$ when $n\ge 3$ and $H^1_0(\Omega)^{\circ}L^{q}(\Omega)$ for any $q\ge 1$ when $n=2$, $H^1_0(\Omega)^{\circ}L^{\infty}(\Omega)$ when $n=1$, we easily obtain that

$$\left|\left(f'(u),|\nabla u|^2\right)\right|\le C\int_{\Omega}\left(1+|u|^{p-1}\right)|\nabla u|^2dx\le C\|\nabla u\|^2+C\|u\|^{p-1}_{p+1}\cdot\|\nabla u\|^2_{p+1}\le C\|\nabla u\|^2+\|\nabla u\|^{p-1}\cdot\|\Delta u\|^2\le C\left(1+\|\Delta u\|^2\right). \tag{A14}$$

It follows that

$$\frac{d}{dt}\left[-(u_t,\Delta u)+\frac{1}{2}\|\Delta u\|^2\right]\le\|\nabla u_t(t)\|^2+\frac{1}{2}\|h(\cdot,t)\|^2+C_1\|\Delta u\|^2+C_2, \tag{A15}$$

for every $t\in[\tau,T]$. Let

$$\Psi(t)=E(u(t),u_t(t))+\varepsilon\left[-(u_t,\Delta u)+\frac{1}{2}\|\Delta u\|^2\right],\qquad \varepsilon>0. \tag{A16}$$

We can choose $\varepsilon_0>0$, such that

$$\Psi(t)\ge C_{R,T,\varepsilon}\left(\|u_t\|^2+\|\Delta u\|^2\right)-C,\quad \forall 0<\varepsilon<\varepsilon_0. \tag{A17}$$

Thus, combing (A3), (A12), and (A15), we have that

$$\frac{d}{dt}\Psi(t)\le C_1\|\Delta u\|^2+C_2\|h(\cdot,t)\|^2+C_3\le C_4\Psi(t)+C_2\|h(\cdot,t)\|^2+C_5. \tag{A18}$$

This implies that

$$\|\Delta u(t)\|^2\le C_{R,T}\left(1+\|\Delta u(\tau)\|^2\right),\quad t\in[\tau,T]. \tag{A19}$$

The above a priori estimates show that (u^N, u^N_t) is bounded in

$$L^{\infty}\left(\tau,T;H^2(\Omega)\cap H^1_0(\Omega)\right)\times\left[L^{\infty}\left(0,T;L^2(\Omega)\right)\cap L^2\left(\tau,T;H^1_0(\Omega)\right)\right], \tag{A20}$$

for every $T>\tau$. Moreover, using the equation for $u^N(t)$, we can show $\int_{\tau}^{T}\|u^N_{tt}\|^2_{-m}\mathrm{d}t\le C_{R,T}$ for some $m\ge\max\{1,n/2\}$. Thus, there exists a subsequence, stilled denoted u^N, and u, such that

$$\begin{aligned} u^N&\longrightarrow u,\quad \text{in } C\left(\tau,T;H^1_0(\Omega)\right),\\ u^N&\longrightarrow u,\quad \text{in } L^{\infty}\left(\tau,T;H^2(\Omega)\right)\text{weak-star},\\ u^N_t&\longrightarrow u_t,\quad \text{in } L^2\left(\tau,T;L^2(\Omega)\right)\cap C\left(\tau,T;H^{-1}(\Omega)\right),\\ u^N_t&\longrightarrow u_t,\quad \text{in } L^2\left(\tau,T;H^1_0(\Omega)\right)\text{ weak,}\end{aligned} \tag{A21}$$

as $N\longrightarrow\infty$. Moreover, by the Lions lemma (see Lemma 1.3 in [37]) we have that

$$f\left(u^N(x,t)\right) \longrightarrow f(u(x,t)), \quad \text{in } L^2([\tau,T]\times\Omega) \text{ weak}, \tag{A22}$$

as $N \longrightarrow \infty$. Then, making a limit transition in the nonlinear term, we prove the existence of a weak solution under the additional condition $u_\tau^0 \in H^2(\Omega)\cap H_0^1(\Omega)$. One can see that this solution u satisfies (14) and (15).

Step 2. Now, let $u^1(t)$ and $u^2(t)$ be weak solutions to (1) with different initial data $(u_0^i, u_1^i) \in \mathscr{H}$ such that

$$\left\|\left(u^i(t), u_t^i(t)\right)\right\|_{\mathscr{H}} + \int_\tau^T \|\nabla u_t(t)\|^2 \mathrm{d}t \le C_R, \quad \forall t \in [\tau, T], \tag{A23}$$

for some $R > 0$. Notice that we do not assume $u_0^i \in H^2(\Omega)$ here. Since $\phi \in C^1$, we conclude from (60) that

$$\left|\phi\left(\|\nabla u\|^2\right)\right|, \left|\phi'\left(\|\nabla u\|^2\right)\right| \le M, t \in [\tau, T]. \tag{A24}$$

We can see that $z(t) = u^1(t) - u^2(t)$ solves the equation

$$\begin{aligned} z_{tt} - \Delta z_t - \frac{1}{2}\phi_{12}(t)\Delta z - \frac{1}{2}[\phi_1(t) - \phi_2(t)]\left(\Delta u^1 + \Delta u^2\right) \\ + f\left(u^1\right) - f\left(u^2\right) = 0, \end{aligned} \tag{A25}$$

where $\phi_{12}(t) = \phi_1(t) + \phi_2(t)$ with $\phi_i(t) = \phi(\|\nabla u^i(t)\|^2)$. By the definition of a weak solution, we can multiply (A25) by z in $L^2(\Omega)$ and reduce that

$$\begin{aligned} \frac{d}{dt}\left[(z, z_t) + \frac{1}{2}\|\nabla z\|^2\right] - \|z_t\|^2 + \frac{1}{2}\phi_{12}(t)\|\nabla z\|^2 \\ + \left(f\left(u^1\right) - f\left(u^2\right), z\right) + \frac{1}{2}[\phi_1(t) - \phi_2(t)]\left(\nabla u^1 + \nabla u^2, \nabla z\right) = 0. \end{aligned} \tag{A26}$$

Using $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \le q < +\infty$ when $n = 1, 2$ and $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ when $n \ge 3$, we have that

$$\begin{aligned} \left|\left(f\left(u^1\right) - f\left(u^2\right), z\right)\right| &\le C\int_\Omega \left(1 + \left|u^1\right|^{p-1} + \left|u^2\right|^{p-1}\right)|z|^2 dx \\ &\le C_R\|\nabla z\|^2. \end{aligned} \tag{A27}$$

Therefore, combining with

$$\begin{aligned} |\phi_1(t) - \phi_2(t)| = \Big| \int_0^1 \phi'\left(\lambda\left\|\nabla u^1(t)\right\|^2 + (1-\lambda)\left\|\nabla u^2(t)\right\|^2\right) d\lambda \\ \cdot \left(\nabla\left(u^1 + u^2\right), \nabla z\right)\Big| \le C\|\nabla z\|, \end{aligned} \tag{A28}$$

we can conclude that

$$\frac{d}{dt}\left[(z, z_t) + \frac{1}{2}\|\nabla z\|^2\right] \le \|z_t\|^2 + C_R\|\nabla z\|^2. \tag{A29}$$

Now consider the multiplier z_t. Since $z \in L^\infty(\tau, T; H_0^1(\Omega))$, $z_t \in L^2(\tau, T; H_0^1(\Omega))$, and $z_{tt} \in L^2(\tau, T; H^{-1}(\Omega))$, we can multiply (A26) by z_t and obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z_t\|^2 + \|\nabla z_t\|^2 + \frac{1}{2}\phi_{12}(t)(\nabla z, \nabla z_t) + \left(f\left(u^1\right) - f\left(u^2\right), z_t\right) \\ - \frac{1}{2}[\phi_1(t) - \phi_2(t)]\left(\Delta\left(u^1 + u^2\right), z_t\right) = 0. \end{aligned} \tag{A30}$$

Similar to (A29), we can get

$$\frac{d}{dt}\|z_t\|^2 + 2\|\nabla z_t\|^2 + 2\left(f\left(u^1\right) - f\left(u^2\right), z_t\right) \le C\|\nabla z\| \cdot \|\nabla z_t\|. \tag{A31}$$

Similar to (A27), we have

$$\left|\left(f\left(u^1\right) - \mathrm{f}\left(u^2\right), z_t\right)\right| \le C_R\|\nabla z\| \cdot \|\nabla z_t\|. \tag{A32}$$

Therefore, we can conclude from Young's inequality that

$$\frac{d}{dt}\|z_t\|^2 + \|\nabla z_t\|^2 \le C\left(\|\nabla z\|^2 + \|\nabla z_t\|^2\right). \tag{A33}$$

Let

$$\Gamma(t) = \|z_t\|^2 + \varepsilon\left[(z, z_t) + \frac{1}{2}\|\nabla z\|^2\right], \tag{A34}$$

for $\varepsilon > 0$ small enough. Then, there exists a positive constants C_i such that

$$C_1\left(\|z_t\|^2 + \|\nabla z\|^2\right) \le \Gamma(t) \le C_2\left(\|z_t\|^2 + \|\nabla z\|^2\right). \tag{A35}$$

From (A29) and (A33), we have the estimation

$$\frac{d}{dt}\Gamma(t) + \|\nabla z_t\|^2 \le C_{\varepsilon,R}\Gamma(t). \tag{A36}$$

Using Gronwall's inequality, we get that

$$\|z_t(t)\|^2 + \|\nabla z(t)\|^2 + \int_\tau^t \|\nabla z_t(s)\|^2 ds \le C_{R,T}\left(\|z_t(\tau)\|^2 + \|\nabla z(\tau)\|^2\right), \tag{A37}$$

for all $t \in [\tau, T]$, which implies the desired conclusion in (20). By this inequality, we can prove the existence of weak solutions for initial data $(u_\tau^0, u_\tau^1) \in \mathscr{H}$. Indeed, we can choose a sequence $\{(u_\tau^{0,n}, u_\tau^{1,n})\} \in (H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega)$ such that $(u_\tau^{0,n}, u_\tau^{1,n}) \longrightarrow (u_\tau^0, u_\tau^1)$ in $\mathscr{H}$. Owing to

(20), the corresponding solutions $(u^n(t), u_t^n(t))$ converge to functions (u, u_t) in $L^\infty(\tau, T; \mathscr{H})$. From the boundedness for $\{u_t^n\}$ in $L^2(\tau, T; H_0^1(\Omega))$ we also have weak convergence of $\{u_t^n\}$ to u_t in the space $L^2(\tau, T; H_0^1(\Omega))$. This implies that $u(t)$ is a weak solution of Problem (1). By (19), this solution is unique.

Step 3. For the proof of smoothness properties stated in (18), we use the same method as [18, 38]. As usual, the argument below can be justified by considering Galerkin approximations. Set $v = u_t$ and differentiate (1) with respect to time. This yields

$$v_{tt} - \Delta v_t - \phi(\|\nabla u\|^2)\Delta v - 2\phi'(\|\nabla u\|^2)\Delta u(\nabla u, \nabla u_t) + f'(u)v = h_t. \quad \text{(A38)}$$

Multiplying the above equation by v, we obtain that

$$\frac{d}{dt}\left[(v, v_t) + \frac{1}{2}\|\nabla v\|^2\right] + \phi(\|\nabla u\|^2)\|\nabla v\|^2 + \left(f'(u)v, v\right) \le \|v_t\|^2 + C_R|(\nabla u, \nabla v)|^2 + (h_t, v). \quad \text{(A39)}$$

This implies that

$$\frac{d}{dt}\left[(v, v_t) + \frac{1}{2}\|\nabla v\|^2\right] \le \|v_t\|^2 + C_1\|\nabla v\|^2 + \|h_t(t)\|^2. \quad \text{(A40)}$$

Multiplying the above equation by $\mathscr{A}^{-1}v_t$ with $\mathscr{A} = -\Delta$ and using Young's inequality, we obtain that

$$\frac{d}{dt}\left\|\mathscr{A}^{-1/2}v_t\right\|^2 + \|v_t\|^2 \le C_R\left(\|\nabla v\|^2 + \|h_t(t)\|^2\right). \quad \text{(A41)}$$

Denote

$$Y(t) = \left\|\mathscr{A}^{-1/2}v_t\right\|^2 + \varepsilon\left[(v, v_t) + \frac{1}{2}\|\nabla v\|^2\right], \quad \text{(A42)}$$

then we have that

$$a_1\left(\left\|\mathscr{A}^{-1/2}v_t\right\|^2 + \|\nabla v\|^2\right) \le Y(t) \le a_2\left(\left\|\mathscr{A}^{-1/2}v_t\right\|^2 + \|\nabla v\|^2\right), \quad \text{(A43)}$$

for some positive constants a_i depending on ε. Due to (A40) and (A41), it is apparent that

$$\frac{dY(t)}{dt} + \frac{1}{2}\|v_t\|^2 \le C_3\|\nabla v\|^2 + C_4\|h_t(t)\|^2 \le \tilde{C}_3 Y(t) + C_4\|h_t(t)\|^2. \quad \text{(A44)}$$

Multiplying (A44) by $(s-\tau)^2$, we get that

$$\frac{d}{ds}\left((s-\tau)^2 Y(s)\right) + \frac{(s-\tau)^2}{2}\|v_t(s)\|^2 \le 2(s-\tau)Y(s) + \tilde{C}_3(s-\tau)^2 Y(s) + C_4(s-\tau)^2\|h_t(s)\|^2. \quad \text{(A45)}$$

It is easy to know

$$2(s-\tau)Y(s) \le 1 + (s-\tau)^2 Y^2(s) \le 1 + (s-\tau)^2 \cdot a_2\left(\left\|\mathscr{A}^{-1/2}v_t\right\|^2 + \|\nabla v\|^2\right)Y(s). \quad \text{(A46)}$$

Since

$$\mathscr{A}^{-1}u_{tt} = -u_t - \phi(\|\nabla u\|^2)u - \mathscr{A}^{-1}(f(u) - h), \quad \text{(A47)}$$

one can see that $\|\mathscr{A}^{-1}v_t\| \le C_R(1 + \|h(\cdot, s)\|)$. Using $\|\mathscr{A}^{-1/2}v_t\|^2 \le C\|\mathscr{A}^{-1}v_t\| \cdot \|v_t\|$ and Young's inequality, we get

$$\frac{d}{ds}\left((s-\tau)^2 Y(s)\right) + \frac{(s-\tau)^2}{4}\|v_t(s)\|^2 \le C_5(s-\tau)^2 \cdot \left(1 + \|\nabla u_t\|^2\right)Y(s) + C_6(s-\tau)^2\left(\|h(\cdot, s)\|^2 + \|h_t(s)\|^2\right). \quad \text{(A48)}$$

By Gronwall's inequality and (16), one can find

$$(t-\tau)^2 Y(t) \le C_{R,T}\left(1 + \|h(\cdot, s)\|^2 + \|h_t(s)\|^2\right). \quad \text{(A49)}$$

This implies (18). The proof is completed.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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