

# ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS

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Motivated by some results of L. Berg (2002), in this paper we find the second member in the asymptotic development of some of the positive solutions of a class of difference equations of second and third orders. The main result in this paper partially solves an open problem by S. Stević (2003), and it is applied to some classes of mathematical biology models, for example, generalized Beverton-Holt stock recruitment model, flour beetle population model, mosquito population equations, and discrete delay logistic difference equation.

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## 1. Introduction

Recently there has been a great interest in studying nonlinear difference equations and systems (cf. [1–36] and the references therein). One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economy, probability theory, genetics, psychology, sociology, and so forth. Such equations also appear naturally as discrete analogues of differential equations which model various biological and economical systems (see, e.g. [7, 13, 16–18, 20, 24, 28, 29] and the references therein).

There are sequences defined by recurrence formulae such that we know their asymptotic behavior, see, for example, [11, 12, 14, 15, 19, 21, 23, 29, 30, 32, 33, 36], that is, we know the first member in their asymptotic behavior.

The following theorem was established in [21, Problem I, page 174]. The proof appearing there is attributed to Jacobsthal.

**THEOREM 1.1.** *Let  $f : (0, \alpha) \rightarrow (0, \alpha)$ , where  $\alpha > 0$ , be a continuous function such that  $0 < f(x) < x$  for every  $x \in (0, \alpha)$  and  $f(x) = x - ax^k + bx^{k+p} + o(x^{k+p})$ , when  $x \rightarrow +0$ , where*

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$k > 1$ ,  $p$ ,  $a$ , and  $b$  are positive numbers. Let  $x_0 \in (0, \alpha)$  and  $x_n = f(x_{n-1})$ ,  $n \geq 1$ . Then

$$x_n \sim \frac{1}{((k-1)an)^{1/(k-1)}}. \quad (1.1)$$

Our version of this proof is somewhat different to its original form, and the idea and structure of this version was the starting point and inspiration for our further investigations, see [23]. In [24] we have noted, by an example, that there are sequences defined by difference equations of second order which behave similarly to those sequences in Theorem 1.1. Motivated by the example in [29], we have found the asymptotics of all positive solutions of a nonlinear difference equation of second order.

A natural problem was whether Theorem 1.1 could be extended in the case of difference equations of order greater than one. The problem was not only of great importance in its own right. This is important because there is a large class of mathematical biology models which are described by equations of this type. For example,

*discrete delay logistic difference equation [20]:*

$$N_{t+1} = \frac{\alpha N_t}{1 + \beta N_{t-k}}, \quad \alpha, \beta > 0, k \in \mathbb{N}; \quad (1.2)$$

*generalized Beverton-Holt stock recruitment model [7, 13]:*

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}, \quad x_0, x_1 > 0, n = 1, 2, 3, \dots, \quad (1.3)$$

where  $a \in (0, 1)$ ,  $b \in \mathbb{R}_+$ , and  $c, d \in \mathbb{R}_+ \cup \{0\}$ , with  $c + d > 0$ ;

*flour beetle population model [18]:*

$$x_{n+3} = ax_{n+2} + bx_n e^{-(cx_{n+2} + dx_n)}, \quad n = 0, 1, 2, \dots, \quad (1.4)$$

with  $a, b, c, d \geq 0$  and  $c + d > 0$ .

*mosquito population equations [16]:*

$$x_{n+1} = (ax_n + bx_{n-1} e^{-x_{n-1}}) e^{-x_n}, \quad x_0, x_1 > 0, n = 1, 2, 3, \dots, \quad (1.5)$$

where  $a \in (0, 1)$ ,  $b \in [0, \infty)$ , and

$$x_{n+1} = (\alpha x_n + \beta x_{n-1}) e^{-x_n}, \quad x_0, x_1 > 0, n = 1, 2, 3, \dots, \quad (1.6)$$

where  $\alpha \in [0, 1)$ ,  $\beta \in (0, \infty)$ .

One of the most interesting cases of these equations is when the sum of the main coefficients of the equations is equal to 1, that is, when  $\alpha = 1$  for (1.2),  $a + b = 1$  for (1.3), (1.4), and (1.5), and  $\alpha + \beta = 1$  for (1.6).

A natural question is to find the first term in the asymptotic behavior of the solutions of (1.3)–(1.5) for  $a + b = 1$  and of (1.6) when  $\alpha + \beta = 1$ .

In [29, 32] we have generalized Theorem 1.1 in the case of sequences defined by a difference equations of order more than one, where among other results we completely described the asymptotic behavior of solutions of (1.3)–(1.6) in these cases.

In [32] we have proved the following theorem.

**THEOREM 1.2.** *Let  $f : [0, +\infty)^k \rightarrow [0, +\infty)$  be a continuous function such that*

- (a)  $0 < f(z_1, \dots, z_k) < \max\{z_1, \dots, z_k\}$  for all  $z_1, \dots, z_k \in [0, \alpha]$ , such that  $(z_1, z_2, \dots, z_k) \neq (0, 0, \dots, 0)$ , where  $\alpha \in (0, \infty]$  and  $k \in \mathbb{N}$ ;
- (b)  $f(z_1, \dots, z_k) = p_1 z_1 + \dots + p_k z_k - \sum_{s=m}^{+\infty} K_s(z_1, \dots, z_k)$  uniformly as  $z_1^2 + \dots + z_k^2 \rightarrow 0$ , where  $m > 1$ ,  $p_1, \dots, p_k \in [0, 1)$ , and  $K_s(z_1, \dots, z_k)$  is a homogeneous polynomial of order  $s$ ;
- (c)  $\sum_{i=1}^k p_i = 1$ ;
- (d)  $f(z_1, \dots, z_k)$  is nondecreasing in each variable in a neighborhood of the origin;
- (e)  $K_m(1, \dots, 1) > 0$ .

Then the sequence defined by

$$x_{n+1} = f(x_n, \dots, x_{n-k+1}) \tag{1.7}$$

with initial conditions  $x_0, x_1, \dots, x_{k-1} \in [0, \alpha)$  satisfies the following asymptotic formula:

$$x_n \sim \left( \frac{P'_k(1)}{(m-1)K_m(1, \dots, 1)} \right)^{1/(m-1)} \frac{1}{m^{-1}\sqrt[n]{n}}, \tag{1.8}$$

where  $P_k(\lambda) = \lambda^k - p_1 \lambda^{k-1} - \dots - p_{k-1} \lambda - p_k$ .

*Remark 1.3.* From the proof of Theorem 1.2, it is easy to see that condition (b) can be replaced by the following condition:

$$(b)' f(z_1, \dots, z_k) = p_1 z_1 + \dots + p_k z_k - K_m(z_1, \dots, z_k) + o\left((z_1^2 + \dots + z_k^2)^{m/2}\right), \tag{1.9}$$

uniformly as  $z_1^2 + \dots + z_k^2 \rightarrow 0$ , where  $m > 1$ ,  $p_1, \dots, p_k \in [0, 1)$ , and  $K_m(z_1, \dots, z_k)$  is a homogeneous polynomial of order  $m$ .

*Remark 1.4.* Note that if  $p_1, \dots, p_k \in (0, 1)$ , then condition (d) is automatically satisfied, since the partial derivatives of the function  $f$  are positive in a neighborhood of the origin.

The convergence of the sequences in Theorem 1.2 follows, for example, from some results, in [25, 27]. For closely related results, see also [1, 8–10, 22, 24–28, 31].

Applying Theorem 1.2 to (1.3)–(1.6), we obtain that the following asymptotics hold.

*Equation (1.3).* If  $a, b \in (0, 1)$  and  $a + b = 1$ , then

$$x_n \sim \frac{2 - a}{(1 - a)(c + d)n}. \tag{1.10}$$

*Equation (1.4).* If  $a, b \in (0, 1)$  and  $a + b = 1$ , then

$$x_n \sim \frac{3 - 2a}{(1 - a)(c + d)n}. \tag{1.11}$$

*Equation (1.5).* If  $a, b \in (0, 1)$  and  $a + b = 1$ , then

$$x_n \sim \frac{1}{n}. \tag{1.12}$$

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Equation (1.6). If  $\alpha, \beta \in (0, 1)$  and  $\alpha + \beta = 1$ , then

$$x_n \sim \frac{2 - \alpha}{n}. \quad (1.13)$$

For the class of difference equations of first order defined in Theorem 1.1, we described in [23], under some additional conditions, a method for finding the next members in asymptotic developments of its solutions. It is a natural question to find the other members in the asymptotic development of the solutions of (1.3)–(1.6). Hence in [32] we have offered the following problem.

*Open problem 1.5.* Find the second member in the asymptotic development of the solutions of (1.3)–(1.6), that is, find the sequence  $y_n$  such that

$$\lim_{n \rightarrow \infty} \frac{x_n - a/n}{y_n} = 1. \quad (1.14)$$

The main object in the paper is to find the second member in the asymptotic development of some of the positive solutions of a class of nonlinear difference equations generalizing (1.2)–(1.6).

### 2. The inclusion theorem

In order to prove the open problem, we need the following result.

**THEOREM 2.1.** *Let  $f : I^k \rightarrow I$  be a continuous and nondecreasing function in each argument on the interval  $I \subseteq \mathbb{R}$ , and let  $(y_n)$  and  $(z_n)$  be sequences in  $I$ , with  $y_n < z_n$  for  $n \geq n_0$  and such that*

$$y_{n+1} \leq f(y_n, \dots, y_{n-k+1}), \quad f(z_n, \dots, z_{n-k+1}) \leq z_{n+1}, \quad n \geq n_0 + k - 1. \quad (2.1)$$

Let further  $x_n$  be a solution of the following difference equation:

$$x_{n+1} = f(x_n, \dots, x_{n-k+1}) \quad (2.2)$$

such that

$$y_n \leq x_n \leq z_n, \quad \text{for } n \in \{n_0, \dots, n_0 + k - 1\}, \quad (2.3)$$

then

$$y_n \leq x_n \leq z_n, \quad \text{for } n \geq n_0. \quad (2.4)$$

*Proof.* We prove the theorem by induction. If  $n = n_0 + k$ , then using (2.1) and the monotonicity of the function  $f$  we have

$$y_{n_0+k} \leq f(y_{n_0+k-1}, \dots, y_{n_0}) \leq f(x_{n_0+k-1}, \dots, x_{n_0}) \leq f(z_{n_0+k-1}, \dots, z_{n_0}) \leq z_{n_0+k}, \quad (2.5)$$

that is,

$$y_{n_0+k} \leq x_{n_0+k} \leq z_{n_0+k}. \quad (2.6)$$

Assume now that it has been proved that

$$y_n \leq x_n \leq z_n, \quad \text{for } n \in \{n_0, \dots, m\}, \quad (2.7)$$

where  $m \geq n_0 + k$ , then using again (2.1), induction hypothesis, and the monotonicity of the function  $f$  we have

$$y_{m+1} \leq f(y_m, \dots, y_{m-k+1}) \leq f(x_m, \dots, x_{m-k+1}) \leq f(z_m, \dots, z_{m-k+1}) \leq z_{m+1}, \quad (2.8)$$

that is,  $y_{m+1} \leq x_{m+1} \leq z_{m+1}$ , finishing the proof.  $\square$

*Remark 2.2.* Some other results of this type and their applications can be found, for example, in [2–6, 34, 35].

*Remark 2.3.* It suffices that the hypotheses concerning  $f$  are satisfied in the strip (2.4).

### 3. Main result

In this section, we formulate and prove the main result in this paper. As a consequence of this result, we find the second member in the asymptotics of some positive solutions of (1.3)–(1.6).

**THEOREM 3.1.** *Consider (1.7), where  $k \in \{2, 3\}$ . Then the following statements are true.*

- (a) *For  $k = 2$  and  $p \in (0, 1)$ , there is a positive solution of (1.7) with the following asymptotics:*

$$x_n = \frac{2-p}{K_2(1,1)n} + b \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right), \quad (3.1)$$

as  $n \rightarrow \infty$ , where

$$f(x, y) = px + (1-p)y - K_2(x, y) - K_3(x, y) + o\left((x^2 + y^2)^{3/2}\right), \quad (3.2)$$

as  $x^2 + y^2 \rightarrow 0$ ,

$$b = \frac{p(s-q) - 2(r+2s)}{K_2(1,1)^2} - \frac{K_3(1,1)}{K_2(1,1)^3} (2-p)^2, \quad (3.3)$$

$K_2(x, y) = qx^2 + rxy + sy^2$  is a positive definite form, and  $K_3(x, y)$  is a homogeneous polynomial in variables  $x$  and  $y$  of third order.

- (b) *For  $k = 3$  and  $p, q, p+q \in (0, 1)$ , there is a positive solution of (1.7) with the following asymptotics:*

$$x_n = \frac{3-q-2p}{K_2(1,1,1)n} + b \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right), \quad (3.4)$$

as  $n \rightarrow \infty$ , where

$$f(x, y, z) = px + qy + (1-p-q)z - K_2(x, y, z) - K_3(x, y, z) + o\left((x^2 + y^2 + z^2)^{3/2}\right), \quad (3.5)$$

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$$as x^2 + y^2 + z^2 \rightarrow 0,$$

$$b = \frac{3(r - 3t - s - v - 2w) - q(3r + s + 2u + v - t) - 2p(2r - 2t + u - w)}{K_2(1, 1, 1)^2} + \frac{K_3(1, 1, 1)}{K_2(1, 1, 1)^3} (3 - 2p - q)^2, \quad (3.6)$$

$K_2(x, y, z) = rx^2 + sy^2 + tz^2 + uxy + vxz + wyz$  is a positive definite form, and  $K_3(x, y, z)$  is a homogeneous polynomial in variables  $x, y,$  and  $z$  of third order.

*Proof.* In both cases, we suppose that there are solutions which have the following asymptotics:

$$x_n = \frac{a}{n} + \frac{b \ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right) = \varphi_n + o\left(\frac{\ln n}{n^2}\right), \quad (3.7)$$

as  $n \rightarrow \infty$ .

We show this by finding the values of the coefficients  $a$  and  $b$ .

(a) Let

$$F(x, y, z) = x - f(y, z). \quad (3.8)$$

Choosing  $a = (2 - p)/(q + r + s) = (2 - p)/(K_2(1, 1))$  and  $b$  arbitrary, comparing coefficients in

$$F(\varphi_{n+1}, \varphi_n, \varphi_{n-1}) = 0 \quad (3.9)$$

by some calculations, we obtain

$$F(\varphi_{n+1}, \varphi_n, \varphi_{n-1}) \sim \frac{(2 - p)(b - b_0)}{n^3}, \quad (3.10)$$

where

$$b_0 = \frac{p(s - q) - 2(r + 2s)}{K_2(1, 1)^2} - \frac{K_3(1, 1)}{K_2(1, 1)^3} (2 - p)^2. \quad (3.11)$$

Let  $b_1$  and  $b_2$  be such that  $b_1 > b_0$  and  $b_2 < b_0$ . With the notations

$$\begin{aligned} y_n &= \frac{a}{n} + \frac{b_2 \ln n}{n^2}, \\ z_n &= \frac{a}{n} + \frac{b_1 \ln n}{n^2}, \end{aligned} \quad (3.12)$$

and from (3.10), we get

$$\begin{aligned} F(y_{n+1}, y_n, y_{n-1}) &\sim \frac{(2 - p)(b_2 - b_0)}{n^3} < 0, \\ F(z_{n+1}, z_n, z_{n-1}) &\sim \frac{(2 - p)(b_1 - b_0)}{n^3} > 0. \end{aligned} \quad (3.13)$$

These relations show that inequalities (2.1) are satisfied for sufficiently large  $n$ , where  $f$  is defined in Theorem 3.1(a) and  $F$  is given by (3.8). Thus, since for sufficiently large  $n$  we can choose  $b$  arbitrary close to  $b_0$ , in view of Theorem 2.1, it follows that in this case there is a solution of (1.7) which has asymptotics (3.4).

(b) Let

$$F(x, y, z, w) = x - f(y, z, w). \quad (3.14)$$

Choosing  $a = (3 - q - 2p)/(K_2(1, 1, 1))$  and  $b$  arbitrary, comparing coefficients in

$$F(\varphi_{n+1}, \varphi_n, \varphi_{n-1}, \varphi_{n-2}) = 0, \quad (3.15)$$

the DERIVE system yields

$$F(\varphi_{n+1}, \varphi_n, \varphi_{n-1}, \varphi_{n-2}) \sim \frac{(3 - q - 2p)(b - b_0)}{n^3}, \quad (3.16)$$

where

$$b_0 = \frac{3(r - 3t - s - v - 2w) - q(3r + s + 2u + v - t) - 2p(2r - 2t + u - w)}{K_2(1, 1, 1)^2} + \frac{K_3(1, 1, 1)}{K_2(1, 1, 1)^3}(3 - 2p - q)^2. \quad (3.17)$$

Let  $b_1$  and  $b_2$  be such that  $b_1 > b_0$  and  $b_2 < b_0$ . With the notations (3.12) and from (3.16), we get

$$F(y_{n+1}, y_n, y_{n-1}, y_{n-2}) \sim \frac{(3 - q - 2p)(b_2 - b_0)}{n^3} < 0, \quad (3.18)$$

$$F(z_{n+1}, z_n, z_{n-1}, z_{n-2}) \sim \frac{(3 - q - 2p)(b_1 - b_0)}{n^3} > 0.$$

These relations show that inequalities (2.1) are satisfied for sufficiently large  $n$ , where  $f$  is given in Theorem 3.1(b) and  $F$  is given by (3.14). Applying Theorem 2.1, we obtain that in this case there is a solution of (1.7) which has asymptotics (3.4).  $\square$

**3.1. Case of discrete delay logistic difference equation.** In the case of a general  $k$ , we only consider the discrete delay logistic difference equation (1.2) with  $\alpha = 1$ , that is,

$$x_{n+1} = \frac{x_n}{1 + \beta x_{n-k}}. \quad (3.19)$$

From [29, Theorem 2], it follows that

$$x_n \sim \frac{1}{\beta n}. \quad (3.20)$$

Let

$$F(x, u, v) = x - \frac{u}{1 + \beta v}. \quad (3.21)$$

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We assume that (3.19) has a positive solution  $x_n$  with the asymptotics

$$x_n = \frac{1}{n} + b \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right), \quad (3.22)$$

and also that  $\beta = 1$ , otherwise we consider the sequence  $y_n = \beta x_n$ . Let

$$\varphi_n = \frac{1}{n} + b \frac{\ln n}{n^2}. \quad (3.23)$$

Then by well-known asymptotics formulae, we have

$$\begin{aligned} F(\varphi_{n+1}, \varphi_n, \varphi_{n-k}) &= \varphi_{n+1} - \varphi_n + \varphi_n \varphi_{n-k} - \varphi_n \varphi_{n-k}^2 + o\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n} \left(1 + \frac{1}{n}\right)^{-1} + b \frac{\ln n}{n^2} \left(1 + \frac{1}{n}\right)^{-2} + b \frac{\ln(1+1/n)}{n^2} \left(1 + \frac{1}{n}\right)^{-2} - \frac{1}{n} - b \frac{\ln n}{n^2} \\ &\quad + \left(\frac{1}{n} + b \frac{\ln n}{n^2}\right) \left(\frac{1}{n} \left(1 - \frac{k}{n}\right)^{-1} + b \frac{\ln n + \ln(1-k/n)}{n^2} \left(1 - \frac{k}{n}\right)^{-2}\right) \\ &\quad - \left(\frac{1}{n} + b \frac{\ln n}{n^2}\right) \left(\frac{1}{n} \left(1 - \frac{k}{n}\right)^{-1} + b \frac{\ln n + \ln(1-k/n)}{n^2} \left(1 - \frac{k}{n}\right)^{-2}\right)^2 + o\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} + b \frac{\ln n}{n^2} - 2b \frac{\ln n}{n^3} + \frac{b}{n^3} - \frac{1}{n} - b \frac{\ln n}{n^2} \\ &\quad + \frac{1}{n^2} + 2b \frac{\ln n}{n^3} + \frac{k}{n^3} - \frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \sim \frac{b+k}{n^3}. \end{aligned} \quad (3.24)$$

Hence, in this case, we have that  $b = -k$  (which implies that  $b = -k/\beta$  if  $\beta \neq 1$ ) in (3.21). Denote  $b_0 = -k/\beta$ . Using (3.12) and choosing  $b_1$  and  $b_2$  such that  $b_1 > b_0$  and  $b_2 < b_0$ , we obtain

$$\begin{aligned} F(y_{n+1}, y_n, y_{n-k}) &\sim \frac{(b_2 - b_0)}{n^3} < 0, \\ F(z_{n+1}, z_n, z_{n-k}) &\sim \frac{(b_1 - b_0)}{n^3} > 0. \end{aligned} \quad (3.25)$$

Now note that from this and by Theorem 2.1 we could conclude that there is a positive solution of (3.19) with the following asymptotics:

$$x_n = \frac{1}{\beta n} - \frac{k}{\beta} \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right), \quad (3.26)$$

as  $n \rightarrow \infty$ , if the monotonicity conditions were not violated.

Hence, for the readers interested in this research area, we leave the following conjecture.

**CONJECTURE 3.2.** *Shows that there is a positive solution of (3.19) with asymptotics (3.22).*



*Remark 3.3.* It is interesting that Conjecture 3.2 cannot be confirmed also by [4, Theorem 2.1] (see also [5]), since for the case of (3.19), two coefficients with the largest moduli in [4, formula (2.3)] have the same moduli so that [4, condition (2.4)] is not satisfied. We would like to point out that [4, Theorem 2.1] was applied with a success at many points, for example, in [4, 34].

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