EXISTENCE CRITERIA AND CLASSIFICATION SCHEMES FOR POSITIVE SOLUTIONS OF SECOND-ORDER NONLINEAR DIFFERENCE SYSTEMS

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Classification schemes for positive solutions of a class of second-order nonlinear difference systems are given in terms of their asymptotic magnitudes; and necessary as well as sufficient conditions for the existence of these solutions are also provided. Finally, some examples are given to illustrate the main results.

1. Introduction

The asymptotic behavior of solutions of nonlinear difference equations is of particular interest in iterative computational schemes and discrete time dynamic models. Therefore, it is the subject of many investigations [1, 2, 4, 5, 8–14, 18, 20].

In this paper, we are concerned with a class of two-dimensional second-order nonlinear difference systems of the form

$$\begin{aligned} \Delta^2 x_n &= a_n f(y_n), \\ \Delta^2 y_n &= -b_n g(x_n), \end{aligned} \tag{1.1}$$

where $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ are real, nontrivial sequences such that $a_n \ge 0$ and $b_n \ge 0$ for $n \ge n_0$, f and g are continuous real-valued and increasing functions on the real line $\mathbb{R}$ and satisfy $xf(x) > 0$, $xg(x) > 0$ for $x \ne 0$.

Existence and uniqueness theorem for solutions of (1.1) is easily established. Indeed, given x_0, x_1, y_0 and y_1, we can calculate

$$x_2 = 2x_1 - x_0 + a_0 f(y_0), \qquad y_2 = 2y_1 - y_0 - b_0 g(x_0), \ldots \tag{1.2}$$

successively in a unique manner. The corresponding sequence $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ will be called a solution of (1.1).

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A solution $\{(x_n, y_n)\}$ of (1.1) is said to be positive if both $\{x_n\}$ and $\{y_n\}$ are positive. Positive solutions of (1.1) are interesting for many reasons. For instance, when $a_n \equiv 1$ and $f(u) = u$, we see from (1.1) that

$$\Delta^2 y_n = \Delta^4 x_n = -b_n g(x_n). \tag{1.3}$$

Therefore, a positive solution of (1.1) yields a positive and strictly concave solution of the fourth-order nonlinear difference equation

$$\Delta^4 x_n + b_n g(x_n) = 0. \tag{1.4}$$

Other difference equations such as

$$\begin{gathered} \Delta^4 x_{n-1} + p_n f(x_n) = 0, \qquad \Delta^4 x_{n-1} + p_n |x_n|^\gamma \operatorname{sign} x_n = 0, \\ \Delta^2(r_{n-1}\Delta^2 x_{n-1}) + p_n f(x_n) = 0 \end{gathered} \tag{1.5}$$

can also be written in the form (1.1), which have been explored to some extent in a number of studies [3, 6, 7, 15–17, 19]. We will be concerned with existence criteria as well as classification schemes for positive solutions of (1.1).

We remark that our system (1.1) is a discrete analog of a second-order differential system of the form

$$x'' = a(t) f(y), \qquad y'' = -b(t) g(x), \tag{1.6}$$

which can be interpreted as the governing equations of the motion of a particle moving in a plane under a nonautonomous plane force field. Therefore the study of (1.1) will also lead to useful complementary information for the differential systems. On the other hand, (1.1) can also be written as a first-order difference system. For results related to these systems, the reader can refer to [7]. We remark, however, that our approach here is more natural and avoids systems with four equations.

Our system (1.1) is naturally classified into four classes: (i) $\sum_{s=n_0}^\infty a_s = \infty$ and $\sum_{s=n_0}^\infty b_s = \infty$; (ii) $\sum_{s=n_0}^\infty a_s = \infty$ and $\sum_{s=n_0}^\infty b_s < \infty$; (iii) $\sum_{s=n_0}^\infty a_s < \infty$ and $\sum_{s=n_0}^\infty b_s = \infty$; and (iv) $\sum_{s=n_0}^\infty a_s < \infty$ and $\sum_{s=n_0}^\infty b_s < \infty$.

For this reason, we will employ the following notations:

$$A_n = \sum_{s=n}^\infty a_s, \quad B_n = \sum_{s=n}^\infty b_s, \quad n \ge n_0. \tag{1.7}$$

In the following section, we will discuss the case $A_{n_0} = \infty$ and $B_{n_0} = \infty$. The cases where $A_{n_0} = \infty$ and $B_{n_0} < \infty$, $A_{n_0} < \infty$ and $B_{n_0} < \infty$, and $A_{n_0} < \infty$ and $B_{n_0} = \infty$ will be studied in Sections 3, 4, and 5, respectively. In Section 6, we give some examples to illustrate our results.

2. The case $A_{n_0} = \infty$ and $B_{n_0} = \infty$

In this section, we always assume that $A_{n_0} = \infty$ and $B_{n_0} = \infty$. We assert that there exist no positive solutions of (1.1).

Theorem 2.1. *Suppose that $A_{n_0} = \infty$ and $B_{n_0} = \infty$. Then there exist no positive solutions of (1.1).*

Proof. Suppose that $\{(x_n, y_n)\}$ is a solution of (1.1) such that $x_n > 0$ and $y_n > 0$ for $n \geq n_0$. Then, from (1.1) we have $\Delta^2 y_n < 0$ for $n \geq n_0$, which implies that $\{\Delta y_n\}$ is decreasing. Therefore, there are two possibilities: (i) $\Delta y_n > 0$ for $n \geq n_0$ and (ii) $\Delta y_n < 0$ for $n \geq n_0$.

If $\Delta y_n > 0$ for $n \geq n_0$, then $\{y_n\}$ is an increasing sequence. Since $y_n > 0$ for $n \geq n_0$, then $y_n \geq y_{n_0} > 0$ for $n \geq n_0$. From the first equation of (1.1) we have $\Delta^2 x_n > 0$ for $n \geq n_0$ and hence

$$\Delta x_n = \Delta x_{n_0} + \sum_{k=n_0}^{n-1} a_k f(y_k) \geq \Delta x_{n_0} + f(y_{n_0}) \sum_{k=n_0}^{n-1}, \quad a_k \longrightarrow \infty \tag{2.1}$$

as $n \to \infty$, which implies that there exists an integer $n_1 \geq n_0$ such that $x_n \geq x_{n_1} > 0$ for $n \geq n_1$. From the second equation of (1.1), we have

$$\Delta y_n = \Delta y_{n_1} - \sum_{k=n_1}^{n-1} b_k g(x_k) \leq \Delta y_{n_1} - g(x_{n_1}) \sum_{k=n_1}^{n-1}, \quad b_k \longrightarrow -\infty \tag{2.2}$$

as $n \to \infty$, which contradicts the fact that $\Delta y_n > 0$ for $n \geq n_0$.

If $\Delta y_n < 0$ for $n \geq n_0$, then from $\Delta^2 y_n < 0$ for $n \geq n_0$, it follows that $\{\Delta y_n\}$ is decreasing, and hence there exists a constant $c > 0$ such that $\Delta y_n \leq -c$ for $n \geq n_2 \geq n_0$, which means that $y_n \leq y_{n_2} - \sum_{k=n_2}^{n-1} c \to -\infty$ as $n \to \infty$, and so there exists an integer $n_3 \geq n_2$ such that $y_n < 0$ for $n \geq n_3$. This is a contradiction and completes the proof. □

3. The case $A_{n_0} = \infty$ and $B_{n_0} < \infty$

Assume that $A_{n_0} = \infty$ and $B_{n_0} < \infty$. If $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is a positive solution of (1.1), that is to say, $x_n > 0$ and $y_n > 0$ for $n \geq n_0$, then, in view of (1.1), we have $\Delta^2 x_n > 0$ and $\Delta^2 y_n < 0$ for $n \geq n_0$, which imply that $\{\Delta x_n\}$ is increasing and $\{\Delta y_n\}$ is decreasing. Hence $\{x_n\}$ and $\{y_n\}$ are monotonic sequences. By the second equation of (1.1), we have

$$\Delta y_n = \Delta y_{n_0} - \sum_{k=n_0}^{n-1} b_k g(x_k), \quad n \geq n_0. \tag{3.1}$$

If there exists an integer $n_1 \geq n_0$ such that $\Delta y_n < \Delta y_{n_1} < 0$ for $n \geq n_1$, then

$$y_n = y_{n_1} + \sum_{k=n_1}^{n-1} \Delta y_k \leq y_{n_1} + \sum_{k=n_1}^{n-1}, \quad \Delta y_{n_1} \longrightarrow -\infty \tag{3.2}$$

as $n \to \infty$, which contradicts the assumption $y_n > 0$ for $n \geq n_0$. Hence $\Delta y_n > 0$ for $n \geq n_0$ and $\lim_{n\to\infty} \Delta y_n = c \geq 0$, which implies that $\lim_{n\to\infty} y_n = \infty$ or $\lim_{n\to\infty} y_n = \beta > 0$.

By the first equation of (1.1), we have

$$\Delta x_n = \Delta x_{n_0} + \sum_{k=n_0}^{n-1} a_k f(y_k) \geq \Delta x_{n_0} + f(y_{n_0}) \sum_{k=n_0}^{n-1}, \quad a_k \longrightarrow \infty \tag{3.3}$$

as $n \to \infty$, and so, $\lim_{n\to\infty} x_n = \infty$.

In view of the above discussions, we may now make the following classification. Let C be the set of all positive solutions of (1.1). Then we have the following result.

Theorem 3.1. *Suppose that $A_{n_0} = \infty$ and $B_{n_0} < \infty$. Then any positive solutions of (1.1) must belong to the following classes:*

$$C(\infty,\alpha) = \Big\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} y_n = \beta > 0\Big\},$$
$$C(\infty,\infty) = \Big\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} y_n = \infty\Big\}. \tag{3.4}$$

In order to further justify our classification schemes, we derive several sufficient conditions for the existence of each type of positive solutions.

Theorem 3.2. *Suppose that $A_{n_0} = \infty$ and $B_{n_0} < \infty$. A sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\infty,\alpha)$ is that*

$$\sum_{k=n_0}^{\infty}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=n_0}^{s-1}\sum_{t=n_0}^{r-1} a_t f(c)\right) < \infty \tag{3.5}$$

for some $c > 0$ and

$$\sum_{k=n_0}^{\infty}\sum_{s=n_0}^{k-1} a_s f(d) = \infty \tag{3.6}$$

for any $d > 0$.

Proof. Choose N so large that

$$\sum_{k=N}^{\infty}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1}\left(\sum_{t=N}^{r-1} a_t f(c)\right)\right) < \frac{c}{2}. \tag{3.7}$$

Let X be the set of all bounded real-valued sequences $\{y_n\}$ with norm $\sup_{n\ge N} |y_n|$. Then X is a Banach space. We define a subset Ω of X as follows:

$$\Omega = \Big\{y_n \in X \mid \frac{c}{2} \le y_n \le c,\ n \ge N\Big\}. \tag{3.8}$$

Then Ω is bounded, convex and closed subset of X. Let us further define an operator $F : \Omega \to X$ as follows:

$$(Fy)_n = c - \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1}\sum_{t=N}^{r-1} a_t f(y_t)\right), \quad n \ge N. \tag{3.9}$$

The mapping F has the following properties. First of all, F maps Ω into Ω. Indeed, if $y \in \Omega$, then

$$\begin{aligned} c \geq (Fy)_n &= c - \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f(y_t) \right) \\ &\geq c - \sum_{k=N}^{\infty} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f(c) \right) \geq \frac{c}{2}. \end{aligned} \tag{3.10}$$

Next, we show that F is continuous. Let $y^{(l)} \in \Omega$ such that $\lim_{l\to\infty} \| y^{(l)} - y \| = 0$. Since Ω is closed, $y \in \Omega$. Then by (3.9), we have

$$\begin{aligned} \left| \left(Fy^{(l)}\right)_n - (Fy)_n \right| &= \left| \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f\left(y_t^{(l)}\right) \right) - \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f(y_t) \right) \right| \\ &\leq \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} b_s \left| g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f\left(y_t^{(l)}\right) \right) - g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f(y_t) \right) \right|. \end{aligned} \tag{3.11}$$

By the continuity of f and g and Lebesgue's dominated convergence theorem, it follows that

$$\limsup_{l\to\infty\, n\geq N} \left| \left(Fy^{(l)}\right)_n - (Fy)_n \right| = 0. \tag{3.12}$$

This shows that $\lim_{l\to\infty} \| Fy^{(l)} - Fy \| = 0$, that is, F is continuous.

Finally, we will show that $F\Omega$ is precompact. Let $y \in \Omega$ and $m, n \geq N$. Then, we have, for $m > n$,

$$\left| (Fy)_m - (Fy)_n \right| \leq \sum_{s=n}^{m-1} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f(y_t) \right) \leq \sum_{s=n}^{\infty} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=N}^{s-1} \sum_{t=N}^{r-1} a_t f(y_t) \right). \tag{3.13}$$

In view of (3.5), this means that $F\Omega$ is precompact.

By Schauder's fixed point theorem, we conclude that there exists a $y \in \Omega$ such that $y = Fy$. Set $x_n = \sum_{r=N}^{n-1} \sum_{t=N}^{r-1} a_t f(y_t)$. Then

$$x_n \geq \sum_{r=N}^{n-1} \sum_{t=N}^{r-1} a_t f\left(\frac{c}{2}\right) \longrightarrow \infty \quad \text{as } n \to \infty, \tag{3.14}$$

and hence $\lim_{n\to\infty} x_n = \infty$. On the other hand,

$$y_n = (Fy)_n = c - \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} b_s g(x_s), \tag{3.15}$$

which implies that $\lim_{n\to\infty} y_n = c$. The proof is complete. □

THEOREM 3.3. *Suppose that $A_{n_0} = \infty$ and $B_{n_0} < \infty$. A sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\infty,\infty)$ is that*

$$\sum_{k=n_0}^{\infty} a_k f(ck) < \infty \tag{3.16}$$

for some $c > 0$ and

$$\sum_{k=n_0}^{\infty} b_k g(dk) < \infty \tag{3.17}$$

for some $d > 0$.

Proof. Suppose that (3.16) and (3.17) hold. Then there exists $N \geq n_0$ such that

$$\sum_{k=N}^{\infty} a_k f(2ck) < d, \qquad \sum_{k=N}^{\infty} {}_k g(2dk) < c. \tag{3.18}$$

Let X be the Banach space of all real-valued sequences $\{(x_n, y_n)\}$ endowed with the norm

$$\|(x,y)\| = \max\left\{\sup_{n\geq N}\left|\frac{x_n}{n}\right|, \sup_{n\geq N}\left|\frac{y_n}{n}\right|\right\} \tag{3.19}$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X as follows:

$$\Omega = \{(x,y) \in X \mid dn \leq x_n \leq 2dn,\ cn \leq y_n \leq 2cn,\ n \geq N\}. \tag{3.20}$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ as follows: for $(x,y) \in X$, $F(x,y) = (u,v)$ and

$$u_n = dn + \sum_{k=N}^{n-1}\sum_{s=N}^{k-1} a_s f(y_s), \quad v_n = cn + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g(x_s), \quad n \geq N. \tag{3.21}$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem [2]: F is increasing and maps into itself. Indeed, if $x \in \Omega$, then

$$\begin{aligned} dn \leq u_n &\leq dn + n\sum_{s=N}^{\infty} a_s f(2cs) \leq 2dn, \quad n \geq N, \\ cn \leq v_n &\leq cn + n\sum_{s=N}^{\infty} b_s g(2ds) \leq 2cn, \quad n \geq N. \end{aligned} \tag{3.22}$$

By Knaster's fixed point theorem [2], we can conclude that there exists $(x,y) \in \Omega$ such that $(x,y) = F(x,y)$. That is,

$$x_n = dn + \sum_{k=N}^{n-1}\sum_{s=N}^{k-1} a_s f(y_s), \quad y_n = cn + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g(x_s), \quad n \geq N. \tag{3.23}$$

Then $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$. Hence, $\{(x_n, y_n)\}$ is a positive solution of (1.1) which belongs to $C(\infty,\infty)$. The proof is complete. $\square$

4. The case $A_{n_0} < \infty$ and $B_{n_0} < \infty$

We first give a classification scheme for positive solutions of (1.1) under the assumption $A_{n_0} < \infty$ and $B_{n_0} < \infty$.

Theorem 4.1. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} < \infty$. Then any positive solutions of (1.1) must belong to the following classes:*

$$\begin{aligned} C(\alpha,\beta) &= \Big\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = \alpha \geq 0,\ \lim_{n\to\infty} y_n = \beta > 0\Big\}, \\ C(\infty,\beta) &= \Big\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} y_n = \beta > 0\Big\}, \\ C(\alpha,\infty) &= \Big\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = \alpha \geq 0,\ \lim_{n\to\infty} y_n = \infty\Big\}, \\ C(\infty,\infty) &= \Big\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} y_n = \infty\Big\}. \end{aligned} \tag{4.1}$$

Proof. Let $\{(x_n, y_n)\}$ be a positive solution of (1.1). Then $\Delta^2 y_n = -b_n g(x_n) < 0$ for $n \geq n_0$. Hence $\{\Delta y_n\}$ is monotonic and either $\Delta y_n > 0$ for $n \geq n_0$ or $\Delta y_n < 0$ for $n \geq n_0$. If the later holds, then $y_n \leq y_{n_0}$ for $n \geq n_0$ and $\Delta y_n \leq \Delta y_{n_0} < 0$ for $n \geq n_0$, and so $y_n \leq y_{n_0} + \sum_{n=n_0}^{n-1} \Delta y_{n_0} \to -\infty$ as $n \to \infty$, which contradicts the assumption that $y_n > 0$ for $n \geq n_0$, and means that $\lim_{n\to\infty} y_n = \infty$ or $\lim_{n\to\infty} y_n = \beta > 0$. On the other hand, it follows from (1.1) that $\Delta^2 x_n > 0$ for $n \geq n_0$, which implies that $\{\Delta x_n\}$ is monotonic and either $\Delta x_n > 0$ for $n \geq n_0$ or $\Delta x_n < 0$ for $n \geq n_0$. If the later holds, then $\lim_{n\to\infty} x_n = \alpha \geq 0$. If the former holds, then $\lim_{n\to\infty} x_n = \infty$ or $\lim_{n\to\infty} x_n = \alpha > 0$. The proof is complete. $\square$

Again, in order to justify our classification schemes, we derive several necessary and/or sufficient conditions for the existence of each type of positive solutions.

Theorem 4.2. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} < \infty$. A necessary and sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\alpha,\beta)$ is that*

$$\sum_{k=n_0}^{\infty} \sum_{s=k}^{\infty} a_s f(c) < \infty, \qquad \sum_{s=k}^{\infty} b_s g(d) < \infty \tag{4.2}$$

for some $c > 0$ and $d > 0$.

Proof. Let $\{(x_n, y_n)\}$ be a solution of (1.1) such that $\lim_{n\to\infty} x_n = \alpha > 0$ and $\lim_{n\to\infty} y_n = \beta > 0$. Then there exist four positive constants c_1, c_2, c_3, c_4 and $N \geq n_0$ such that $c_1 \leq x_n \leq c_2$, $c_3 \leq y_n \leq c_4$ for $n \geq N$. In view of the first equation of (1.1) and $\lim_{n\to\infty} x_n = \alpha > 0$, we have $x_n = \alpha + \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} a_s f(y_s)$, and so

$$\sum_{k=n_0}^{\infty} \sum_{s=k}^{\infty} a_s f(c_3) < \infty. \tag{4.3}$$

Furthermore, we see from the second equation of (1.1) that $\Delta y_n = \sum_{k=n}^{\infty} b_k g(x_k)$, and $y_n = \beta - \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} b_s g(x_s) > 0$. Thus,

$$\sum_{k=n_0}^{\infty}\sum_{s=k}^{\infty} b_s g(c_1) < \beta < \infty. \tag{4.4}$$

Conversely, suppose that (4.2) holds. Then there exists $N \geq n_0$ such that

$$\sum_{k=N}^{\infty}\sum_{s=k}^{\infty} a_s f(2c) < d, \qquad \sum_{k=N}^{\infty}\sum_{s=k}^{\infty} b_s g(2d) < c. \tag{4.5}$$

Let X be the Banach space of all real-valued sequences $\{(x_n, y_n)\}$ endowed with the norm

$$||(x,y)|| = \max\left\{\sup_{n\geq N} |x_n|, \sup_{n\geq N} |y_n|\right\} \tag{4.6}$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X as follows:

$$\Omega = \{(x,y) \in X \mid d \leq x_n \leq 2d,\, c \leq y_n \leq 2c,\, n \geq N\}. \tag{4.7}$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ as follows: for $(x,y) \in X$, let $F(x,y) = (u,v)$ and

$$u_n = d + \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} a_s f(y_s), \quad v_n = c + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g(x_s), \quad n \geq N. \tag{4.8}$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem [2]: F is increasing and maps into itself. Indeed, if $x \in \Omega$, then

$$\begin{aligned} d \leq u_n &= d + \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} a_s f(y_s) \leq d + \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} a_s f(2c) \leq 2d, \quad n \geq N, \\ c \leq v_n &= c + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g(x_s) \leq c + \sum_{k=N}^{\infty}\sum_{s=k}^{\infty} b_s g(2d) \leq 2c, \quad n \geq N. \end{aligned} \tag{4.9}$$

By Knaster's fixed point theorem [2], we can conclude that there exists $(x,y) \in \Omega$ such that $(x,y) = F(x,y)$. That is,

$$x_n = d + \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} a_s f(y_s), \quad y_n = c + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g(x_s), \quad n \geq N. \tag{4.10}$$

Then

$$\lim_{n\to\infty} x_n = d, \qquad \lim_{n\to\infty} \Delta y_n = \lim_{n\to\infty} \sum_{k=n}^{\infty} b_k g(x_k) = 0, \tag{4.11}$$

and so $\lim_{n\to\infty} y_n = \beta \geq 0$. In view of $\Delta y_n = \sum_{k=n}^{\infty} b_s g(x_s) > 0$, it follows that $\beta > 0$. Hence, $\{(x_n, y_n)\}$ is a positive solution of (1.1) which belongs to $C(\alpha,\beta)$. The proof is complete. □

By means of similar reasoning used in the proof of Theorems 3.2 and 4.2, we may prove the following three theorems.

THEOREM 4.3. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} < \infty$. A sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\infty,\beta)$ is that*

$$\sum_{k=n_0}^{\infty} \sum_{s=n_0}^{k-1} a_s f(c) = \infty \tag{4.12}$$

for any $c > 0$ and

$$\sum_{k=n_0}^{\infty} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=n_0}^{s-1} \sum_{t=n_0}^{r-1} a_s f(d) \right) < \infty \tag{4.13}$$

for some $d > 0$.

THEOREM 4.4. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} < \infty$. A sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\alpha,\infty)$ is that*

$$\sum_{k=n_0}^{\infty} \sum_{s=k}^{\infty} b_s g(c) = \infty \tag{4.14}$$

for any $c > 0$ and

$$\sum_{k=n_0}^{\infty} \sum_{s=k}^{\infty} a_s f\left(\sum_{r=n_0}^{s-1} \sum_{t=n_0}^{r-1} b_s g(d) \right) < \infty \tag{4.15}$$

for some $d > 0$.

THEOREM 4.5. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} < \infty$. A sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\infty,\infty)$ is that*

$$\sum_{k=n_0}^{\infty} a_k f(ck) < \infty \tag{4.16}$$

for some $c > 0$ and

$$\sum_{k=n_0}^{\infty} b_k g(dk) < \infty \tag{4.17}$$

for some $d > 0$.

5. The case $A_{n_0} < \infty$ and $B_{n_0} = \infty$

In this section, we consider the classification and existence for positive solutions of (1.1) under the assumption $A_{n_0} < \infty$ and $B_{n_0} = \infty$.

Theorem 5.1. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} = \infty$. Then any positive solutions of (1.1) must belong to the following classes:*

$$\begin{aligned} C(0,\beta) &= \left\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = 0,\ \lim_{n\to\infty} y_n = \beta > 0\right\}, \\ C(0,\infty) &= \left\{(x_n, y_n) \in C \mid \lim_{n\to\infty} x_n = 0,\ \lim_{n\to\infty} y_n = \infty\right\}. \end{aligned} \tag{5.1}$$

Proof. Let $\{(x_n, y_n)\}$ be a positive solution of (1.1). Then $\Delta^2 y_n = -b_n g(x_n) < 0$ for $n \ge n_0$. Hence $\{\Delta y_n\}$ is monotonic and either $\Delta y_n > 0$ for $n \ge n_0$ or $\Delta y_n < 0$ for $n \ge n_0$. If the later holds, then $y_n \le y_{n_0}$ for $n \ge n_0$ and $\Delta y_n \le \Delta y_{n_0} < 0$ for $n \ge n_0$, and so $y_n \le y_{n_0} + \sum_{n=n_0}^{n-1} \Delta y_{n_0} \to -\infty$ as $n \to \infty$, which contradicts the assumption $y_n > 0$ for $n \ge n_0$, and means that $\lim_{n\to\infty} y_n = \infty$ or $\lim_{n\to\infty} y_n = \beta > 0$. On the other hand, it follows from (1.1) that $\Delta^2 x_n > 0$ for $n \ge n_0$, which implies that $\{\Delta x_n\}$ is monotonic and either $\Delta x_n > 0$ for $n \ge n_0$ or $\Delta x_n < 0$ for $n \ge n_0$. If the former holds, then $x_n \ge x_{n_0}$ for $n \ge n_0$. By the second equation of (1.1) we have

$$\Delta y_n = \Delta y_{n_0} - \sum_{k=n_0}^{n-1} b_k g(x_k) \le \Delta y_{n_0} - g(x_{n_0}) \sum_{k=n_0}^{n-1},\quad b_k \longrightarrow -\infty \tag{5.2}$$

as $n \to \infty$, which implies that $\lim_{n\to\infty} \Delta y_n = -\infty$ and hence $\lim_{n\to\infty} y_n = -\infty$, this contradicts the assumption $y_n > 0$ for $n \ge n_0$. If the later holds, then $\lim_{n\to\infty} x_n = \alpha \ge 0$. Since $\Delta x_n < 0$ for $n \ge n_0$, then $x_n \ge \alpha \ge 0$ for $n \ge n_0$. If $\alpha > 0$, then

$$\Delta y_n = \Delta y_{n_0} - \sum_{k=n_0}^{n-1} b_k g(x_k) \le \Delta y_{n_0} - g(\alpha) \sum_{k=n_0}^{n-1},\quad b_k \longrightarrow \infty \tag{5.3}$$

as $n \to \infty$, which also contradicts the assumption $y_n > 0$ for $n \ge n_0$. The proof is complete. □

In the following, in order to justify our classification schemes, we derive several necessary and/or sufficient conditions for the existence of each type of positive solutions.

Theorem 5.2. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} = \infty$. A necessary and sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(0,\beta)$ is that*

$$\sum_{k=n_0}^{\infty} \sum_{s=k}^{\infty} a_s f(c) < \infty \tag{5.4}$$

for some $c > 0$ and

$$\sum_{k=n_0}^{\infty} \sum_{s=n_0}^{k-1} b_s g\left(\sum_{r=s}^{\infty} \sum_{t=r}^{\infty} a_s f(d)\right) < \infty \tag{5.5}$$

for some $d > 0$.

Proof. Let $\{(x_n, y_n)\}$ be a solution of (1.1) such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = \beta > 0$. Then there exist two positive constants c_1, c_2 and $N \ge n_0$ such that $c_1 \le y_n \le c_2$ for $n \ge N$. In view of the first equation of (1.1) and $\lim_{n\to\infty} x_n = 0$, we have $\Delta x_n = -\sum_{k=n}^{\infty} a_k f(y_k)$, and so

$$\infty > x_n = \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} a_s f(y_s) \ge \sum_{k=n}^{\infty}\sum_{s=k}^{\infty} a_s f(c_1). \tag{5.6}$$

Furthermore, we see from the second equation of (1.1) that $\Delta y_n = \sum_{k=n}^{\infty} b_k g(x_k)$, and

$$\begin{aligned}\infty > y_n = y_N + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g(x_s) &\ge \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(y_t)\right)\\ &\ge \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(c_1)\right).\end{aligned} \tag{5.7}$$

Conversely, suppose that (5.4) and (5.5) hold. Then there exists $N \ge n_0$ such that

$$\sum_{k=N}^{\infty}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(2c)\right) \le c. \tag{5.8}$$

Let X be the Banach space of all real-valued sequences $\{y_n\}$ endowed with the norm $\|y\| = \sup_{n\ge N} |y_n|$ and with the usual pointwise ordering $\le$. Define a subset Ω of X as follows:

$$\Omega = \{y \in X \mid c \le y_n \le 2c,\ n \ge N\}. \tag{5.9}$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ as follows:

$$(Fy)_n = c + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(y_t)\right), \quad n \ge N. \tag{5.10}$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem [2]: F is increasing and maps into itself. Indeed, if $y \in \Omega$, then

$$\begin{aligned}c \le (Fy)_n &= c + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(y_t)\right)\\ &\le c + \sum_{k=N}^{\infty}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(2c)\right) \le 2c, \quad n \ge N.\end{aligned} \tag{5.11}$$

By Knaster's fixed point theorem [2], we can conclude that there exists a $y \in \Omega$ such that $y = Fy$. That is,

$$y_n = c + \sum_{k=N}^{n-1}\sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty}\sum_{t=r}^{\infty} a_t f(y_t)\right), \quad n \ge N. \tag{5.12}$$

Set $x_n = \sum_{r=n}^{\infty} \sum_{t=r}^{\infty} a_t f(y_t)$, then $\lim_{n\to\infty} x_n = 0$ and

$$y_n = c + \sum_{k=N}^{n-1} \sum_{s=k}^{\infty} b_s g(x_s), \quad n \geq N, \tag{5.13}$$

and so $\lim_{n\to\infty} \Delta y_n = \lim_{n\to\infty} \sum_{k=n}^{\infty} b_k g(x_k) = 0$. Hence $\lim_{n\to\infty} y_n = \beta \geq c > 0$ and $\{(x_n, y_n)\}$ is a positive solution of (1.1) which belongs to $C(0,\beta)$. The proof is complete. □

Theorem 5.3. *Suppose that $A_{n_0} < \infty$ and $B_{n_0} = \infty$. A sufficient condition for (1.1) to have a positive solution $\{(x_n, y_n)\}$ which belongs to $C(0,\infty)$ is that*

$$\sum_{k=n_0}^{\infty} b_k g\left(\sum_{s=k}^{\infty} \sum_{r=s}^{\infty} a_r f(cr)\right) < \infty \tag{5.14}$$

for some $c > 0$.

Proof. Suppose that (5.14) holds. Then there exists N so large that

$$\sum_{k=N}^{\infty} b_k g\left(\sum_{s=k}^{\infty} \sum_{r=s}^{\infty} a_r f(2cr)\right) < c. \tag{5.15}$$

Let X be the set of all real-valued sequences $\{y_n\}$ with norm $\sup_{n\geq N} |y_n/n|$. Then X is a Banach space. We define a subset Ω of X as follows:

$$\Omega = \{y_n \in X \mid cn \leq y_n \leq 2cn,\, n \geq N\}. \tag{5.16}$$

Then Ω is a bounded, convex and closed subset of X. Let us further define an operator $F : \Omega \to X$ as follows:

$$(Fy)_n = cn + \sum_{k=N}^{n-1} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty} \left(\sum_{t=r}^{\infty} a_t f(y_t)\right)\right), \quad n \geq N. \tag{5.17}$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem [2]: F is increasing and maps into itself. Indeed, if $x \in \Omega$, then

$$cn \leq (Fy)_n \leq cn + \sum_{k=N}^{n-1} \sum_{s=k}^{\infty} b_s g\left(\sum_{r=s}^{\infty} \sum_{t=r}^{\infty} a_t f(2ct)\right) \leq 2cn, \quad n \geq N. \tag{5.18}$$

By Knaster's fixed point theorem, we can conclude that there exists a $y \in \Omega$ such that $y = Fy$. Set $x_n = \sum_{k=n}^{\infty} \sum_{s=k}^{\infty} a_s f(y_s)$, then

$$y_n = cn + \sum_{k=N}^{n-1} \sum_{s=k}^{\infty} b_s g(x_s) \tag{5.19}$$

and $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = \infty$. The proof is complete. □

6. Examples

In this section, we will give some examples to illustrate our results.

Example 6.1. Consider the following system:

$$\begin{aligned} \Delta^2 x_n &= a_n y_n, \\ \Delta^2 y_n &= -b_n x_n, \quad n \geq n_0, \end{aligned} \tag{6.1}$$

where $a_n = 6(n+1)$ and $b_n = 0$. It is easy to see that

$$\begin{gathered} A_{n_0} = \sum_{n=n_0}^{\infty} a_n = \sum_{n=n_0}^{\infty} 6(n+1) = \infty, \qquad B_{n_0} = \sum_{n=n_0}^{\infty} b_n = 0 < \infty, \\ \sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} b_s \left(\sum_{r=n_0}^{s-1} \sum_{t=n_0}^{r-1} c a_t \right) = 0 < \infty, \quad \text{for some } c > 0, \\ \sum_{n=n_0}^{\infty} \sum_{s=n_0}^{n-1} d a_s = \infty, \quad \text{for any } d > 0. \end{gathered} \tag{6.2}$$

By Theorem 3.2, (6.1) has a positive solution $\{(x_n, y_n)\}$ which belongs to $C(\infty, \alpha)$. In fact, it is easy to verify that $x_n = n^3$ and $y_n = 1$ is such a positive solution, that is to say, $\Delta^2 x_n = 6(n+1)$ and $\Delta^2 y_n = 0$.

Example 6.2. Consider the following system:

$$\begin{aligned} \Delta^2 x_n &= 2\left(\sqrt{n}\right)^3 \left(\frac{1}{y_n}\right)^3, \\ \Delta^2 y_n &= -\frac{\sqrt{n+2} - \sqrt{n}}{\left(\sqrt{n+2} + \sqrt{n+1}\right)\left(\sqrt{n+1} + \sqrt{n}\right) n^2} x_n, \quad n \geq n_0. \end{aligned} \tag{6.3}$$

Obviously,

$$\begin{aligned} A_{n_0} &= \sum_{n=n_0}^{\infty} a_n = \sum_{n=n_0}^{\infty} 2\left(\sqrt{n}\right)^3 = \infty, \\ B_{n_0} &= \sum_{n=n_0}^{\infty} b_n = \sum_{n=n_0}^{\infty} \frac{\sqrt{n+2} - \sqrt{n}}{\left(\sqrt{n+2} + \sqrt{n+1}\right)\left(\sqrt{n+1} + \sqrt{n}\right) n^2} < \infty. \end{aligned} \tag{6.4}$$

Furthermore,

$$\begin{aligned} \sum_{n=n_0}^{\infty} a_n f(cn) &= \frac{2}{c^3} \sum_{n=n_0}^{\infty} \left(\sqrt{n}\right)^3 \frac{1}{n^3} = \frac{2}{c^3} \sum_{n=n_0}^{\infty} n^{-3/2} < \infty, \quad \text{for some } c > 0, \\ \sum_{n=n_0}^{\infty} b_n g(cn) &= c \sum_{n=n_0}^{\infty} \frac{\sqrt{n+2} - \sqrt{n}}{\left(\sqrt{n+2} + \sqrt{n+1}\right)\left(\sqrt{n+1} + \sqrt{n}\right) n} < \infty, \quad \text{for some } c > 0. \end{aligned} \tag{6.5}$$

By Theorem 3.3, (6.3) has a positive solution which belongs to $C(\infty,\infty)$. In fact, $x_n = n^2$ and $y_n = \sqrt{n}$ is such a positive solution.

Similarly, we can provide some examples to illustrate our results in Sections 4, 5, and 6.

As a final remark, our results can be extended without too much difficulty to the following two-dimensional delay difference system:

$$\begin{aligned} \Delta^2 x_n &= a_n f(y_{n-\tau}), \\ \Delta^2 y_n &= -b_n g(x_{n-\delta}), \end{aligned} \tag{6.6}$$

where τ and δ are positive integers.

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