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Research Article

Some Exact Solutions to Generalized Kadomtsev-Petviashvili Equation

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Most of the papers have explored the interactions between solitons with a zero background, while reports about exact solutions for nonzero background are rare. Hence, this paper is aimed at exploring the breather, lump, and interaction solutions with a small perturbation to $(2+1)$-dimensional generalized Kadomtsev-Petviashvili (gKP) equation. General high-order periodic breather solutions are obtained using Hirota's bilinear method with a small perturbation. At the same time, combining the use of long wave limit methods and module resonance constraints, general lump solutions and mixed solutions to gKP equation are generated. Finally, the space-time structures of the breather solutions, lump solutions, and interaction solutions are investigated and discussed.

1. Introduction

The soliton, also known as a solitary wave, is a special form of ultrashort pulse, or a pulse-like traveling wave whose shape, amplitude, and velocity remain constant during its propagation [1]. So far, soliton phenomena have been discovered in many subject areas, for example, laser self-focusing in media, acoustic and electromagnetic waves in plasma, motion of domain walls in liquid crystals, vortex in fluid, dislocation of crystals, magnetic flux in superconductors, and signal transmission in the nervous system [2–4].

In mathematics, the progress of soliton theory is embodied in the discovery of a large number of nonlinear partial differential equations with soliton solutions [5, 6] and has gradually established a more systematic mathematical and physical partial differential equations and the theory of soliton [7, 8]. In order to solve these nonlinear mathematical physical equations, scholars working in the field of solitons have developed a series of solution methods, such as the inverse scattering method [8], Hirota bilinear method [7], numerical method, and symbolic calculations [9].

In this paper, we consider the $(2+1)$-dimensional generalized Kadomtsev-Petviashvili equation [10] as follows:

$$\left(u_t + c\mathrm{uu}_x + b\mathrm{u}_{\mathrm{xxx}}\right)_x + \frac{c_0}{2} u_{\mathrm{yy}} = 0, \tag{1}$$

where $u = u(x, y, t)$ denotes a scalar function of the space variables x, y and time variable t, the parameters c is the nonlinear term coefficient, b is the dispersion coefficient along the x-axis, c_0 is the velocity of the linear wave, and $c_0/2$ is the dispersion coefficient along the y-axis.

When $c = 6$, $b = 1$, and $c_0/2 = -1$, Equation (1) is reduced to the KPI equation:

$$u_{\mathrm{xt}} + 6\left(\mathrm{uu}_x\right)_x + u_{\mathrm{xxxx}} - u_{\mathrm{yy}} = 0. \tag{2}$$

And the exact solutions with a zero background including N-soliton solution and lump solution to the standard Kadomtsev-Petviashvili equation has been studied systematically in Refs. [7, 11, 12]. In recent years, the research on the lump solution of Equation (3) has been very hot, mainly focusing on the normal scattering of lump waves [13], anomalous scattering between lump waves [14–18], and bound states of lump waves [19]. The reports on anomalous scattering focus on the diversity of scattering patterns, such as triangular patterns, and polygonal patterns [17]. At the same time, the resonance phenomenon between lump chains (we called breather waves in this paper) and lump waves has been fully studied [20–23].

When $c=6$, $b=1$, and $c_0/2=1$, Equation (1) is reduced to the KPII equation:

$$u_{xt}+6(uu_x)_x+u_{xxxx}+u_{yy}=0. \quad (3)$$

This equation can be used to describe some nonlinear phenomena in shallow water [24]. For the KP2 system, Kodama has made a very outstanding contribution in the field about the resonance phenomena between line waves [25].

Considering that the above-described solutions are all obtained on the zero background, the solutions with a nonzero background in the actual system are more general and can describe the objective world more accurately. In this paper, applying Hirota's bilinear method with a perturbation parameter u_0 to Equation (1), we obtain periodic breather wave solutions. Meanwhile, we generate lump and interaction solutions with a nonzero background to Equation (1) from solitons by taking long wave limits [12]. The arrangement of this paper is organized as follows: in Section 2, under the variable transformation, we construct bilinear formalism with a perturbation parameter. In Section 3, we mainly investigate general higher-order breather and lump solutions of Equation (1). In Section 4, we describe how to obtain mixed solutions and interaction solutions.

2. Bilinear Formalism with a Perturbation Parameter

Under the variable transformation,

$$u(x,y,t)=u_0+\frac{12b}{c}(\ln f)_{xx}, \quad (4)$$

where $f(x,y,t)$ is a complex function and u_0 is a free real number.

Substituting Equation (4) into Equation (1), then Equation (1) becomes the following equation:

$$\begin{aligned}(\ln f)_{xxt}+cu_0(\ln f)_{xxx}+12b(\ln f)_{xx}(\ln f)_{xxx}\\+b(\ln f)_{xxxx}+\frac{c_0}{2}(\ln f)_{xyy}=0.\end{aligned} \quad (5)$$

When Equation (5) integrates once with respect to x, we can obtain the following equation:

$$(\ln f)_{xt}+cu_0(\ln f)_{xx}+6b(\ln f)_{xx}^2+b(\ln f)_{xxxx}+\frac{c_0}{2}(\ln f)_{yy}=0. \quad (6)$$

The bilinear form of Equation (1) with a small perturbation parameter u_0 is generated as

$$\left(D_xD_t+cu_0D_x^2+bD_x^4+\frac{c_0}{2}D_y^2\right)f\cdot f=0. \quad (7)$$

The operator D is the Hirota's bilinear differential operator defined by

$$D_x^mD_t^nf\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,y,t)\cdot g\left(x',y',t'\right)\Big|_{x'=x,t'=t}. \quad (8)$$

3. General Higher-Order Breather and Lump Solutions

In this section, we mainly investigate general high-order breather and lump solutions of Equation (1).

3.1. First-Order Breather and Lump Solutions. We first assume f in Equation (4) as the following formal form to derive first-order breather solutions in Equation (1):

$$f=1+\varepsilon f_1+\varepsilon^2 f_2, \quad (9)$$

with

$$\begin{aligned}f_1&=e^{\eta_1}+e^{\eta_2},\\ f_2&=e^{\eta_1+\eta_2+A_{12}},\end{aligned} \quad (10)$$

where

$$\eta_s=w_st+k_sx+p_sy+\phi_s, s=1,2, \quad (11)$$

and w_s, k_s, p_s, and ϕ_s are freely complex parameters. Substituting f defined in Equation (9) into Equation (7) and collecting the power order of ε, one can obtain the following equations at the ascending power order of ε:

$$\begin{aligned}\varepsilon^0&:\left(D_xD_t+cu_0D_x^2+bD_x^4+\frac{c_0}{2}D_y^2\right)(1\cdot 1)=0,\\ \varepsilon^1&:\left(D_xD_t+cu_0D_x^2+bD_x^4+\frac{c_0}{2}D_y^2\right)(1\cdot f_1+f_1\cdot 1)=0,\\ \varepsilon^2&:\left(D_xD_t+cu_0D_x^2+bD_x^4+\frac{c_0}{2}D_y^2\right)(1\cdot f_2+f_1\cdot f_1+f_2\cdot 1)=0,\end{aligned} \quad (12)$$

namely,

$$f_{1xt}+cu_0f_{1xx}+bf_{1xxxx}+\frac{c_0}{2}f_{1yy}=0, \quad (13)$$

$$\begin{aligned}cu_0\left(f_{1xx}f_1-f_{1x}^2+f_{2xx}\right)+b\left(f_{1xxxx}f_1-4f_{1xxx}f_{1x}+3f_{1xx}^2+f_{2xxxx}\right)\\+f_{1xt}f_1-f_{1x}f_{1t}+f_{2xt}=0.\end{aligned} \quad (14)$$

Substituting functions f_1 and f_2 defined in Equation (10)

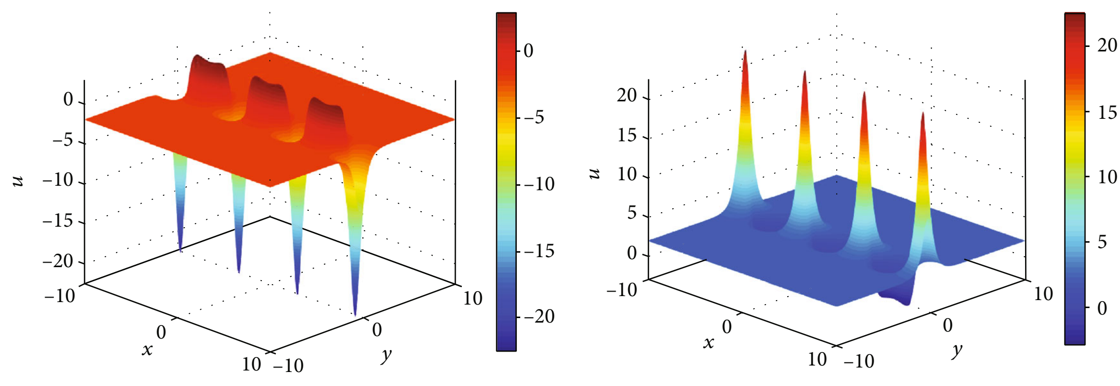


Figure 1: Two types of breather solutions in Equation (1) with parameters $c_0 = 1$, $b = -1$, $k = 1$, $p = 2$, and $\phi_0 = 0$ at $t = 0$. (a) Bright-type breather with $u_0 = 2$ and $c = -1$. (b) Dark-type breather with $u_0 = -2$ and $c = 1$.

into Equation (13) and Equation (14), we have

$$\left(2\mathrm{bk}_1^4 + 2\mathrm{ck}_1^2 u_0 + c_0 p_1^2 + 2k_1 w_1\right) e^{\eta_1} + \left(2\mathrm{bk}_2^4 + 2\mathrm{ck}_2^2 u_0 + c_0 p_2^2 + 2k_2 w_2\right) e^{\eta_2} = 0, \tag{15}$$

and

$$\begin{aligned}&\left[2\left(w_1 - w_2\right)\left(k_1 - k_2\right) + 2\,\mathrm{cu}_0\left(k_1 - k_2\right)^2 + 2\,b\left(k_1 - k_2\right)^4 + c_0\left(p_1 - p_2\right)^2\right] \mathrm{e}^{\eta_1+\eta_2} \\ &+ \left[2\left(k_1 + k_2\right)\left(w_1 + w_2\right) + 2\,\mathrm{cu}_0\left(k_1 + k_2\right)^2 + 2\,b\left(k_1 + k_2\right)^4 + c_0\left(p_1 + p_2\right)^2\right] \mathrm{e}^{\eta_1+\eta_2+A_{12}} = 0.\end{aligned} \tag{16}$$

By solving Equations (15) and (16), we can obtain the following formulas:

$$e^{A_{12}} = -\frac{2\left(k_1 - k_2\right)^4 b + 2\,u_0\left(k_1 - k_2\right)^2 c + \left(p_1 - p_2\right)^2 c_0 + 2\left(w_1 - w_2\right)\left(k_1 - k_2\right)}{2\left(k_1 + k_2\right)^4 b + 2\,u_0\left(k_1 + k_2\right)^2 c + \left(p_1 + p_2\right)^2 c_0 + 2\left(w_1 + w_2\right)\left(k_1 + k_2\right)}, \tag{17}$$

$$2\mathrm{bk}_s^4 + 2\mathrm{ck}_s^2 + c_0 p_s^2 + 2k_s w_s = 0, s = 1, 2. \tag{18}$$

Under the constraints of Equations (17) and (18), when $\varepsilon = 1$, f can be written as

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_2+A_{12}}, \tag{19}$$

which corresponds to the two-soliton solution of Equation (1). To guarantee the corresponding breather solutions being real functions, there are two restrictions for a valid calculation: (1) $\eta_1 = \bar{\eta}_2$, so η_1 and η_2 are conjugates of each other. (2) Parameters $k_s, w_s, p_s, s = 1, 2$ must satisfy the constraint of Equation (18).

In particular, the following parameter constrains may be used to facilitate the calculation:

$$\begin{aligned}k_1 &= -k_2 = i \cdot k, \\ p_1 &= p_2 = p, \\ \phi_1 &= \phi_2 = \phi_0,\end{aligned} \tag{20}$$

where k, p, and ϕ_0 are freely real parameters. Then, we can obtain $\eta_1 = \bar{\eta}_2$, and the function f in Equation (19) can be rewritten as

$$f = H(y)\left[\sqrt{M}\cosh(\theta) + \cos(\mathrm{wt} + \mathrm{kx})\right], \tag{21}$$

where

$$\begin{aligned}H(y) &= 2\,\mathrm{e}^{\mathrm{py}+\phi_0}, \\ w &= \frac{2\,\mathrm{bk}^4 - 2\,\mathrm{cu}_0\,k^2 + c_0\,p^2}{2k}, \\ \theta &= \mathrm{py} + \phi_0 + \ln\left(\sqrt{M}\right), \\ M &= -\frac{6\,\mathrm{bk}^4 - c_0\,p^2}{c_0\,p^2}.\end{aligned} \tag{22}$$

The first-order breather solutions in Equation (1) in the (x,y)-plane are shown in Figure 1. It is seen that there are dark-type and bright-type breather solutions in Equation (1).

To generate rational solution, we take a long wave limit with the provision in Equation (19).

$$\begin{aligned}p_s &= P_s \varepsilon, \\ k_s &= K_s\,\varepsilon, \varepsilon \mapsto 0, \\ \mathrm{e}^{\phi_s} &= -1, \\ s &= 1, 2.\end{aligned} \tag{23}$$

Then, the expansions of f in Equation (19) are given as follows:

$$f = \left(\theta_1\theta_2 + a_{12}\right)\varepsilon^2 + o\left(\varepsilon^2\right), \tag{24}$$

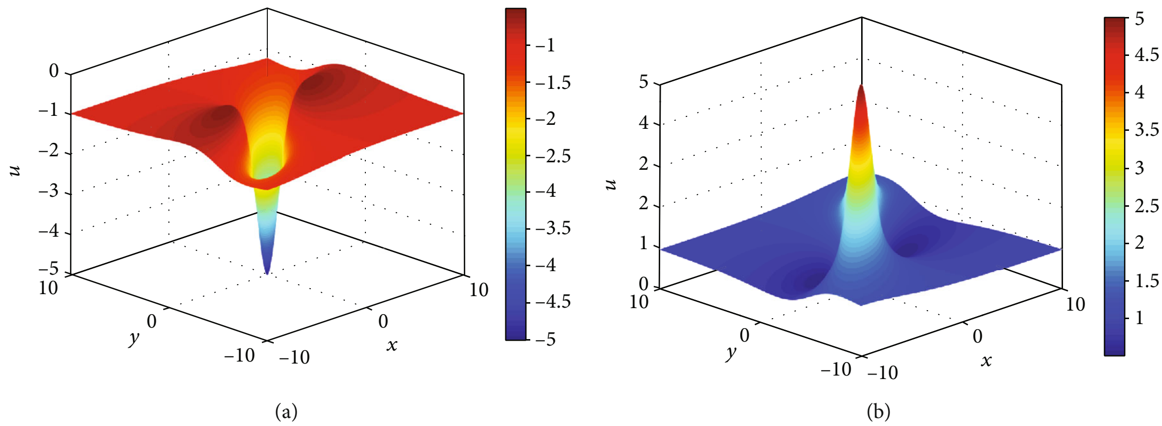


FIGURE 2: Two types of lump solutions in Equation (1) with parameter $c_0 = 1$, $b = -1$, $K_1 = 1 - i$, $K_2 = 1 + i$, $P_1 = -2$, and $P_2 = -2$ at $t = 0$. (a) Bright-type lump solution with $u_0 = 1$ and $c = -1$. (b) Dark-type lump solution with $u_0 = -1$ and $c = 1$.

where

$$\theta_s = -K_s c u_0 t - \frac{P_s{}^2 c_0}{2K_s} t + K_s x + P_s y, s = 1, 2,$$
$$a_{12} = \frac{24 b K_1{}^3 K_2{}^3}{c_0 (K_1 P_2 - K_2 P_1)^2}. \tag{25}$$

In order to get rational solutions in Equation (1), divide both sides of Equation (24) byε^2, and then, attempt to compute the limiting value asεapproaches0. For convenience, let us still call the limit that we just obtained f. Then, the f is given as follows:

$$f = \theta_1 \theta_2 + a_{12}. \tag{26}$$

To guarantee the corresponding rational solutions being lump solutions, where $K_1 = \bar{K}_2$ and $P_1 = \bar{P}_2$, then u in Equation (4) can be written as

$$u = u_0 - \frac{12 b \left[\theta_1^2 ((\partial/\partial x)\theta_2)^2 + \theta_2^2 ((\partial/\partial x)\theta_1)^2 - 2 a_{12} ((\partial/\partial x)\theta_1)((\partial/\partial x)\theta_2)\right]}{c(\theta_1\theta_2 + a_{12})^2}. \tag{27}$$

This lump solution u in Equation (27) possesses three critical points:

$$A_1 = \left(\frac{2 c K_1 K_2 u_0 - P_1 P_2 c_0}{2K_2 K_1} t, \frac{(K_1 P_2 + K_2 P_1) c_0 t}{2 K_2 K_1} \right),$$
$$A_2 = \left(\frac{c_0 t (K_1 P_2 - K_2 P_1)(2 c K_1 K_2 u_0 - P_1 P_2 c_0) + 12\sqrt{2} K_1{}^2 K_2{}^2 \sqrt{c_0 b}}{2 c_0 (K_1 P_2 - K_2 P_1) K_1 K_2}, \frac{(K_1 P_2 + K_2 P_1) c_0 t}{2 K_2 K_1} \right),$$
$$A_3 = \left(\frac{c_0 t (K_1 P_2 - K_2 P_1)(2 c K_1 K_2 u_0 - P_1 P_2 c_0) - 12\sqrt{2} K_1{}^2 K_2{}^2 \sqrt{c_0 b}}{2 c_0 (K_1 P_2 - K_2 P_1) K_1 K_2}, \frac{(K_1 P_2 + K_2 P_1) c_0 t}{2 K_2 K_1} \right), \tag{28}$$

which are derived by solving $u_x = 0$ and $u_y = 0$. Based on the analysis of these critical points at the second-order derivatives in

$$u_{xx} u_{yy} - (u_{xy})^2, \tag{29}$$

to determine whether the aforementioned critical points are local maximum points or local minimum points.

In order to describe the properties of lump solutions more clearly and facilitate discussion, let us set $b = -1$. Then, the lump solution can be classified into two patterns:

(a) *Bright lump*. $u_0 > 0, c < 0$: u has one local maximum (point A_1) and two minimum points (points A_2 and A_3) (see Figure 2(a))

(b) *Dark lump*. $u_0 < 0, c > 0$: u has two local maximum (points A_2 and A_3) and one minimum point (point A_1) (see Figure 2(b))

Two different patterns of lump solutions, namely, bright-type and dark-type lump solutions, are shown in Figure 2.

3.2. Second-Order Breather and Lump Solutions. In order to obtain the general high-order breather solutions and lump solutions in Equation (1), we assume that the auxiliary function f in Equation (7) has higher-order expansions in terms of ε:

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots + \varepsilon^n f_n. \tag{30}$$

Again, substituting Equation (30) into bilinear Equation (7) and then collecting the coefficient of ε, $2n + 1$ equations would be yielded corresponding to different orders of ε. Maybe it is tedious and troublesome to solve these $2n + 1$ equations. According to the work of Hirota, Kaur and Wazwaz, and Singh et al., [7, 26, 27], we calculate and verify that f has the following form:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j<s}^{N} \mu_j \mu_s A_{js} + \sum_{j=1}^{N} \mu_j \eta_j \right), \tag{31}$$

where the sum of μ is the sum over all the possibilities of $\mu_j = 0, 1, (j = 1, 2 \cdots)$.

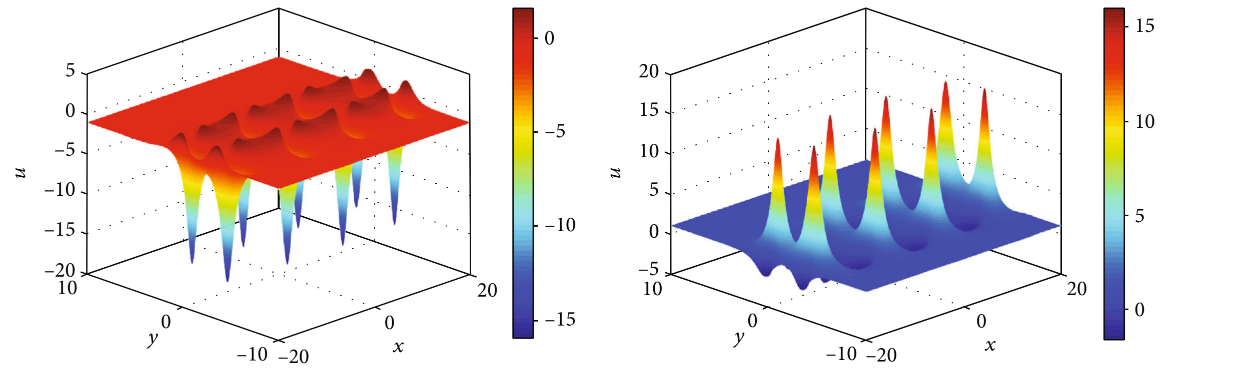


FIGURE 3: Two types of second-order breather solutions in Equation (1) with parameters $c_0 = 1$, $b = -1$, $k_1 = -1/2$, $k_2 = i/2$, $p_1 = -1$, $p_2 = -1$, $k_3 = -3/5\,i$, $k_4 = 3/5\,i$, $p_3 = -1$, $p_4 = -1$, $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$, and $\phi_4 = 0$ at $t = 0$. (a) Bright-type breather with $u_0 = 1$ and $c = -1$. (b) Dark-type breather with $u_0 = -1$ and $c = 1$.

The above coefficients and parameters are given explicitly as follows:

$$\eta_j = k_j x + \omega_j t + p_j y + \phi_j,$$
$$e^{A_{js}} = -\frac{2b\left(k_j - k_s\right)^4 + 2\mathrm{cu}_0\left(k_j - k_s\right)^2 + c_0\left(p_j - p_s\right)^2 + 2\left(w_j - w_s\right)\left(k_j - k_s\right)}{2b\left(k_j + k_s\right)^4 + 2\mathrm{cu}_0\left(k_j + k_s\right)^2 + c_0\left(p_j + p_s\right)^2 + 2\left(w_j + w_s\right)\left(k_j + k_s\right)}, \tag{32}$$

$$2\,\mathrm{bk}_s^{\,4} + 2\,\mathrm{ck}_s^{\,2}u_0 + c_0 p_s^{\,2} + 2\,k_s w_s = 0. \tag{33}$$

In order to obtain the second-order breather solutions in Equation (1) and for a valid calculation, there are also some restrictions as the first-order breather solution: (1) $\eta_1 = \bar{\eta}_2$, and $\eta_3 = \bar{\eta}_4$, so η_1 and η_2 and $\eta_3$and η_4 are conjugates of each other; (2) k_s, w_s, and p_s have to satisfy the constraint of Equation (33); and (3) $N = 4$ in Equation (31). Then, the evolution of second-breather solutions in Equation (1) are shown in the Figure 3.

The way to get second-order lump solutions is roughly the same as the way to get first-order lump solutions. We take $N = 4$ and $\exp(\phi_s) = -1, s = 1, 2, 3, 4$ in Equation (31) and take a long wave limit with the provision in Equation (31) and eliminate the $o(\varepsilon^4)$. Then, f is given as follows:

$$f = \prod_j \theta_j + \sum_{j<s} a_{\mathrm{js}} \prod_{k\neq \mathrm{js}} \theta_k + \sum_{j<s} \prod a_{\mathrm{js}}, \tag{34}$$

with

$$\theta_s = \frac{\left(-2\,K_s^{\,2}\mathrm{cu}_0 - P_s^{\,2}c_0\right)t}{2K_s} + K_s x + P_s y,$$
$$a_{\mathrm{js}} = \frac{24\mathrm{bK}_j^{\,3}K_s^{\,3}}{c_0\left(K_j P_s - K_s P_j\right)^2}, \tag{35}$$

where $j = 1, 2, 3, 4$, $s = 1, 2, 3, 4$, and $j < s$, K_j, P_j are complex parameters. To guarantee the corresponding rational solutions being lump solutions, there is a restriction for a valid calculation: $K_1 = \bar{K}_2$, $K_3 = \bar{K}_4$, $P_1 = \bar{P}_2$, and $P_3 = \bar{P}_4$.

Since there are too many parameters involved, in order to directly describe the properties of the second-order lump solutions, it is advisable to assign values to the following parameters: $b = -1$, $c_0 = 1$, $K_1 = -2 - i$, $K_2 = -2 + i$, $P_1 = 2$, $P_2 = 2$, $K_3 = -2 + i$, $K_4 = -2 - i$, $P_3 = -2$, and $P_4 = -2$. If using parameters$u_0 = 1$and$c = -1$, then we obtain the$f$ which can lead to bright-type lump solutions in Equation (1); if using parameters$u_0 = -1$and$c = 1$, then we can get dark-type lump solutions. After calculation, it can be found that f corresponding to bright-type lump solutions is the same as f corresponding to dark-type lump solutions. Then, f in Equation (34) can be written as

$$f = \frac{289\,t^4}{25} + \frac{1292\,t^3x}{25} + \frac{2294\,t^2x^2}{25} + \frac{104\,t^2y^2}{25} + 76\,\mathrm{tx}^3 - 16\,\mathrm{txy}^2 + 25\,x^4 - 24\,x^2y^2 + 16\,y^4 + \frac{70341\,t^2}{100} + \frac{3207\,\mathrm{tx}}{2} + \frac{4725\,x^2}{4} + 1791\,y^2 + \frac{2480625}{64}. \tag{36}$$

According to Equation (4), the bright-type lump is explicitly as follows:

$$u = 1 + 12(\ln f)_{\mathrm{xx}}, \tag{37}$$

and the dark-type lump is explicitly as follows:

$$u = -1 - 12(\ln f)_{\mathrm{xx}}, \tag{38}$$

where f is Equation (36). The second-order lump solutions in Equation (1) in the (x,y)-plane are shown in the Figure 4.

3.3. Higher-Order Breather and Lump Solutions. The similar procedures described previously could be generalized to the higher-order breather and lump solutions. To guarantee the nth-order breather solutions being real functions, there are two restrictions for a valid calculation: (1) take $\eta_1 = \bar{\eta}_2, \eta_3 = \bar{\eta}_4 \cdots \eta_{2n-1} = \bar{\eta}_{2n}$ in Equation (31). (2)

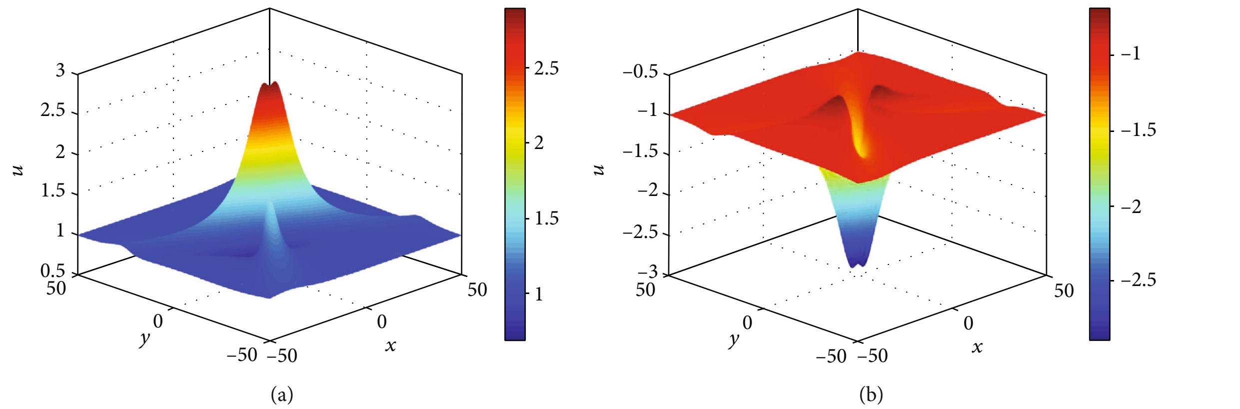


FIGURE 4: Two types of second-order lump solutions in Equation (1) with parameters $b=-1$, $c_0=1$, $K_1=-2-i$, $K_2=-2+i$, $P_1=2$, $P_2=2$, $K_3=-2+i$, $K_4=-2-i$, $P_3=-2$, and $P_4=-2$ at $t=0$. (a) Bright-type lump with $u_0=1$ and $c=-1$. (b) Dark-type lump with $u_0=-1$ and $c=1$.

Parameters k_s, p_s, and w_s must satisfy the constrain of Equation (33). Then, we obtain the nth-order breather solutions in Equation (1).

For example, if we want to obtain the third-order breather solutions, according to the description in the previous paragraph, we will take $\eta_1=\bar{\eta}_2$, $\eta_3=\bar{\eta}_4$, and $\eta_5=\bar{\eta}_6$ in Equation (31). In order to describe the properties of the third-breather solution more clearly and facilitate the calculation, the parameters can be assigned as follows: $c_0=1$, $b=-1$, $k_1=-i$, $k_2=i$, $p_1=2$, $p_2=2$, $k_3=i$, $k_4=-i$, $p_3=-7/4$, $p_4=-7/4$, $k_5=i$, $k_6=-i$, $p_5=-3/2$, $p_6=-3/2$, $\phi_s=0$, and $s=1,2,3,4,5,6$. If using parameters $u_0=1$ and $c=-1$, then we can derive bright-type breather solutions; if using parameters $u_0=-1$ and $c=1$, we can derive dark-type breather solutions. Under the conditions of these parameters, bright-type breather solutions and dark-type breather solutions have the same f. The third-breather solutions in Equation (1) in the (x,y)-plane are shown in the Figure 5.

The process of obtaining nth-order lump solutions is roughly similar to that of obtaining first-order lump solutions and second-order lump solutions. In order to obtain nth-order lump solutions, we take $N=2n$ in Equation (31). Then, we take a long wave limit with the provision in Equation (31):

$$\begin{aligned} p_s &= P_s\varepsilon, \\ k_s &= K_s\varepsilon, \varepsilon \mapsto 0, \\ e^{\phi_s} &= -1, \\ s &= 1,2\cdots N. \end{aligned} \tag{39}$$

And just like we did with Equations (24) and (26), we get rid of the higher-order terms of $o(\varepsilon^N)$, and then, we get a polynomial f. General higher-order rational solutions in Equation (1) can be presented in the following forms:

$$u = u_o + \frac{12b}{c}(\ln f_N)_{xx}, \tag{40}$$

where

$$f_N = \prod_{s=1}^{N}\theta_s + \frac{1}{2}\sum_{j,s}^{N} a_{js}\prod_{p\neq j,s}^{N}\theta_p + \cdots + \frac{1}{M!2^M}\sum_{l,s\cdots m,n}^{N} a_{ls}a_{jk}\overset{M}{\widehat{\cdots}}a_{mn}\prod_{q\neq l,s\cdots m,n}^{N}\theta_q + \cdots, \tag{41}$$

with

$$\begin{aligned} \theta_s &= \frac{\left(-2K_s^{\,2}\mathrm{cu}_0 - P_s^{\,2}c_0\right)t}{2K_s} + K_s x + P_s y, \\ a_{js} &= \frac{24bK_j^{\,3}K_s^{\,3}}{c_0\left(K_jP_s - K_sP_j\right)^2}. \end{aligned} \tag{42}$$

To guarantee the corresponding rational solutions being lump solutions, there are some restrictions for a valid calculation: $K_1=\bar{K}_2, K_3=\bar{K}_4, \cdots K_{2n-1}=\bar{K}_{2n}$ and $P_1=\bar{P}_2, P_3=\bar{P}_4, \cdots P_{2n-1}=\bar{P}_{2n}$.

For example, if we want to get third-lump solutions, we have to set N equal to 6 in Equation (41). In order to describe the properties of third-lump solution more clearly, we assign the following values to the following parameters: $c_0=1$, $b=-1$, $K_1=3i$, $K_2=-3i$, $P_1=1$, $P_2=1$, $K_3=2i$, $K_4=-2i$, $P_3=-2$, $P_4=-2$, $K_5=3i$, $K_6=-3i$, $P_5=2$, and $P_6=2$. And if using $u_0=1$ and $c=-1$, then we obtain bright-type lump solutions; if using $u_0=-1$ and $c=1$, then we obtain dark-type lump solutions. Under the parameter constraints above, two types of lump solutions have the same f, as shown below:

$$\begin{aligned} f = {} & \frac{\left(9t^2+12\,\mathrm{tx}+4x^2+4y^2\right)\left(121t^2+198\,\mathrm{tx}+81x^2+36y^2\right)\left(361t^2+684\,\mathrm{tx}+324x^2+36y^2\right)}{324} \\ & + \frac{4996352617\,t^4}{5400} + \frac{222120217\,t^3x}{75} + \frac{88451544\,t^2x^2}{25} + \frac{2053073\,t^2y^2}{15} + \frac{46738512\,\mathrm{tx}^3}{25} \\ & + \frac{5841228\,\mathrm{txy}^2}{25} + \frac{9216018\,x^4}{25} + \frac{2409264\,x^2y^2}{25} - \frac{828282\,y^4}{25} + \frac{196632775107\,t^2}{1250} \\ & + \frac{154136915604\,\mathrm{tx}}{625} + \frac{60286704219\,x^2}{625} + \frac{16102685466\,y^2}{625} + \frac{1976312557827}{1250}. \end{aligned} \tag{43}$$

And then from Equation (40), we can get two types of

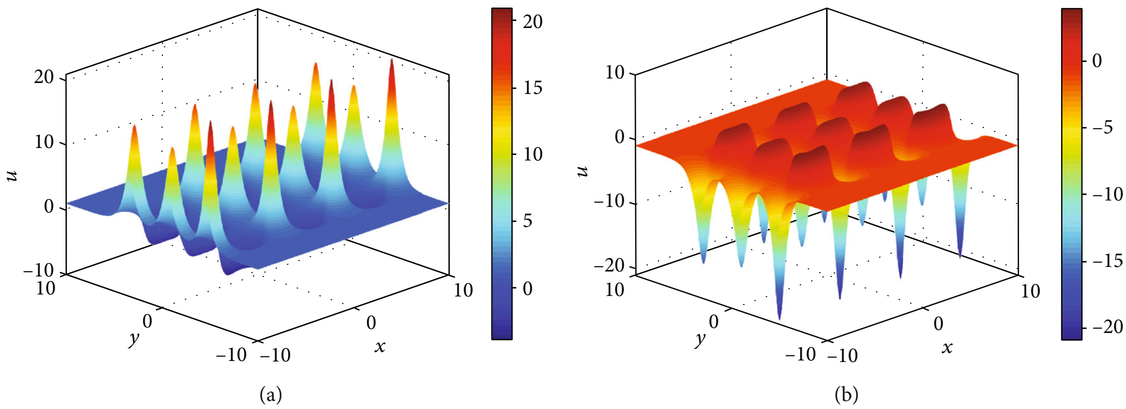


Figure 5: Two types of breather solutions in Equation (1) with parameters $c_0 = 1$, $b = -1$, $k_1 = -i$, $k_2 = i$, $p_1 = 2$, $p_2 = 2$, $k_3 = i$, $k_4 = -i$, $p_3 = -7/4$, $p_4 = -7/4$, $k_5 = i$, $k_6 = -i$, $p_5 = -3/2$, $p_6 = -3/2$, $\phi_s = 0$, and $s = 1, 2, 3, 4, 5, 6$ at $t = 0$. (a) Bright-type breather solution with $u_0 = 1$ and $c = -1$. (b) Dark-type breather solution with $u_0 = -1$ and $c = 1$.

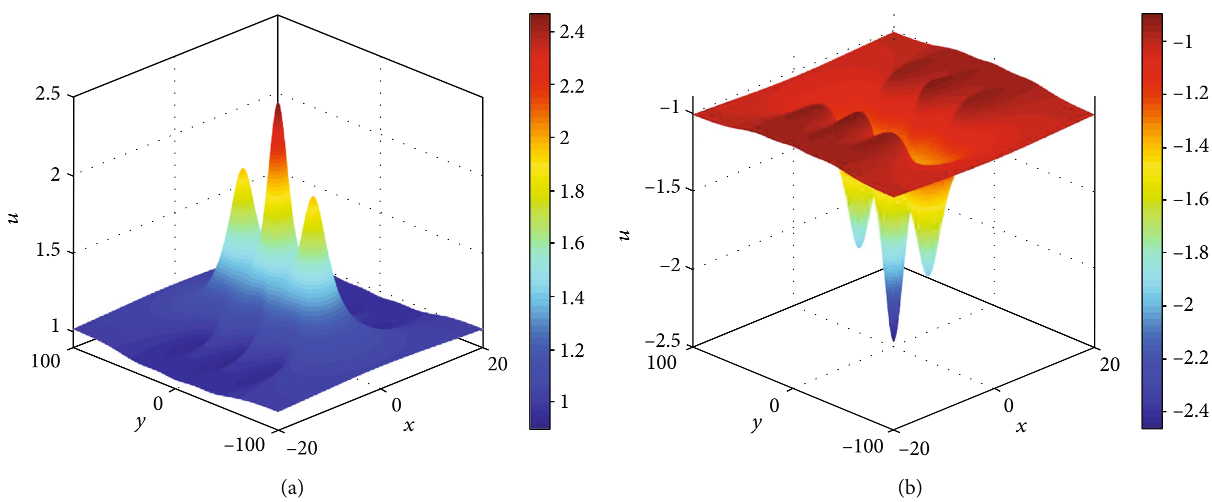


Figure 6: Two types of third-lump solution in Equation (1) at $t = 0$. (a) Bright-type lump with $u_0 = 1$ and $c = -1$. (b) Dark-type lump with $u_0 = -1$ and $c = 1$.

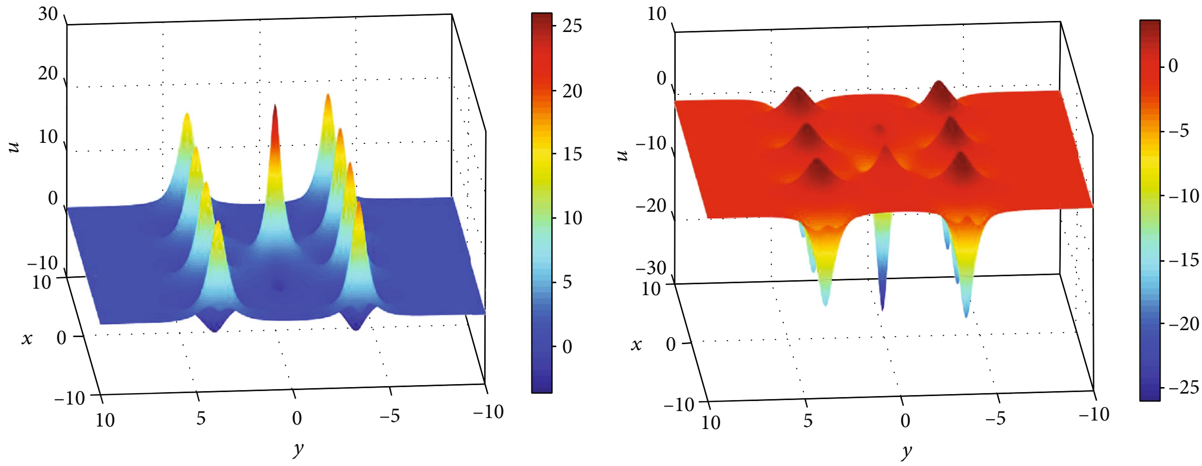
lump solutions. The third-lump solutions in Equation (1) in the (x,y)-plane are shown in Figure 6.

In addition, interaction solutions between breather solutions and lump solutions also can be obtained from solitons by taking long wave limits. If using $N = 6$, $p_1 = P_1\varepsilon$, $p_2 = P_2\varepsilon$, $k_1 = K_1\varepsilon$, $k_2 = K_2\varepsilon$, $P_1 = \bar{P}_2$, $K_1 = \bar{K}_2$, $\phi_1 = i\pi$, $\phi_2 = i\pi$, $\eta_3 = \bar{\eta}_4$, and $\eta_5 = \bar{\eta}_6$ in Equation (31), then we can obtain interactions between a lump and two breathers after taking a long wave limit. In order to better describe the structure of interactions, we assign the parameters as follows: $c_0 = 1$, $b = -1$, $K_1 = 4i$, $K_2 = -4i$, $P_1 = 10$, $P_2 = 10$, $k_3 = i$, $k_4 = -i$, $p_3 = -5/3$, $p_4 = -5/3$, $k_5 = i$, $k_6 = -i$, $p_5 = 11/6$, $p_6 = 11/6$, $\phi_1 = i\pi$, $\phi_2 = i\pi$, $\phi_3 = 0$, $\phi_4 = 0$, $\phi_5 = 0$, and $\phi_6 = 0$. If using $u_0 = 1$ and $c = -1$, then we get the bright-type interactions; if using $u_0 = -1$ and $c = 1$, then we can obtain dark-type interactions. Two types of interactions between a lump and two breathers in Equation (1) in the (x,y)-plane are shown in Figure 7.

The idea and process of obtaining interactions between two lumps and a breather are roughly the same as that of obtaining interaction between a lump and two breathers. If using $N = 6$, $p_s = P_s\varepsilon$, $k_s = K_s\varepsilon$, $\phi_s = i\pi$, $s = 1, 2, 3, 4$, $P_{2j-1} = \bar{P}_{2j}$, $K_{2j-1} = \bar{K}_{2j}$, $j = 1, 2$, and $\eta_5 = \bar{\eta}_6$ in Equation (31), then we can obtain interactions between two lumps and a breather after taking a long wave limit. In order to better describe the structure of interactions between a breather and two lumps in Equation (1), we assign the parameters as follows: $c_0 = 1$, $b = -1$, $K_1 = 3i$, $K_2 = -3i$, $P_1 = 5$, $P_2 = 5$, $K_3 = 3i$, $K_4 = -3i$, $P_3 = 6$, $P_4 = 6$, $k_5 = i$, $k_6 = -i$, $p_5 = 5/4$, $p_6 = 5/4$, $\phi_1 = i\pi$, $\phi_2 = i\pi$, $\phi_3 = i\pi$, $\phi_4 = i\pi$, $\phi_5 = 0$, and $\phi_6 = 0$, then we get the bright-type interactions; if using $u_0 = -1$ and $c = 1$, then we can obtain dark-type interactions. Two types of interactions between a breather and two lumps in Equation (1) in the (x,y)-plane are shown in Figure 8.

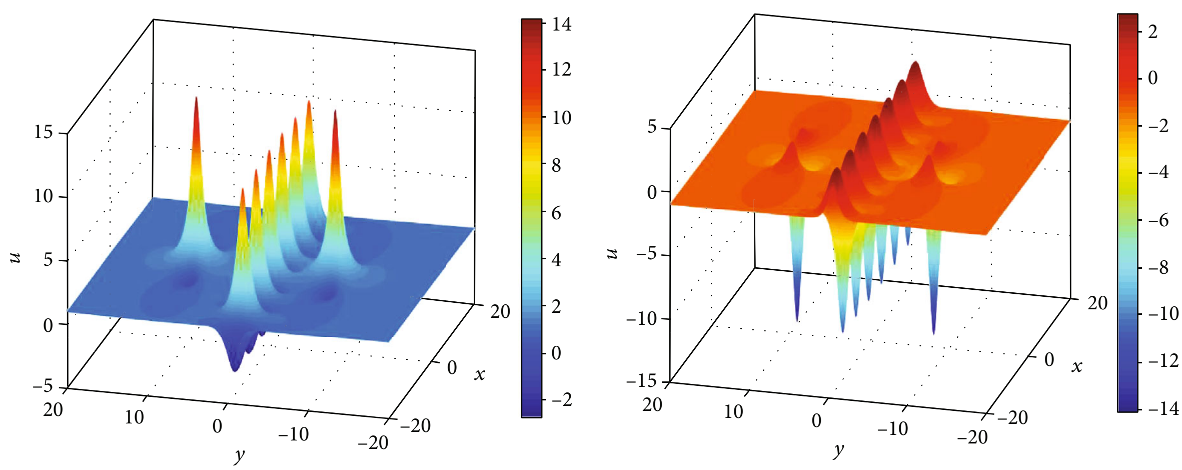
4. Interaction between Lumps and a Stripe

The method of obtaining interaction solutions is roughly similar to that of obtaining rational solutions, but slightly



(a) (b)

Figure 7: Two types of interactions between a lump and two breathers in Equation (1) at $t = 0$. (a) Bright-type interaction solution with $u_0 = 1$ and $c = -1$. (b) Dark-type interaction solution with $u_0 = -1$ and $c = 1$.



(a) (b)

Figure 8: Two types of interactions between a lump and two breathers in Equation (1) at $t = 0$. (a) Bright-type interaction solution with $u_0 = 1$ and $c = -1$. (b) Dark-type interaction solution with $u_0 = -1$ and $c = 1$.

different. In this part, we will describe how to obtain interaction solutions.

4.1. Interaction between a Lump and a Stripe. To obtain the interaction between a lump and a stripe of Equation (1), we substitute $N = 3$, $k_1 = K_1\varepsilon$, $k_2 = K_2\varepsilon$, $p_1 = P_1\varepsilon$, $p_2 = P_2\varepsilon$, $\phi_1 = i\pi$, and $\phi_2 = i\pi$ into Equation (31), and then, we expand the resulting f in terms of ε at $\varepsilon = 0$. Similar to Equations (24) and (26), we also want to get rid of $o(\varepsilon^2)$. In other words, we divide the expansion by ε^2, and then, we take the limit as ε is equal to 0. For convenience, the expression after we obtain the limit is still called f, as follows:

$$f = \theta_1\theta_2 + a_{12} + e^{\eta_3}(a_{13}a_{23} + \theta_2 a_{13} + \theta_1 a_{23} + \theta_1\theta_2 + a_{12}), \tag{44}$$

with

$$a_{js} = \begin{cases} \dfrac{24bK_j^3K_s^3}{c_0(K_jP_s - K_sP_j)^2} & s < 3, \\ -\dfrac{24bK_j^3k_3^3}{6bK_j^2k_3^4 - c_0K_j^2p_3^2 + 2c_0P_jp_3k_3K_j - c_0P_j^2k_3^2} & s = 3, \end{cases} \tag{45}$$

$$\theta_s = \frac{(-2K_s^2cu_0 - P_s^2c_0)t}{2K_s} + K_sx + P_sy\, s = 1, 2, 3, \quad 2bk_3^4 + 2ck_3^2u_0 + c_0p_3^2 + 2k_3w_3 = 0. \tag{46}$$

In order for the mixed solution in Equation (44) to become interaction between a lump and a stripe, there are some restrictions for a valid calculation: $K_1 = \bar{K}_2$, $P_1 = \bar{P}_2$, and parameters k_3, p_3, ϕ_3 must be real parameters. In order to describe the

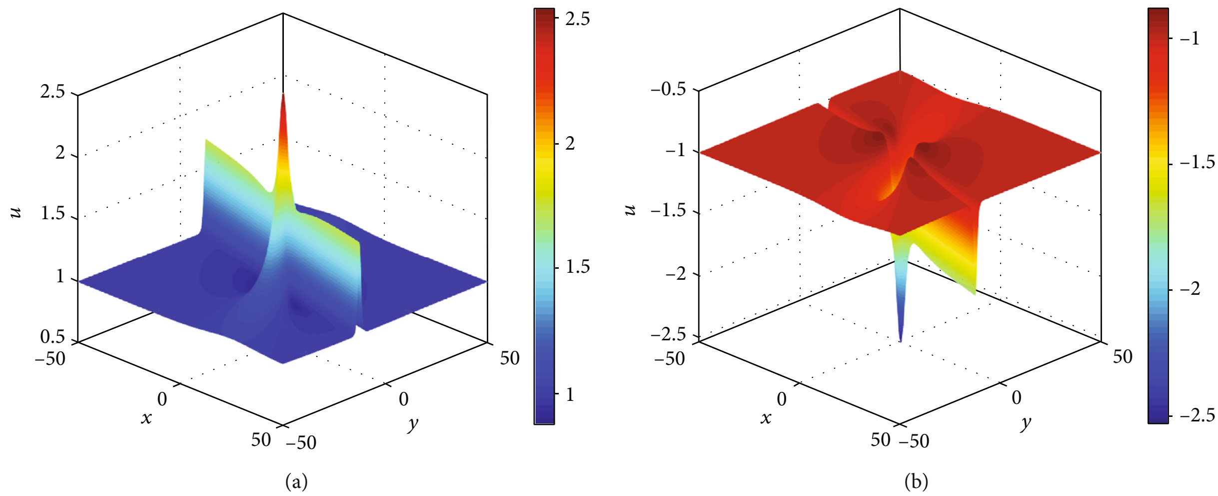


Figure 9: Two types of interactions between a lump and a stripe in Equation (1). (a) Bright-type interaction solutions with $u_0 = 1$ and $c = -1$. (b) Dark-type interaction solutions with $u_0 = -1$ and $c = 1$.

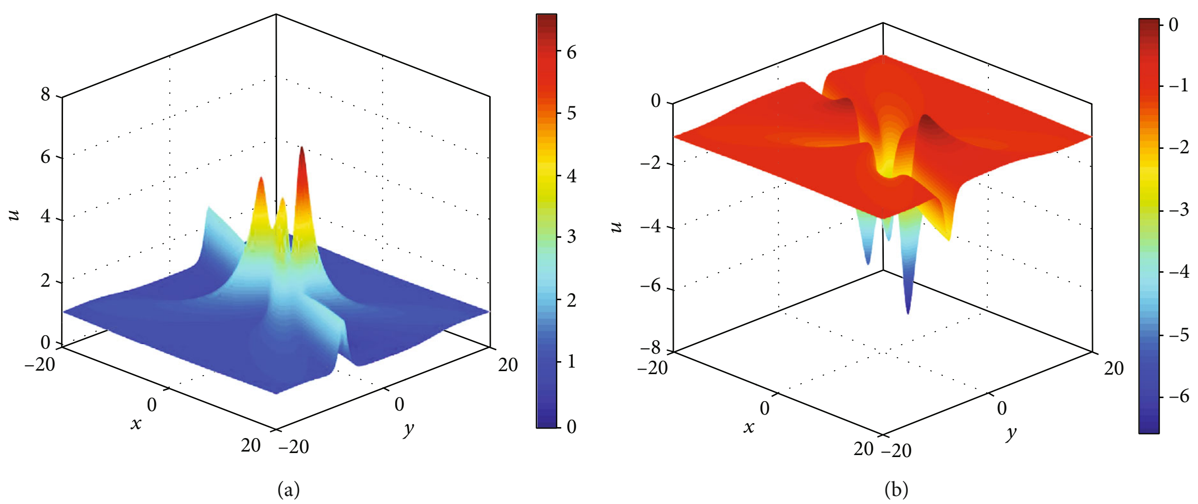


Figure 10: Two types of interactions between two lumps and a stripe in Equation (1). (a) Bright-type interaction solution with $u_0 = 1$ and $c = -1$. (b) Dark-type interaction solution with $u_0 = -1$ and $c = 1$.

properties of mixed solutions more directly and concretely, the parameters of Equation (45) are assigned as follows: $K_1 = 1 + i$, $K_2 = 1 - i$, $P_1 = 1$, $P_2 = 1$, $k_3 = 1/2$, $p_3 = 2$, $\phi_3 = 0$, $c_0 = 1$, and $b = -1$. If using $u_0 = 1$ and $c = -1$, we will get bright-type interaction solutions; if using $u_0 = -1$ and $c = 1$, we are going to get dark-type interaction solutions. After calculation, two types of interaction solutions have the same f under the above parameters. The interactions between a lump and a stripe in the (x,y)-plane are shown in the Figure 9.

$$f = \left(4\,tx + 2x^2 + y^2 + \frac{17t^2}{8} + \frac{3}{2}\,ty + 2\,xy + \frac{19248}{389} - \frac{1212\,t}{389} - \frac{1296\,x}{389} - \frac{816\,y}{389}\right)e^{-27t/8 + x/2 + 2y} + 4\,tx + 2x^2 + y^2 + \frac{17t^2}{8} + \frac{3}{2}\,ty + 2\,xy + 48. \tag{47}$$

4.2. Interaction between Two Lumps and a Stripe. The method and idea of obtaining interaction between two lumps and a stripe are roughly the same as the process of obtaining interaction between a lump and a stripe, but the calculation is more complicated. We substitute $k_s = K_s\varepsilon$, $p_s = P_s\varepsilon$, $\exp(\phi_s) = -1$, and $s = 1, 2, 3, 4$ into Equation (31), and then, we expand the expression at $\varepsilon = 0$. And then, we are going to divide this by ε^4, and we are going to take the limit as ε is equal to 0. For convenience, let us call this final result f.

$$\begin{aligned} f = {} & \theta_1\theta_2\theta_3\theta_4 + a_{12}\theta_3\theta_4 + a_{13}\theta_2\theta_4 + a_{14}\theta_2\theta_3 + a_{23}\theta_1\theta_4 + a_{24}\theta_1\theta_3 + a_{34}\theta_1\theta_2 \\ & + a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23} + e^{\eta_5}(a_{15}a_{25}a_{35}a_{45} + a_{15}a_{25}a_{35}\theta_4 + a_{15}a_{25}a_{45}\theta_3 \\ & + a_{15}a_{25}\theta_3\theta_4 + a_{15}a_{35}a_{45}\theta_2 + a_{15}a_{35}\theta_2\theta_4 + a_{15}a_{45}\theta_2\theta_3 + a_{15}\theta_2\theta_3\theta_4 \\ & + a_{25}a_{35}a_{45}\theta_1 + a_{25}a_{35}\theta_1\theta_4 + a_{25}a_{45}\theta_1\theta_3 + a_{25}\theta_1\theta_3\theta_4 + a_{35}a_{45}\theta_1\theta_2 \\ & + a_{35}\theta_1\theta_2\theta_4 + a_{45}\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_3\theta_4 + a_{12}a_{35}a_{45} + a_{12}a_{35}\theta_4 + a_{12}a_{45}\theta_3 \\ & + a_{12}\theta_3\theta_4 + a_{13}a_{25}a_{45} + a_{13}a_{25}\theta_4 + a_{13}a_{45}\theta_2 + a_{13}\theta_2\theta_4 + a_{14}a_{25}a_{35} \\ & + a_{14}a_{25}\theta_3 + a_{14}a_{35}\theta_2 + a_{14}\theta_2\theta_3 + a_{15}a_{23}a_{45} + a_{15}a_{23}\theta_4 \\ & + a_{15}a_{24}a_{35} + a_{15}a_{24}\theta_3 + a_{15}a_{25}a_{34} + a_{15}a_{34}\theta_2 + a_{23}a_{45}\theta_1 \\ & + a_{23}\theta_1\theta_4 + a_{24}a_{35}\theta_1 + a_{24}\theta_1\theta_3 + a_{25}a_{34}\theta_1 + a_{34}\theta_1\theta_2 \\ & + a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23}), \end{aligned} \tag{48}$$

with

$$a_{js} = \begin{cases} \dfrac{24\mathrm{bK}_j^3 K_s^3}{c_0 \left(K_j P_s - K_s P_j\right)^2} & s < 5, \\ \dfrac{-24\mathrm{bK}_j^3 k_5^3}{6\,\mathrm{bK}_j^2 k_5^4 - K_j^2 c_0 p_5^2 + 2\,K_j P_j c_0 k_5 p_5 - P_j^2 c_0 k_5^2} & s = 5, \end{cases} \tag{49}$$

and

$$\theta_s = \frac{\left(-2\,K_s^2 \mathrm{cu}_0 - P_s^2 c_0\right) t}{2K_s} + K_s x + P_s y \, s = 1, 2, 3, 4, \qquad 2\,\mathrm{bk}_5^4 + 2\,\mathrm{ck}_5^2 u_0 + c_0 p_5^2 + 2\,k_5 w_5 = 0. \tag{50}$$

In order to ensure that mixed solution is an interaction solution between two lumps and a stripe, we make some restrictions in Equation (48). $N = 5$, $K_1 = \bar{K}_2$, $K_3 = \bar{K}_4$, $P_1 = \bar{P}_2$, $P_3 = \bar{P}_4$, and parameters k_5, p_5, ϕ_5 must be real parameters. To intuitively describe the properties of mixed solutions of five soliton, the parameters of Equation (48) are assigned as follows: $K_1 = 1 + i$, $K_2 = 1 - i$, $P_1 = -2$, $P_2 = -2$, $K_3 = 1 - i$, $K_4 = 1 + i$, $P_3 = 2$, $P_4 = 2$, $k_5 = 3/4$, $p_5 = 2$, $\phi_5 = 0$, $c_0 = 1$, and $b = -1$. If using $u_0 = 1$ and $c = -1$, we will obtain bright-type interaction solutions; if using $u_0 = -1$ and $c = 1$,we will obtain dark-type interaction solutions. Under the above parameter constraints, two types of mixed solutions of five solitons correspond to the same f. The interaction solutions in the (x,y)-plane are shown in the Figure 10.

5. Conclusion

In this manuscript, applying Hirota's bilinear method with a perturbation parameter u_0 to generalized Kadomtsev-Petviashvili equation, we obtain a periodic breather wave solution. Meanwhile, lump solutions and interaction solutions are generated from solitons by taking long wave limits. The exact solutions contain some free parameters u_0, b, c, c_0, so some new and interesting space structures of breather, lump, and interaction solutions are found and investigated, which include structures of bright type and structures of dark type. Our results show the diversity of the spatial and space-time structures of solitary waves in nonlinear dynamic systems. Meanwhile, we also hope that our results will provide some valuable information in the field of nonlinear science.

Data Availability

All the data and formulas are in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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