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TRIGONOMETRIC APPROXIMATIONS FOR SOME BESSEL FUNCTIONS

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Formulas are obtained for approximating the tabulated Bessel functions $J_n(x)$, $n = 0-9$ in terms of trigonometric functions. These formulas can be easily integrated and differentiated and are convenient for personal computers and pocket calculators.

Keywords: Bessel functions; trigonometric functions; personal computers; pocket calculators; engineering applications

INTRODUCTION

Ordinary Bessel functions of the first kind $J_n(x)$ arise in many physical and engineering applications. These functions must often be approximated by appropriate formulas suitable for implementation using personal computers and pocket calculators. Although main-frame computer routines are available for evaluating Bessel functions they are not suitable for implementation using pocket calculators and personal computers. In an attempt to obtain formulas for approximate computation of Bessel functions, Waldron [1] obtained trigonometric approximations for $J_0(x)$ and $J_1(x)$ from which $J_n(x), n > 1$ can be obtained using the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{1}$$

However, (1) can yield accurate results only if $n < x$, otherwise severe accumulation of rounding errors will result [2]. Moreover, the factor

$2n/x$ can prevent the use of the resulting approximations for finding simple expressions for integrals involving Bessel functions [3]. Blachman and Mousavinezhad [3] obtained the approximate expression of (2) for $J_0(x)$.

$$J_0(x) = 1/6 + (1/3)\cos(x/2) + (1/3)\cos(\sqrt{3}x/2) + (1/6)\cos x \quad (2)$$

Using (2) and the recurrence relations of (3) and (4) the Bessel functions of any order n can be obtained.

$$J_1(x) = -\frac{d}{dx}J_0(x) \quad (3)$$

and

$$J_{n+1}(x) = J_{n-1}(x) - 2\frac{d}{dx}J_n(x) \quad (4)$$

This procedure yields

$$J_1(x) = (1/6)\sin(x/2) + (1/6)\sin x + (\sqrt{3}/6)\sin(\sqrt{3}x/2) \quad (5)$$

$$J_2(x) = (1/6) + (1/6)\cos(x/2) - (1/6)\cos(\sqrt{3}x/2) - (1/6)\cos x \quad (6)$$

$$J_3(x) = (1/3)\sin(x/2) - (1/6)\sin x \quad (7)$$

$$J_4(x) = (1/6) - (1/6)\cos(x/2) - (1/6)\cos(\sqrt{3}x/2) + (1/6)\cos x \quad (8)$$

$$J_5(x) = (1/6)\sin(x/2) + (1/6)\sin x - (\sqrt{3}/6)\sin(\sqrt{3}x/2) \quad (9)$$

$$J_6(x) = (1/6) - (1/3)\cos(x/2) - (1/6)\cos x + (1/3)\cos(\sqrt{3}x/2) \quad (10)$$

$$J_7(x) = -(1/6)\sin(x/2) - (1/6)\sin x + (\sqrt{3}/6)\sin(\sqrt{3}x/2) \quad (11)$$

$$J_8(x) = (1/6) - (1/6)\cos(x/2) + (1/6)\cos x - (1/6)\cos(\sqrt{3}x/2) \quad (12)$$

and

$$J_9(x) = -(1/3)\sin(x/2) + (1/3)\sin x \quad (13)$$

Using (2), (5) – (13), the Bessel functions $J_n(x)$, $n = 0-9$ were calculated and the results are shown in Tables I–X together with the corresponding values obtained from mathematical tables [2]. From columns 2 and 4 of Tables I–X one can easily see that the accuracy of the

TABLE I Comparison of $J_0(x)$ with its approximations

x	$J_0(x)$ *Tables* [2]	$J_0(x)$ *Eq.* (14)	$J_0(x)$ *Eq.* (2)
0	1	1	1
1	0.76519	0.76505	0.76520
2	0.22389	0.22399	0.22389
3	−0.26005	−0.25971	−0.26005
4	−0.39714	−0.39679	−0.39714
5	−0.17759	−0.17741	−0.17744
6	0.15064	0.15059	0.15174
7	0.30007	0.29797	0.30539
8	0.17165	0.17163	0.19090
9	−0.09033	−0.09015	−0.03555
10	−0.24593	−0.24566	−0.11919
11	−0.17119	−0.17099	0.07201
12	0.04768	0.04773	0.43825
13	0.20692	0.20686	0.73002
14	0.17107	0.17104	0.74205
15	−0.01422	−0.01412	0.45941
16	−0.17489	−0.17469	0.05092
17	−0.16985	−0.16966	−0.26400

TABLE II Comparison of $J_1(x)$ with its approximations

x	$J_1(x)$ *Tables* [2]	$J_1(x)$ *Eq.* (14)	$J_1(x)$ *Eq.* (5)
0	0	0	0
1	0.44005	0.43977	0.44005
2	0.57672	0.57406	0.57672
3	0.33905	0.33889	0.33906
4	−0.06604	−0.06594	−0.06608
5	−0.32757	−0.32736	−0.32791
6	−0.27668	−0.27653	−0.27860
7	−0.00468	−0.00470	−0.01225
8	0.23463	0.23444	0.21231
9	0.24531	0.24514	0.19392
10	0.04347	0.04360	−0.05067
11	−0.17678	−0.17664	−0.31350
12	−0.22344	−0.22328	−0.37372
13	−0.07031	−0.07025	−0.17289
14	0.13337	0.13328	0.15111
15	0.20510	0.20497	0.38348
16	0.09039	0.09035	0.39428
17	−0.09766	−0.09756	0.21348

TABLE III Comparison of $J_2(x)$ with its approximations

x	$J_2(x)$ *Tables* [2]	$J_2(x)$ *Eq.* (14)	$J_2(x)$ *Eq.* (6)
0	0	0	0
1	0.11490	0.11497	0.11490
2	0.35283	0.35273	0.35283
3	0.48609	0.48588	0.48610
4	0.36412	0.36396	0.36432
5	0.04656	0.04657	0.04804
6	−0.24287	−0.24270	−0.23588
7	−0.30141	−0.30123	−0.27767
8	−0.11299	−0.11293	−0.05121
9	0.14484	0.14472	0.27344
10	0.25463	0.25445	0.47407
11	0.13904	0.13894	0.44985
12	−0.08493	−0.08486	0.28061
13	−0.21774	−0.21759	0.13490
14	−0.15201	−0.15191	0.11888
15	0.04157	0.04154	0.19915
16	0.18619	0.18606	0.25585
17	0.15836	0.15826	0.20426

TABLE IV Comparison of $J_3(x)$ with its approximations

x	$J_3(x)$ *Tables* [2]	$J_3(x)$ *Eq.* (14)	$J_3(x)$ *Eq.* (7)
0	0	0	0
1	0.01956	0.01969	0.01956
2	0.12894	0.12903	0.12894
3	0.30906	0.30896	0.30898
4	0.43017	0.42987	0.42923
5	0.36483	0.36794	0.35931
6	0.11477	0.11471	0.09361
7	−0.16756	−0.16732	−0.22643
8	−0.29113	−0.29078	−0.41716
9	−0.18094	−0.18074	−0.39453
10	0.05838	0.05827	−0.22897
11	0.22735	0.22704	−0.06852
12	0.19514	0.19489	−0.00371
13	0.00332	0.00333	0.00168
14	−0.17681	−0.17656	0.05389
15	−0.19402	−0.19376	0.20429
16	−0.04385	−0.04380	0.37777
17	0.13493	0.13472	0.42640
18	0.18632	0.18607	0.26254
19	0.07249	0.07244	−0.05003
20	−0.09890	−0.09709	−0.33350

TABLE V Comparison of $J_4(x)$ with its approximations

x	$J_4(x)$ *Tables* [2]	$J_4(x)$ *Eq.* (14)	$J_4(x)$ *Eq.* (8)
0	0	0	0
1	0.00248	0.00251	0.00248
2	0.03400	0.03409	0.03402
3	0.13203	0.13209	0.13253
4	0.28113	0.28105	0.28516
5	0.39123	0.39100	0.40964
6	0.35764	0.35738	0.41417
7	0.15780	0.15769	0.28578
8	−0.10536	−0.10520	0.11818
9	−0.26547	−0.26517	0.03999
10	−0.21960	−0.21936	0.09983
11	−0.01504	−0.01504	0.21510
12	0.18250	0.18225	0.24184
13	0.21928	0.21900	0.11185
14	0.07624	0.07616	−0.08684
15	−0.11918	−0.11900	−0.16963
16	−0.20264	−0.20237	−0.01487
17	−0.11074	−0.11060	0.31321
18	0.06964	−0.06954	0.59405
19	0.18065	0.18042	0.61999
20	0.13067	0.13193	0.36757

TABLE VI Comparison of $J_5(x)$ with its approximations

x	$J_5(x)$ *Tables* [2]	$J_5(x)$ *Eq.* (14)	$J_5(x)$ *Eq.* (9)
0	0	0	0
1	0.00025	0.00029	0.00025
2	0.00704	0.00708	0.00686
3	0.04303	0.04310	0.04048
4	0.13209	0.13213	0.11691
5	0.26114	0.26108	0.20776
6	0.36209	0.36190	0.23250
7	0.34790	0.34767	0.11432
8	0.18577	0.18564	−0.13480
9	−0.05504	−0.05494	−0.38240
10	−0.23406	−0.23380	−0.45031
11	−0.23829	−0.23804	−0.25501
12	−0.07347	−0.07340	0.10172
13	0.13162	0.13144	0.38465
14	0.22038	0.22010	0.39808
15	0.13046	0.13030	0.14595
16	−0.05747	−0.05737	−0.16046
17	−0.18704	−0.18679	−0.26778
18	−0.15537	−0.15517	−0.09090
19	0.00357	0.00353	0.20849
20	0.15117	0.14990	0.34991

TABLE VII Comparison of $J_6(x)$ with its approximations

x	$J_6(x)$ *Tables* [2]	$J_6(x)$ *Eq.* (14)	$J_6(x)$ *Eq.* (10)
0	0	0	0
1	0.00002	0.00002	0.00004
2	0.00120	0.00121	0.00240
3	0.01139	0.01143	0.02279
4	0.04909	0.04915	0.09818
5	0.13105	0.13108	0.26210
6	0.24584	0.24579	0.49167
7	0.33920	0.33904	0.67839
8	0.33758	0.33737	0.67516
9	0.20432	0.20417	0.40869
10	−0.01446	−0.01442	−0.02861
11	−0.20158	−0.20137	−0.40191
12	−0.24372	−0.24346	−0.48315
13	−0.11803	−0.11790	−0.22352
14	0.08117	0.08106	0.19387
15	0.20615	0.20589	0.48155
16	0.16672	0.16652	0.46714
17	0.00072	0.00074	0.22906
18	−0.15596	−0.15576	0.02937
19	−0.17877	−0.17858	0.08959
20	−0.05509	−0.05695	0.39225

TABLE VIII Comparison of $J_7(x)$ with its approximations

x	$J_7(x)$ *Tables* [2]	$J_7(x)$ *Eq.* (14)	$J_7(x)$ *Eq.* (11)
0	0	0	0
1	0	0	−0.00025
2	0.000174	0.000176	−0.00686
3	0.00255	0.00256	−0.04048
4	0.01518	0.01521	−0.11691
5	0.05338	0.05344	−0.20776
6	0.12959	0.12962	−0.23250
7	0.23358	0.23354	−0.11432
8	0.32059	0.32046	0.13480
9	0.32746	0.32727	0.38240
10	0.21671	0.21657	0.45031
11	0.01838	0.01839	0.25500
12	−0.17025	−0.17007	−0.10172
13	−0.24057	−0.24033	−0.38465
14	−0.15080	−0.15066	−0.39808
15	0.03446	0.03441	−0.14595
16	0.18251	0.18229	0.16046
17	0.18755	0.18733	0.26778
18	0.05140	0.05135	0.09090
19	−0.11648	−0.11633	−0.02085
20	−0.18422	−0.18410	−0.34991

TABLE IX Comparison of $J_8(x)$ with its approximations

x	$J_8(x)$ *Tables* [2]	$J_8(x)$ *Eq.* (14)	$J_8(x)$ *Eq.* (12)
0	0	0	0
1	0	0	0.00248
2	0.0000222	0.0000252	0.03402
3	0.000493	0.000497	0.13253
4	0.00403	0.00404	0.28516
5	0.01841	0.01843	0.40964
6	0.05653	0.05658	0.41417
7	0.12797	0.12798	0.28578
8	0.22345	0.22341	0.11818
9	0.30507	0.30495	0.03999
10	0.31785	0.31769	0.09983
11	0.22497	0.22483	0.21510
12	0.04510	0.04508	0.24184
13	−0.14105	−0.14091	0.11185
14	−0.23197	−0.23174	−0.08684
15	−0.17398	−0.17381	−0.16963
16	−0.00702	−0.00702	−0.01487
17	0.15374	0.15354	0.31321
18	0.19593	0.19571	0.59405
19	0.09294	0.09288	0.61999
20	−0.07387	−0.07200	0.36757

TABLE X Comparison of $J_9(x)$ with its approximations

x	$J_9(x)$ *Tables* [2]	$J_9(x)$ *Eq.* (14)	$J_9(x)$ *Eq.* (13)
0	0	0	0
1	0	0	0.12068
2	0	0	0.02261
3	0.0000844	0.0000874	−0.28546
4	0.00094	0.00094	−0.55537
5	0.00552	0.00554	−0.51913
6	0.02117	0.02120	−0.14018
7	0.05892	0.05896	0.33592
8	0.12632	0.12633	0.58205
9	0.21488	0.21484	0.46322
10	0.29186	0.29176	0.13830
11	0.30886	0.30871	−0.09815
12	0.23038	0.23024	−0.08572
13	0.06698	0.06695	0.06835
14	−0.11431	−0.11419	0.11121
15	−0.22005	−0.21984	−0.09590
16	−0.18953	−0.18935	−0.42575
17	−0.04286	−0.04283	−0.58663
18	0.12276	0.12261	−0.38770
19	0.19474	0.19455	0.07501
20	0.12513	0.12649	0.48566

approximations of (5)–(13) deteriorates as the order, n, of the Bessel function $J_n(x)$ increases and also as the argument, x, increases.

The major intention of this paper is, therefore, to present new trigonometric approximate expressions for the Bessel functions of any order n. These expressions are obtained by approximating the tabulated values of Bessel functions using Fourier-series representation. Comparisons with tabulated values of Bessel functions show that the proposed approximations enjoy excellent accuracies.

PROPOSED APPROXIMATIONS

Here we propose to approximate the Bessel function $J_n(x)$, in the range $x \leq B$, using the Fourier-series of (14).

$$J_n(x) = \sum_{m=0}^{\infty} a_{nm} \cos\left(\frac{m\pi}{B}x\right) \tag{14}$$

In general, the parameters a_{n0} and a_{nm} can be obtained using standard curve-fitting or discrete Fourier transfer (DFT) techniques. These techniques invariably demand extensive computing facilities and well-developed software. Alternatively, by using short-cut methods [4, 5], the parameters a_{nm} can be obtained without recourse to the DFT or standard curve-fitting routines. However, these methods result in Fourier-series having only a finite number of terms. This raises the question of the order of the Fourier-series as this will significantly affect the accuracy of the resulting approximations. In general, a high-order Fourier-series requires a proportionally large number of data points input to the short-cut methods as well as the DFT and the standard curve-fitting techniques. However, using the procedure described in [6] it is possible, at least in theory, to obtain an infinite-order Fourier-series model using a finite number of data points. Thus the number of terms of the Fourier-series can be arbitrarily increased until any desired degree of accuracy is achieved. In principle the procedure described in [6] is based on a short-cut approach in which the function $J_n(x)$ is mirror-imaged and then interpolated between the tabulated data points by using straight line segments joined end-to-end as shown in Figure 1. Using the slopes of the straight-line segments of Figure 1, it is easy

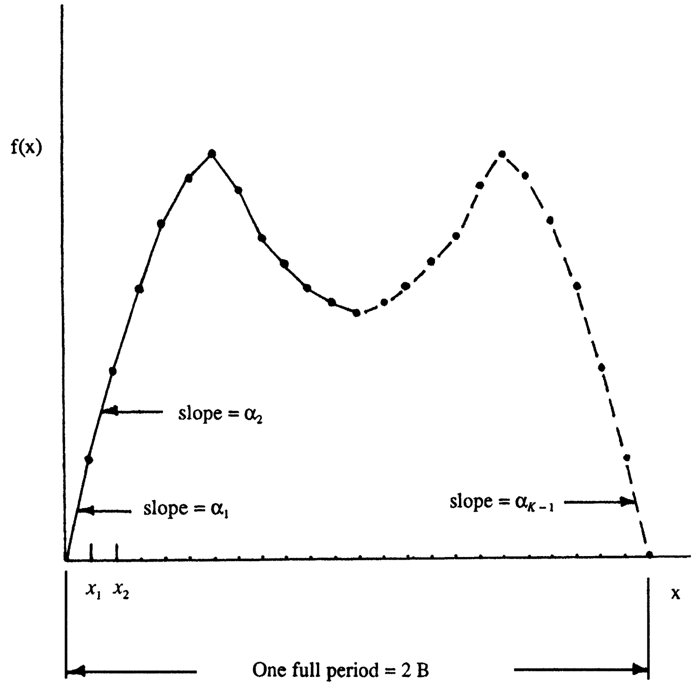

FIGURE 1 Tabulated data after mirror imaging and approximation by straight line segments.

using the procedure described by Kreyszig [7] to show that the parameters a_{nm} can be expressed by

$$a_{nm} = \frac{-B}{(m\pi)^2}\left(\alpha_1 - \alpha_{K-1} + \sum_{k=1}^{K-2}(\alpha_{k+1} - \alpha_k)\cos\left(\frac{m\pi}{B}x_{k+1}\right)\right) \qquad (15)$$

The results obtained for the Bessel functions $J_n(x)$, $n = 0-9$, with $B = 17.5$ for $J_0(x) - J_1(x)$ and $B = 20$ for $J_3(x) - J_9(x)$, are shown in Tables XI and XII.

Using the parameters of Tables XI and XII and Eq. (14) calculations were made and the results are shown in Tables I–X. Comparing columns 2 and 3 of Tables I–X one can easily see that the proposed approximations for the Bessel functions $J_n(x)$, $n = 0-9$ are in excellent agreement with their tabulated values.

TABLE XI Parameters a_{nm} of the Bessel function $J_n(x)$

m	n = 0	1	2	3	4
0	0.045975	−0.11742	0.070450	0.069537	0.10891
1	0.13467	0.10201	0.088666	0.074446	0.069250
2	0.10100	0.12789	0.11301	0.077580	0.043844
3	0.16036	0.097121	0.030011	−0.008820	−0.058767
4	0.12909	0.14014	0.033279	−0.031927	−0.097518
5	0.33356	0.044599	−0.24124	−0.18851	−0.17230
6	0.11367	−0.22829	−0.11277	−0.11337	0.11823
7	−0.032948	−0.059191	0.033792	0.11266	0.054726
8	0.017887	−0.053971	−0.018578	−0.006727	−0.021027
9	−0.011752	−0.024999	0.012281	0.020823	0.012360
10	0.0084883	−0.027894	−0.008901	−0.005692	−0.008452

TABLE XII Parameters a_{nm} of the Bessel function $J_n(x)$

m	n = 5	6	7	8	9
0	0.038932	0.036490	0.49863	0.063432	0.060823
1	0.082409	0.077591	0.045243	0.011402	−0.000127
2	−0.014968	−0.055757	−0.064547	−0.066475	−0.084917
3	−0.073121	−0.087796	−0.10956	−0.10571	−0.052421
4	−0.14365	−0.10642	0.0084468	0.11960	0.15094
5	0.0026941	0.15197	0.15016	0.034399	−0.076617
6	0.15561	0.023082	−0.091267	−0.087210	−0.010134
7	−0.056002	−0.059775	0.0091717	0.047930	0.026255
8	0.0095681	0.025285	0.0025383	−0.021244	−0.016787
9	−0.009506	−0.015442	−0.000837	0.013327	0.010785
10	0.0053299	0.010772	0.0007886	−0.009451	−0.007801

CONCLUSION

In this paper new formulas for approximating Bessel functions $J_n(x)$, $n = 0-9$ have been presented. These formulas, in terms of trigonometric functions, can be easily integrated and differentiated and are, therefore, convenient for further mathematical processing. Although the approximations presented here are valid for values of $x \leq 20$, their extension to cover wider ranges of arguments x is straight forward. The proposed approximations do not use any recurrence relation and, therefore, avoid the severe accumulation of rounding errors. Finally, it is worth mentioning that the algorithm used for obtaining the present approximations for the Bessel functions $J_n(x)$ is general and can be used for obtaining trigonometric approximations for any tabulated functions.

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