ON BOUNDEDNESS OF THE SOLUTIONS OF THE DIFFERENCE EQUATION $x_{n+1} = x_{n-1}/(p + x_n)$

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We study the difference equation $x_{n+1} = x_{n-1}/(p + x_n)$, $n = 0,1,\ldots$, where initial values $x_{-1}, x_0 \in (0,+\infty)$ and $0 < p < 1$, and obtain the set of all initial values $(x_{-1}, x_0) \in (0,+\infty) \times (0,+\infty)$ such that the positive solution $\{x_n\}_{n=-1}^{\infty}$ is bounded. This answers the Open Problem 2 proposed by Kulenović and Ladas.

Kulenović and Ladas in [2] (also see [1]) studied the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{p + x_n}, \quad n = 0,1,\ldots, \tag{1}$$

where initial values $x_{-1}, x_0 \in (0,+\infty)$ and $p \in (0,+\infty)$, and obtained the following theorem.

Theorem 1. (i) If $p > 1$, then the unique equilibrium 0 of (1) is globally asymptotically stable.

(ii) If $p = 1$, then every positive solution of (1) converges to a period-two solution.

(iii) If $0 < p < 1$, then 0 and $\overline{x} = 1 - p$ are the only equilibrium points of (1), and every positive solution $\{x_n\}_{n=-1}^{\infty}$ of (1) with $(x_N - \overline{x})(x_{N+1} - \overline{x}) < 0$ for some $N \geq -1$ is unbounded.

They proposed the following open problem.

Open Problem 2. Assume that $0 < p < 1$. Determine the set of initial values $x_{-1}, x_0 \in (0, +\infty)$ for which the solution $\{x_n\}_{n=-1}^{\infty}$ of (1) is bounded.

In this note, we will answer the above open problem.

Write $D = (0,+\infty) \times (0,+\infty)$ and define $f : D \to D$ by, for all $(x,y) \in D$,

$$f(x,y) = \left(y, \frac{x}{p+y}\right). \tag{2}$$

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It is easy to see that if $\{x_n\}_{n=-1}^{\infty}$ is a solution of (1), then $f^n(x_{-1},x_0)=(x_{n-1},x_n)$ for any $n\geq 0$. From Theorem 1, we have the following corollary.

Corollary 3. *Let $0<p<1$, $(x_{-1},x_0)\in D$, and $(x_{n-1},x_n)=f^n(x_{-1},x_0)$ for any $n\geq 0$. If there exists $N\geq -1$ such that $(x_N-\overline{x})(x_{N+1}-\overline{x})<0$, then $\{x_n\}_{n=-1}^{\infty}$ is a unbounded solution of (1).*

Let

$$\begin{aligned} A_1&=(0,\overline{x})\times(0,\overline{x}), & A_2&=(\overline{x},+\infty)\times(\overline{x},+\infty),\\ A_3&=(0,\overline{x})\times(\overline{x},+\infty), & A_4&=(\overline{x},+\infty)\times(0,\overline{x}),\\ R_0&=\{\overline{x}\}\times(0,\overline{x}), & L_0&=\{\overline{x}\}\times(\overline{x},+\infty),\\ R_1&=(0,\overline{x})\times\{\overline{x}\}, & L_1&=(\overline{x},+\infty)\times\{\overline{x}\}. \end{aligned} \tag{3}$$

Then $D=(\cup_{i=1}^{4}A_i)\cup L_0\cup L_1\cup R_0\cup R_1\cup\{(\overline{x},\overline{x})\}$.

Lemma 4. *The following statements are true.*

(i) *f is a homeomorphism.*
(ii) *$f(L_1)=L_0$ and $f(L_0)\subset A_4$.*
(iii) *$f(R_1)=R_0$ and $f(R_0)\subset A_3$.*
(iv) *$f(A_3)\subset A_4$ and $f(A_4)\subset A_3$.*
(v) *$A_2\cup L_1\subset f(A_2)\subset A_2\cup L_1\cup A_4$ and $A_1\cup R_1\subset f(A_1)\subset A_1\cup R_1\cup A_3$.*

Proof. (i) Since $f(x_1,y_1)\neq f(x_2,y_2)$ for any $(x_1,y_1),(x_2,y_2)\in D$ with $(x_1,y_1)\neq(x_2,y_2)$ and $f^{-1}(u,v)=(v(p+u),u)$ is continuous, f is a homeomorphism.

(ii) Let $(x,y)\in L_1$ and $(u,v)=f(x,y)=(y,x/(p+y))$, then $y=\overline{x}$ and $x>\overline{x}$, it follows

$$u=y=\overline{x},\qquad v=\frac{x}{(p+y)}>\frac{\overline{x}}{(p+\overline{x})}=\overline{x}, \tag{4}$$

which implies $f(L_1)\subset L_0$.

On the other hand, let $(u,v)\in L_0$ and $(x,y)=f^{-1}(u,v)=(v(p+u),u)$, then $u=\overline{x}$ and $v>\overline{x}$, it follows

$$y=u=\overline{x},\qquad x=v(p+u)>\overline{x}(p+\overline{x})=\overline{x}, \tag{5}$$

which implies $f^{-1}(L_0)\subset L_1$. Thus $f(L_1)=L_0$.

Now let $(x,y)\in L_0$ and $(u,v)=f(x,y)=(y,x/(p+y))$, then $x=\overline{x}$ and $y>\overline{x}$, it follows

$$u=y>\overline{x},\qquad v=\frac{x}{(p+y)}<\overline{x}, \tag{6}$$

which implies $f(L_0)\subset A_4$.

The proof of (iii) is similar to that of (ii).

(iv) Let $(x,y) \in A_3$ and $(u,v) = f(x,y) = (y, x/(p+y))$, then $\overline{x} < y$ and $0 < x < \overline{x}$, from which it follows

$$v = \frac{x}{(p+y)} < \frac{\overline{x}}{(p+\overline{x})} = \overline{x}, \quad u > \overline{x}. \tag{7}$$

Thus $(u,v) \in A_4$. In a similar fashion, we may show $f(A_4) \subset A_3$.

(v) Let $(x,y) \in A_2$ and $(u,v) = f(x,y) = (y, x/(p+y))$, then $y > \overline{x}$ and $x > \overline{x}$, from which it follows $u > \overline{x}$. Since f is a homeomorphism and $L_0 \cup L_1 \cup \{(\overline{x},\overline{x})\}$ is the boundary of A_2 with $f(L_1) = L_0$ and $f(L_0) \subset A_4$, we obtain $A_2 \cup L_1 \subset f(A_2) \subset A_2 \cup L_1 \cup A_4$. We similarly have $A_1 \cup R_1 \subset f(A_1) \subset A_1 \cup R_1 \cup A_3$. Lemma 4 is proven. □

Lemma 5. *If $0 < p < 1$ and $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (1) with $x_n \geq \overline{x} = 1-p$ for all $n \geq -1$ (or $x_n \leq \overline{x} = 1-p$ for all $n \geq -1$), then $\lim_{n\to\infty} x_n = \overline{x}$.*

Proof. We will prove the lemma for $x_n \geq \overline{x} = 1-p$ for all $n \geq -1$. The case for $x_n \leq \overline{x} = 1-p$ for all $n \geq -1$ is similar. From $x_n \geq \overline{x}$ for all $n \geq -1$ and

$$x_{n+1} - x_{n-1} = \frac{\overline{x} - x_n}{p + x_n} x_{n-1}, \tag{8}$$

it follows that the sequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ are monotone decreasing. Let $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n+1} = b$. By (8), we have $a = b = \overline{x}$. Lemma 5 is proven. □

Set

$$x = g_2(y) = (p+y)\overline{x} \quad (y > 0), \tag{9}$$

then $y = h_2(x) = g_2^{-1}(x) = x/\overline{x} - p$ is an increasing and differentiable function which maps $(p\overline{x}, +\infty)$ onto $(0, +\infty)$. Let

$$x = g_3(y) = (p+y)h_2(y) \quad (y > p\overline{x}), \tag{10}$$

then $y = h_3(x) = g_3^{-1}(x)$ is an increasing and differentiable function which maps $(0, +\infty)$ onto $(p\overline{x}, +\infty)$.

Assume that for some positive integer n we already define increasing and differentiable functions $h_{2n}(x)$ and $h_{2n+1}(x)$ such that h_{2n} maps $(p^n\overline{x}, +\infty)$ onto $(0, +\infty)$ and h_{2n+1} maps $(0, +\infty)$ onto $(p^n\overline{x}, +\infty)$. Set

$$x = g_{2n+2}(y) = (p+y)h_{2n+1}(y) \quad (y > 0), \tag{11}$$

then $y = h_{2n+2}(x) = g_{2n+2}^{-1}(x)$ is an increasing and differentiable function which maps $(p^{n+1}\overline{x}, +\infty)$ onto $(0, +\infty)$. Set

$$x = g_{2n+3}(y) = (p+y)h_{2n+2}(y) \quad (y > p^{n+1}\overline{x}), \tag{12}$$

then $y = h_{2n+3}(x) = g_{2n+3}^{-1}(x)$ is an increasing and differentiable function which maps $(0, +\infty)$ onto $(p^{n+1}\overline{x}, +\infty)$. In such a way, we construct a family of increasing and differentiable functions $y = h_n(x)$.

Let $P_0 = A_2$ and $Q_0 = A_1$. For any $n \geq 1$, write

$$P_n = f^{-1}(P_{n-1}), \qquad Q_n = f^{-1}(Q_{n-1}), \qquad L_n = f^{-1}(L_{n-1}), \qquad R_n = f^{-1}(R_{n-1}). \tag{13}$$

From Lemma 4 we have that $L_2 = f^{-1}(L_1) \subset P_0$, $R_2 = f^{-1}(R_1) \subset Q_0$, $P_1 = f^{-1}(P_0) \subset P_0$ and $Q_1 = f^{-1}(Q_0) \subset Q_0$, which implies that for any $n \geq 1$,

$$L_{n+1} \subset P_{n-1}, \qquad R_{n+1} \subset Q_{n-1}, \qquad P_n \subset P_{n-1}, \qquad Q_n \subset Q_{n-1}. \tag{14}$$

Let $(x,y) \in L_2$. Since $f(L_2) = L_1$ and $(u,v) = f(x,y) = (y, x/(p+y))$, it follows that

$$\frac{x}{(p+y)} = v = \overline{x}, \quad y = u > \overline{x}. \tag{15}$$

Thus $x = g_2(y) = (p+y)\overline{x} > \overline{x}$ $(y > \overline{x})$ and $L_2 = \{(x,y) : y = h_2(x),\ x > \overline{x}\}$. In a similar fashion, we may show $R_2 = \{(x,y) : y = h_2(x),\ p\overline{x} < x < \overline{x}\}$.

Since f is a homeomorphism, $f(P_1) = P_0$, and $L_0 \cup L_1 \cup \{(\overline{x},\overline{x})\}$ is the boundary of P_0 with $f(L_2) = L_1$ and $f(L_1) = L_0$, we have

$$P_1 = \{(x,y) : \overline{x} < y < h_2(x),\ x > \overline{x}\}. \tag{16}$$

In a similar fashion, we may show

$$Q_1 = \{(x,y) : 0 < y < \overline{x},\ 0 < x \leq p\overline{x}\} \cup \{(x,y) : h_2(x) < y < \overline{x},\ p\overline{x} < x < \overline{x}\}. \tag{17}$$

Let $(x,y) \in L_3$. Since $f(L_3) = L_2$ and $(u,v) = f(x,y) = (y, x/(p+y)) \in L_2$, it follows that

$$\frac{x}{(p+y)} = v = h_2(u) = h_2(y), \quad y = u > \overline{x}. \tag{18}$$

Thus $x = g_3(y) = (p+y)h_2(y) > \overline{x}$ $(y > \overline{x})$ and $L_3 = \{(x,y) : y = h_3(x),\ x > \overline{x}\}$. In a similar fashion, we may show $R_3 = \{(x,y) : y = h_3(x),\ 0 < x < \overline{x}\}$.

Since f is a homeomorphism, $f(P_2) = P_1$, and $L_1 \cup L_2 \cup \{(\overline{x},\overline{x})\}$ is the boundary of P_2 with $f(L_3) = L_2$ and $f(L_2) = L_1$, we have

$$P_2 = \{(x,y) : h_3(x) < y < h_2(x),\ x > \overline{x}\}. \tag{19}$$

In a similar fashion, we may show

$$Q_2 = \{(x,y) : 0 < y < h_3(x),\ 0 < x \leq p\overline{x}\} \cup \{(x,y) : h_2(x) < y < h_3(x),\ p\overline{x} < x < \overline{x}\}. \tag{20}$$

Using induction, one can easily show that for any $n \geq 2$,

$$L_n = \{(x,y) : y = h_n(x),\ x > \overline{x}\}, \tag{21}$$

and for any $n \geq 1$,

$$\begin{aligned}
R_{2n} &= \{(x,y) : y = h_{2n}(x),\ p^n\overline{x} < x < \overline{x}\},\\
R_{2n+1} &= \{(x,y) : y = h_{2n+1}(x),\ 0 < x < \overline{x}\},\\
Q_{2n} &= \{(x,y) : 0 < y < h_{2n+1}(x),\ 0 < x \leq p^n\overline{x}\}\\
&\quad \cup \{(x,y) : h_{2n}(x) < y < h_{2n+1}(x),\ p^n\overline{x} < x < \overline{x}\},\\
Q_{2n+1} &= \{(x,y) : 0 < y < h_{2n+1}(x),\ 0 < x \leq p^{n+1}\overline{x}\}\\
&\quad \cup \{(x,y) : h_{2n+2}(x) < y < h_{2n+1}(x),\ p^{n+1}\overline{x} < x < \overline{x}\},\\
P_{2n} &= \{(x,y) : h_{2n+1}(x) < y < h_{2n}(x),\ x > \overline{x}\},\\
P_{2n+1} &= \{(x,y) : h_{2n+1}(x) < y < h_{2n+2}(x),\ x > \overline{x}\}.
\end{aligned} \tag{22}$$

By (14), it follows that for $x > \overline{x}$,

$$\overline{x} < h_3(x) \leq h_5(x) \leq \cdots \leq h_4(x) \leq h_2(x) \tag{23}$$

and for $0 < x \leq \overline{x}$,

$$\overline{x} \geq h_3(x) \geq h_5(x) \geq \cdots, \tag{24}$$

and for any $n \geq 2$ and $p^n\overline{x} < x \leq \overline{x}$

$$h_{2n-1}(x) \geq h_{2n}(x) \geq h_{2n-2}(x). \tag{25}$$

From (23), (24), and (25) we may assume that for every $x > 0$,

$$F(x) = \lim_{n\to\infty} h_{2n+1}(x), \quad G(x) = \lim_{n\to\infty} h_{2n}(x) \quad \left(n > \log_p\left(\frac{x}{\overline{x}}\right)\right). \tag{26}$$

Then $F(x) \leq G(x)$ if $x > \overline{x}$ and $F(x) \geq G(x)$ if $0 < x \leq \overline{x}$.

LEMMA 6. *$F(x)$ and $G(x)$ are continuous.*

Proof. We first show that $F(x)$ is continuous. Let $x, x_0 \in (0,+\infty)$. Choosing $N > 0$ such that $x, x_0 \in (p^N\overline{x}, +\infty)$, then for every $n > N+1$, there exists c_n between x and x_0 such that

$$\left|h_{2n+1}(x) - h_{2n+1}(x_0)\right| = \left|h'_{2n+1}(c_n)\right| \left|x - x_0\right|. \tag{27}$$

Let $\xi_n = h_{2n+1}(c_n)$, then $h'_{2n}(\xi_n) \ge 0$ and

$$\begin{aligned}
h_{2n}(\xi_n) + (p+\xi_n)h'_{2n}(\xi_n) &\ge h_{2n}(\xi_n) = h_{2n}(h_{2n+1}(c_n)) \\
&\ge h_{2n}(h_{2n+1}(p^N\overline{x})) \ge h_{2N}(h_{2N+2}(p^N\overline{x})), \\
|h_{2n+1}(x) - h_{2n+1}(x_0)| &= \left| \frac{1}{(h_{2n}(\xi_n) + (p+\xi_n)h'_{2n}(\xi_n))} \right| |x - x_0| \\
&\le \left| \frac{1}{h_{2N}(h_{2N+2}(p^N\overline{x}))} \right| |x - x_0|.
\end{aligned} \tag{28}$$

Thus

$$|F(x) - F(x_0)| = \lim_{n\to\infty} |h_{2n+1}(x) - h_{2n+1}(x_0)| \le \left| \frac{1}{h_{2N}(h_{2N+2}(p^N\overline{x}))} \right| |x - x_0|, \tag{29}$$

which implies $F(x)$ is continuous. In a similar fashion, we may show that $G(x)$ is also continuous. □

Let S be the set of initial values $(x_{-1},x_0) \in D$ such that the positive solution $\{x_n\}_{n=-1}^{\infty}$ of (1) is bounded. Then we have the following theorem.

Theorem 7. *Let* $0 < p < 1$, *then* $S = W_1 \cup \{(\overline{x},\overline{x})\} \cup W_2$, *where* $W_1 = \{(x,y) : F(x) \le y \le G(x),\ \overline{x} < x\}$ *and* $W_2 = \{(x,y) : G(x) \le y \le F(x),\ 0 < x < \overline{x}\}$. *Moreover, every positive solution* $\{x_n\}_{n=-1}^{\infty}$ *of (1) with initial value* $(x_{-1},x_0) \in S$ *converges to* $\overline{x}$.

Proof. Let $(x_{-1},x_0) \in W_1 \cup \{(\overline{x},\overline{x})\} \cup W_2$ and $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (1) with initial value (x_{-1},x_0).

If $(x_{-1},x_0) = (\overline{x},\overline{x})$, then $\{x_n\}_{n=-1}^{\infty}$ is a trivial solution of (1), which implies $\lim_{n\to\infty} x_n = \overline{x}$ and $(x_{-1},x_0) \in S$.

If $(x_{-1},x_0) \in W_1$, then $(x_{-1},x_0) \in P_n$ for any $n \ge 0$, which implies $f^n(x_{-1},x_0) = (x_{n-1}, x_n) \in A_2$ for any $n \ge 0$. Thus it follows from Lemma 5 that $\lim_{n\to\infty} x_n = \overline{x}$ and $(x_{-1},x_0) \in S$. In a similar fashion, we may show that if $(x_{-1},x_0) \in W_2$, then $\lim_{n\to\infty} x_n = \overline{x}$ and $(x_{-1},x_0) \in S$.

Now let $(x_{-1},x_0) \in D - W_1 \cup \{(\overline{x},\overline{x})\} \cup W_2$ and $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (1) with initial value (x_{-1},x_0).

If $(x_{-1},x_0) \in A_3 \cup A_4 \cup R_0 \cup R_1 \cup L_0 \cup L_1$, then by Lemma 4 we have $f^2(x_{-1},x_0) = (x_1,x_2) \in \{(x,y) : (x-\overline{x})(y-\overline{x}) < 0\}$, it follows from Corollary 3 that $(x_{-1},x_0) \notin S$.

If $(x_{-1},x_0) \in A_2 - W_1$, then there exists $n \ge 0$ such that

$$(x_{-1},x_0) \in P_n - P_{n+1} = f^{-n}(A_2) - f^{-n-1}(A_2), \tag{30}$$

from which it follows

$$f^n(x_{-1},x_0) = (x_{n-1},x_n) \in A_2 - f^{-1}(A_2). \tag{31}$$

By Lemma 4, we have $f^{n+1}(x_{-1},x_0) \in A_4 \cup L_1$, which implies $f^{n+3}(x_{-1},x_0) = (x_{n+2},x_{n+3}) \in A_4$, it follows from Corollary 3 that $(x_{-1},x_0) \notin S$. In a similar fashion, we may show that if $(x_{-1},x_0) \in A_1 - W_2$, then it follows that $(x_{-1},x_0) \notin S$. Theorem 7 is proven. □

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