TWO PERIODIC SOLUTIONS OF NEUTRAL DIFFERENCE EQUATIONS MODELLING PHYSIOLOGICAL PROCESSES

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We establish existence, multiplicity, and nonexistence of periodic solutions for a class of first-order neutral difference equations modelling physiological processes and conditions. Our approach is based on a fixed point theorem in cones as well as some analysis techniques.

1. Introduction

The existence of periodic solutions for difference equations has been extensively considered by many authors [1, 4, 8, 9, 12, 16]. Recently, existence of multiple solutions of functional differential equations has been studied and some results have been obtained [6, 14, 18]. Wang [14] investigated existence, multiplicity, and nonexistence of positive periodic solutions for the equation

$$\frac{d}{dt}x(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t-\tau(t))), \tag{1.1}$$

where λ is a positive parameter. Chow [2], Smith and Kuang [13], and many others studied the type of equations or their generalized forms. This type of equations has been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias [11, 15].

To our best knowledge, few papers are on multiplicity of periodic solutions of neutral functional difference systems. In this paper, we consider the following first-order neutral difference equation:

$$\Delta(x(n) - cx(n-\delta)) = a(n)g(x(n))x(n) - \lambda b(n)f(x(n-\tau(n))), \quad n \in \mathbb{Z}, \tag{1.2}$$

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where $\mathbb{Z}$ is the set of integers, $\Delta x(n) = x(n+1) - x(n)$, λ is a positive parameter, c is a constant, and $|c| \neq 1$, δ is a positive integer, $a(n)$, $b(n)$, and $\tau(n)$ are positive T-periodic sequences, $T \in \mathbb{N}$.

Let $N^* = \{0,1,2,\dots,T-1\}$ and

$$f_0 = \lim_{u\to 0^+}\frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty}\frac{f(u)}{u},$$
$$i_0 = \text{number of zeros in the set } \{f_0, f_\infty\},$$
$$i_\infty = \text{number of infinities in the set } \{f_0, f_\infty\}. \tag{1.3}$$

It is clear that $i_0, i_\infty = 0, 1$, or 2. Then we should show that (1.2) has i_0 or i_∞ periodic solution(s) for some certain λ, respectively. In what follows, we set

$$X = \{x \mid x(n),\ x(n+T) \equiv x(n),\ n \in \mathbb{Z}\} \tag{1.4}$$

with the norm defined by $\|x\|_X = \max\{|x(n)| : n \in N^*\}$. Then X is a Banach space. Let $A : X \to X$ be defined by $(Ax)(n) = x(n) - cx(n-\delta)$.

Lemma 1.1. *If $|c| \neq 1$, then A has continuous bounded inverse A^{-1} on X and for all $x \in X$,*

$$(A^{-1}x)(n) = \begin{cases} \sum_{j\geq 0} c^j x(n - j\delta), & \text{if } |c| < 1, \\ -\sum_{j\geq 1} c^{-j} x(n + j\delta), & \text{if } |c| > 1,\ n \in \mathbb{Z}, \end{cases}$$
$$\|A^{-1}x\|_X \leq \frac{\|x\|_X}{|1 - |c||}. \tag{1.5}$$

Proof. According to [10, 17], we can get equality (1.5) and then verify the results of Lemma 1.1.

We consider the following assumptions.

(E_1) $a(n)$, $b(n)$ are positive T-periodic sequences, $\tau(n)$ is a positive T-periodic integer sequence.

(E_2) $f, g \in \mathbb{C}([0,\infty),[0,\infty))$ and there exist two positive constants l, L such that $0 < l \leq g(u) \leq L < +\infty$ for $u \in \mathbb{R}$; $f(u) > 0$ for $u > 0$.

Define

$$A_1 = \frac{1}{\prod_{r=n}^{n+T-1}[a(r)L+1] - 1}, \qquad B = \frac{\prod_{r=n}^{n+T-1}[a(r)L+1]}{\prod_{r=n}^{n+T-1}[a(r)l+1] - 1}, \tag{1.6}$$

and $\alpha = A_1/B$, for any $r > 0$, we denote

$$M(r) = \max\left\{ f(t) : 0 \le t \le \frac{r}{1-|c|} \right\},$$
$$m(r) = \min\left\{ f(t) : \frac{\alpha - |c|}{1-c^2} r \le t \le \frac{r}{1-|c|} \right\}, \tag{1.7}$$
$$k = \min\left\{ \alpha, \frac{1}{1 + BL\Sigma_{s=0}^{T-1} a(s)} \right\}.$$

We aim to establish existence, multiplicity, and nonexistence of positive T-periodic solutions for first-order neutral difference equation (1.2). Our approach is based on a fixed point theorem in cones as well as some analysis techniques which are used by Wang [14]. The rest of this paper is organized as follows. Section 2 is about statement of the method (a fixed point theorem in cones) and some lemmas which play important roles in proofs of main results; in Section 3, we establish our main results and give an example to illustrate the applicability of our results. □

2. Preliminaries

We first state the following well-known result. For the proof, we refer to the classical works [3, 5, 7].

Lemma 2.1 (Deimling [3], Guo and Lakshmikantham [5], and Krasnosel'skiĭ [7]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \ne x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(i) *If $\|Tx\| \ge \|x\|$ for any $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*

(ii) *If $\|Tx\| \le \|x\|$ for any $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

Next, we transfer existence of positive T-periodic solutions of (1.2) into existence of positive fixed points of some fixed point mapping.

In order to establish existence, multiplicity, and nonexistence of positive T-periodic solutions for (1.2), we first consider the following equation:

$$\Delta y(n) = a(n) g\big((A^{-1}y)(n)\big)(A^{-1}y)(n) - \lambda b(n) f\big((A^{-1}y)(n - \tau(n))\big), \tag{2.1}$$

where A^{-1} is defined by (1.5). By Lemma 1.1 and the definition of A and A^{-1}, we conclude the following.

Lemma 2.2. *$y(n)$ is a T-periodic solution of (2.1) if and only if $(A^{-1}y)(n)$ is a T-periodic solution of (1.2).*

Aiming to apply Lemma 2.1 to (2.1), we rewrite (2.1) as

$$\Delta y(n) = a(n) g\big((A^{-1}y)(n)\big) y(n) - \big[a(n) G(y(n)) + \lambda b(n) f\big((A^{-1}y)(n - \tau(n))\big)\big], \tag{2.2}$$

where

$$G(y(n)) = -cg((A^{-1}y)(n))(A^{-1}y)(n-\tau). \tag{2.3}$$

A cone K in X is defined by

$$K = \{u \in X : u(n) \geq \alpha\|u\|_X,\ n \in \mathbb{Z}\}. \tag{2.4}$$

For $r > 0$, define Ω_r by $\Omega_r = \{u \in K : \|u\|_X < r\}$ and $\partial\Omega_r = \{u \in K : \|u\|_X = r\}$. Let the operator $Q : K \to X$ be defined by

$$Qu(n) = \Sigma_{s=n}^{n+T-1} K_u(n,s)[a(s)G(u(s)) + \lambda b(s)f((A^{-1}u)(s-\tau(s)))], \quad n \in \mathbb{Z}, \tag{2.5}$$

where

$$K_u(n,s) = \frac{\prod_{r=s+1}^{n+T-1}[a(r)g((A^{-1}u)(r)) + 1]}{\prod_{r=n}^{n+T-1}[a(r)g((A^{-1}u)(r)) + 1] - 1}, \quad n,s \in \mathbb{Z},\ n \leq s \leq n+T-1. \tag{2.6}$$

Assumption (E_2) implies that

$$0 < A_1 \leq K_u(n,s) \leq B, \quad n,s \in \mathbb{Z},\ n \leq s \leq n+T-1. \tag{2.7}$$

Lemma 2.3. *The positive T-periodic solution of (2.1) is equivalent to the fixed point of Q in K.*

Lemma 2.4. *If assumptions (E_1) and (E_2) hold, $c \in (-\alpha, 0]$, and $y \in K$, then*
(a) $((\alpha - |c|)/(1-c^2))\|y\|_X \leq (A^{-1}y)(n) \leq (1/(1-|c|))\|y\|_X$,
(b) $l|c|((\alpha - |c|)/(1-c^2))\|y\|_X \leq G(y(n)) \leq (L|c|/(1-|c|))\|y\|_X$, $n \in N^*$.

Proof
Part (a). Since $-\alpha < c \leq 0$, it follows from Lemma 1.1 that

$$\begin{aligned}
(A^{-1}y)(n) &= \sum_{j\geq 0} c^j y(n - j\delta) \\
&= \sum_{j\geq 0} c^{2j} y(n-2j\delta) - \sum_{j\geq 1} |c|^{2j-1} y(n-(2j-1)\delta) \\
&\geq \frac{\alpha - |c|}{1-c^2}\|y\|_X, \quad n \in N^*, \\
(A^{-1}y)(n) &\leq \frac{1}{1-|c|}\|y\|_X.
\end{aligned} \tag{2.8}$$

Part (b). From part (a) and the assumption (E_2), for any $n \in \mathbb{Z}$, we get

$$l|c|\frac{\alpha - |c|}{1-c^2}\|y\|_X \leq G(y(n)) \leq \frac{L|c|}{1-|c|}\|y\|_X. \tag{2.9}$$

□

Lemma 2.5. *If assumptions (E_1) and (E_2) hold and $c \in (-\alpha, 0]$, then $Q(K) \subset K$ and $Q : K \to K$ is completely continuous.*

Proof. By Lemma 1.1, similar to the proof of Lemma 2.2 in [7], we can prove Lemma 2.5. □

LEMMA 2.6. *If assumptions* (E_1) *and* (E_2) *hold and* $c \in (-\alpha, 0]$, *then* $y(n)$ *is the fixed point of* Q *in* K *if and only if* $(A^{-1}y)(n)$ *is a positive* T*-periodic solution of (1.2).*

Proof. If $y(n)$ is the fixed point of Q in K, $y(n)$ is a positive T-periodic solution of (2.1) and $y \in K$ by Lemma 2.3. It follows from Lemmas 2.2 and 2.4 that $(A^{-1}y)(n)$ is a T-periodic solution of (1.2) and $(A^{-1}y)(n) \geq ((\alpha - |c|)/(1-c^2))\|y\|_X > 0$. Therefore, $(A^{-1}y)(n)$ is a positive T-periodic solution of (1.2).

If there exists $y(n)$ such that $(A^{-1}y)(n)$ is a positive T-periodic solution of (1.2), then $y(n)$ is a T-periodic solution of (2.1) by Lemma 2.2. From the definition of A^{-1} and $c \in (-\alpha, 0]$, $y(n) = (A^{-1}y)(n) - c(A^{-1}y)(n-\delta) > 0$. Lemmas 2.3 and 2.5 imply that $y(n)$ is the fixed point of Q in K. □

LEMMA 2.7. *Assumptions* (E_1) *and* (E_2) *hold and* $c \in (-\alpha, 0]$, $\eta > 0$. *If* $f((A^{-1}y)(n - \tau(n))) \geq (A^{-1}y)(n-\tau(n))\eta$ *for any* $y \in K$ *and* $n \in \mathbb{Z}$, *then*

$$\|Qy\|_X \geq \lambda A_1 \eta \Sigma_{s=0}^{T-1} b(s) \frac{\alpha - |c|}{1-c^2}\|y\|_X. \tag{2.10}$$

Proof. By Lemma 2.4, for any $y \in K$ and $n \in \mathbb{Z}$, $G(y(n)) \geq 0$ as $c \in (-\alpha, 0]$. Therefore,

$$\begin{aligned} Qy(n) &\geq \lambda A_1 \Sigma_{s=n}^{n+T-1} b(s) f\big((A^{-1}y)(s-\tau(s))\big) = \lambda A_1 \Sigma_{s=0}^{T-1} b(s) f\big((A^{-1}y)(s-\tau(s))\big) \\ &\geq \lambda A_1 \eta \Sigma_{s=0}^{T-1} b(s)(A^{-1}y)(s-\tau(s)) \geq \lambda A_1 \eta \Sigma_{s=0}^{T-1} b(s) \frac{\alpha-|c|}{1-c^2}\|y\|_X. \end{aligned} \tag{2.11}$$

That is,

$$\|Qy\|_X \geq \lambda A_1 \eta \Sigma_{s=0}^{T-1} b(s) \frac{\alpha - |c|}{1-c^2}\|y\|_X. \tag{2.12}$$

□

LEMMA 2.8. *Assumptions* (E_1) *and* (E_2) *hold and* $c \in (-\alpha, 0]$. *For any* $n \in \mathbb{Z}$, *if there exists* $\varepsilon > 0$ *such that* $f((A^{-1}y)(n-\tau(n))) \leq (A^{-1}y)(n-\tau(n))\varepsilon$, *then*

$$\|Qy\|_X \leq \frac{B\Sigma_{s=0}^{T-1}[L|c|a(s) + \lambda\varepsilon b(s)]}{1-|c|}\|y\|_X. \tag{2.13}$$

Proof. From Lemmas 1.1 and 2.4, we have

$$\begin{aligned} \|Qy\|_X &\leq B\Sigma_{s=0}^{T-1}\big[a(s)G(y(s)) + \lambda b(s) f\big((A^{-1}y)(s-\tau(s))\big)\big] \\ &\leq B\Sigma_{s=0}^{T-1}\left[a(s)\frac{L|c|}{1-|c|}\|y\|_X + \lambda b(s)\varepsilon(A^{-1}y)(s-\tau(s))\right] \\ &\leq \frac{B\Sigma_{s=0}^{T-1}[L|c|a(s) + \lambda\varepsilon b(s)]}{1-|c|}\|y\|_X. \end{aligned} \tag{2.14}$$

□

Lemma 2.9. *Assumptions (E_1) and (E_2) hold and $c \in (-\alpha, 0]$. For $y \in \partial\Omega_r$, $r > 0$, one can obtain*

$$\|Qy\|_X \geq \lambda A_1 m(r)\Sigma_{s=0}^{T-1} b(s). \tag{2.15}$$

Proof. Since $y \in \partial\Omega_r$, by Lemma 2.4, $((\alpha - |c|)/(1 - c^2))r \leq (A^{-1}y)(n - \tau(n)) \leq r/(1 - |c|)$. So $f((A^{-1}y)(n - \tau(n))) \geq m(r)$ for $y \in \partial\Omega_r$ and $n \in \mathbb{Z}$. Similar to the proof of Lemma 2.7, we can obtain Lemma 2.9. □

Lemma 2.10. *Assumptions (E_1) and (E_2) hold and $c \in (-\alpha, 0]$. If $y \in \partial\Omega_r$, $r > 0$, then*

$$\|Qy\|_X \leq B\Sigma_{s=0}^{T-1}\left[\lambda b(s)M(r) + \frac{L|c|a(s)r}{1 - |c|}\right]. \tag{2.16}$$

Proof. By $y \in \partial\Omega_r$ and Lemma 1.1, $0 \leq (A^{-1}y)(n - \tau(n)) \leq r/(1 - |c|)$. So $f((A^{-1}y)(n - \tau(n))) \leq M(r)$ for any $y \in \partial\Omega_r$ and $n \in \mathbb{Z}$. From The proof of Lemma 2.8, we can similarly prove Lemma 2.10. □

3. Main results

We state our main results as follows.

Theorem 3.1. *Suppose that assumptions (E_1), (E_2) hold and $-k < c \leq 0$.*

(a) *If $i_0 = 1$ or 2, then (1.2) has i_0 positive T-periodic solution(s) for $\lambda > 1/A_1 m(1)\Sigma_{s=0}^{T-1} b(s) > 0$.*

(b) *If $i_\infty = 1$ or 2, then (1.2) has i_∞ positive T-periodic solution(s) for $0 < \lambda < (1 - |c| - BL|c|\Sigma_{s=0}^{T-1} a(s))/BM(1)\Sigma_{s=0}^{T-1} b(s)(1 - |c|)$.*

(c) *If $i_\infty = 0$ or $i_0 = 0$, then (1.2) has no positive T-periodic solution for sufficiently small or large $\lambda > 0$, respectively.*

Theorem 3.2. *Suppose that assumptions (E_1), (E_2) hold and $-k < c \leq 0$.*

(a) *If there exists a constant $c_1 > 0$ such that $f(u) \geq c_1 u$ for $u \in [0, +\infty)$, then (1.2) has no positive T-periodic solution for $\lambda > (1 - c^2)/A_1 c_1(\alpha - |c|)\Sigma_{s=0}^{T-1} b(s)$.*

(b) *If there exists a constant $c_2 > 0$ such that $f(u) \leq c_2 u$ for $u \in [0, +\infty)$, then (1.2) has no positive T-periodic solution for $0 < \lambda < (1 - |c| - BL|c|\Sigma_{s=0}^{T-1} a(s))/Bc_2\Sigma_{s=0}^{T-1} b(s)$.*

Theorem 3.3. *Suppose that assumptions (E_1), (E_2) hold and $-k < c \leq 0$. If $i_0 = i_\infty = 0$ and*

$$\frac{1 - c^2}{\max\{f_\infty, f_0\}A_1(\alpha - |c|)\Sigma_{s=0}^{T-1} b(s)} < \lambda < \frac{1 - |c| - BL|c|\Sigma_{s=0}^{T-1} a(s)}{\min\{f_0, f_\infty\}B\Sigma_{s=0}^{T-1} b(s)}, \tag{3.1}$$

then (1.2) has one positive T-periodic solution.

Proof of Theorem 3.1

Part (a). Take $r_1 = 1$ and $\lambda_0 = 1/A_1 m(r_1)\Sigma_{s=0}^{T-1} b(s) > 0$. For any $y \in \partial\Omega_{r_1}$ and $\lambda > \lambda_0$, it follows from Lemma 2.9 that

$$\|Qy\|_X > \|y\|_X, \quad y \in \partial\Omega_{r_1}. \tag{3.2}$$

From Lemma 2.1, $i(Q, \Omega_{r_1}, K) = 0$.

Case 1. If $f_0 = 0$, then for any $\varepsilon > 0$, we can choose $0 < \overline{r}_2 < r_1$ such that $f(u) \le \varepsilon u$ for $0 \le u \le \overline{r}_2$. Since $-k < c \le 0$, $1 > BL|c|\Sigma_{s=0}^{T-1}a(s)/(1-|c|)$. Take $\varepsilon > 0$ satisfying

$$\frac{\lambda B\varepsilon\Sigma_{s=0}^{T-1}b(s)}{1-|c|} < 1 - \frac{BL|c|\Sigma_{s=0}^{T-1}a(s)}{1-|c|}. \tag{3.3}$$

Let $r_2 = (1-|c|)\overline{r}_2$. If $y \in \partial\Omega_{r_2}$, then $0 \le (A^{-1}y)(n-\tau(n)) \le 1/(1-|c|)\|y\|_X \le \overline{r}_2$. So $f((A^{-1}y)(n-\tau(n))) \le \varepsilon(A^{-1}y)(n-\tau(n))$ for any $y \in \partial\Omega_{r_2}$ and $n \in \mathbb{Z}$. By Lemma 2.8 and inequality (3.3), for all $y \in \partial\Omega_{r_2}$, we have

$$\|Qy\|_X \le \frac{\lambda B\varepsilon\Sigma_{s=0}^{T-1}b(s) + BL|c|\Sigma_{s=0}^{T-1}a(s)}{1-|c|}\|y\|_X < \|y\|_X. \tag{3.4}$$

Lemma 2.1 implies that $i(Q,\Omega_{r_2},K) = 1$. Thus $i(Q,\Omega_{r_1}\setminus\overline{\Omega}_{r_2},K) = -1$ and Q has a fixed point $y(n)$ in $\Omega_{r_1}\setminus\overline{\Omega}_{r_2}$. It follows from Lemma 2.6 that (1.2) has at least one positive T-periodic solution $(A^{-1}y)(n)$ for $\lambda > \lambda_0$.

Case 2. If $f_\infty = 0$, then there exists a constant $\widetilde{H} > 0$ for any $\varepsilon > 0$ such that $f(u) \le \varepsilon u$ for all $u \ge \widetilde{H}$. $-k < c \le 0$ shows that $1 > BL|c|\Sigma_{s=0}^{T-1}a(s)/(1-|c|)$. So we can choose $\varepsilon > 0$ satisfying inequality (3.3).

Take $r_3 = \max\{2r_1, ((1-c^2)/(\alpha-|c|))\widetilde{H}\}$. For any $y \in \partial\Omega_{r_3}$, since $(A^{-1}y)(n-\tau(n)) \ge ((\alpha-|c|)/(1-c^2))\|y\|_X \ge \widetilde{H}$, $f((A^{-1}y)(n-\tau(n))) \le \varepsilon(A^{-1}y)(n-\tau(n))$. From Lemma 2.8 and inequality (3.3), for each $y \in \partial\Omega_{r_3}$, we get

$$\|Qy\|_X \le \frac{\lambda B\varepsilon\Sigma_{s=0}^{T-1}b(s) + BL|c|\Sigma_{s=0}^{T-1}a(s)}{1-|c|}\|y\|_X < \|y\|_X. \tag{3.5}$$

It follows from Lemma 2.1 that $i(Q,\Omega_{r_3},K) = 1$. Therefore, $i(Q,\Omega_{r_3}\setminus\overline{\Omega}_{r_1},K) = 1$ and Q has at least one fixed point $y(n)$ in $\Omega_{r_3}\setminus\overline{\Omega}_{r_1}$. By Lemma 2.6, we conclude that (1.2) has at least one positive T-periodic solution $(A^{-1}y)(n)$ for $\lambda > \lambda_0$.

Case 3. If $f_\infty = f_0 = 0$, from the above arguments, there exist r_1, r_2, and r_3 with $0 < r_2 < r_1 < r_3$ such that Q has fixed points $y_1(n)$ and $y_2(n)$ in $\Omega_{r_1}\setminus\overline{\Omega}_{r_2}$ and $\Omega_{r_3}\setminus\overline{\Omega}_{r_1}$, respectively. By Lemma 2.6, for any $\lambda > \lambda_0$, (1.2) has at least two positive T-periodic solutions $(A^{-1}y_1)(n)$ and $(A^{-1}y_2)(n)$.

Part (b). $-k < c \le 0$ implies that $1 > BL|c|\Sigma_{s=0}^{T-1}a(s)/(1-|c|)$. Let $r_1 = 1$ and $\lambda_1 = (1-|c| - BL|c|\Sigma_{s=0}^{T-1}a(s))/BM(r_1)\Sigma_{s=0}^{T-1}b(s)(1-|c|) > 0$. From Lemma 2.10, for any $y \in \partial\Omega_{r_1}$ and $0 < \lambda < \lambda_1$, we have

$$\|Qy\|_X < \|y\|_X. \tag{3.6}$$

By Lemma 2.1, $i(Q,\Omega_{r_1},K) = 1$.

Case 1. If $f_0 = \infty$, then for any $\eta > 0$, there exists $0 < \overline{r}_2 < r_1$ such that $f(u) \ge \eta u$ for each $0 \le u \le \overline{r}_2$. Take $\eta > 0$ satisfying

$$\lambda A_1\eta\frac{\alpha-|c|}{1-c^2}\Sigma_{s=0}^{T-1}b(s) > 1. \tag{3.7}$$

Let $r_2 = (1-|c|)\overline{r}_2$. For any $y \in \partial\Omega_{r_2}$, $0 \le (A^{-1}y)(n-\tau(n)) \le (1/(1-|c|))\|y\|_X \le \overline{r}_2$. Thus $f((A^{-1}y)(n-\tau(n))) \ge \eta(A^{-1}y)(n-\tau(n))$ for $y \in \partial\Omega_{r_2}$ and $n \in \mathbb{Z}$. By Lemma 2.7 and inequality (3.7), for any $y \in \partial\Omega_{r_2}$, we get

$$\|Qy\|_X \ge \lambda A_1 \eta \frac{\alpha-|c|}{1-c^2}\Sigma_{s=0}^{T-1} b(s)\|y\|_X > \|y\|_X. \tag{3.8}$$

Lemma 2.1 tells that $i(Q,\Omega_{r_2},K) = 0$. So $i(Q,\Omega_{r_1} \setminus \overline{\Omega}_{r_2},K) = 1$ and Q has at least one fixed point $y(n)$ in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$. From Lemma 2.6, $(A^{-1}y)(n)$ is a positive T-periodic solution of (1.2) for $\lambda \in (0,\lambda_1)$.

Case 2. If $f_\infty = \infty$, then for any $\eta > 0$, we can find $\widetilde{H} > 0$ satisfying that $f(u) \ge \eta u$ for each $u \ge \widetilde{H}$. Take $\eta > 0$ such that inequality (3.7) holds.

Let $r_3 = \max\{2r_1,((1-c^2)/(\alpha-|c|))\widetilde{H}\}$. As $y \in \partial\Omega_{r_3}$, $(A^{-1}y)(n-\tau(n)) \ge ((\alpha-|c|)/(1-c^2))\|y\|_X \ge \widetilde{H}$. Then $f((A^{-1}y)(n-\tau(n))) \ge \eta(A^{-1}y)(n-\tau(n))$ for any $y \in \partial\Omega_{r_3}$. For any $y \in \partial\Omega_{r_3}$, it follows from Lemma 2.7 and inequality (3.7) that

$$\|Qy\|_X \ge \lambda A_1 \eta \frac{\alpha-|c|}{1-c^2}\Sigma_{s=0}^{T-1} b(s)\|y\|_X > \|y\|_X. \tag{3.9}$$

By Lemma 2.1, we obtain $i(Q,\Omega_{r_3},K) = 0$. Thus, $i(Q,\Omega_{r_3} \setminus \overline{\Omega}_{r_1},K) = -1$ and Q has at least one fixed point $y(n)$ in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$. Lemma 2.6 shows that $(A^{-1}y)(n)$ is a positive T-periodic solution of (1.2) for $\lambda \in (0,\lambda_1)$.

Case 3. If $f_\infty = f_0 = \infty$, from the arguments of Cases 1 and 2 in Part (b), there exist constants $0 < r_2 < r_1 < r_3$ such that Q has one fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$, respectively, denoting $y_1(n)$ and $y_2(n)$. That is, for any $\lambda \in (0,\lambda_1)$, (1.2) has at least two positive T-periodic solutions $(A^{-1}y_1)(n)$ and $(A^{-1}y_2)(n)$.

Part (c)

Case 1. If $i_0 = 0$, then $f_0 > 0$ and $f_\infty > 0$. Letting $c_1 = \min\{(f(u)/u) : u > 0\} > 0$, we have

$$f(u) \ge c_1 u, \quad u \in [0,+\infty). \tag{3.10}$$

Take $\lambda_2 = (1-c^2)/(A_1 c_1(\alpha-|c|)\Sigma_{s=0}^{T-1} b(s))$ and suppose that $u(n)$ is the positive T-periodic solution of (1.2) for $\lambda > \lambda_2$. For any $n \in \mathbb{Z}$, $f(A^{-1}u(n-\tau(n))) \ge c_1 A^{-1}u(n-\tau(n)) \ge (c_1(\alpha-|c|)/(1-c^2))\|u\|_X$ and $Qu(n) = u(n)$. From Lemma 2.7, for $\lambda > \lambda_2$, we obtain

$$\|u\|_X = \|Qu\|_X \ge \lambda A_1 c_1 \frac{\alpha-|c|}{1-c^2}\Sigma_{s=0}^{T-1} b(s)\|u\|_X > \|u\|_X, \tag{3.11}$$

which is a contradiction. Thus, when $i_0 = 0$ and $\lambda > \lambda_2$, (1.2) has no positive T-periodic solution.

Case 2. $i_\infty = 0$ implies that $f_0 < \infty$ $f_\infty < \infty$. Since $-k < c \le 0$, $1-|c| > BL|c|\Sigma_{s=0}^{T-1} a(s)$. Letting $c_2 = \max\{f(u)/u : u > 0\} > 0$, we get

$$f(u) \le c_2 u, \quad u \in [0,+\infty). \tag{3.12}$$

Take $\lambda_3 = (1-|c|-BL|c|\Sigma_{s=0}^{T-1} a(s))/Bc_2\Sigma_{s=0}^{T-1} b(s)$. Suppose that $u(n)$ is the positive T-periodic solution of (1.2) corresponding to $\lambda \in (0,\lambda_3)$. For any $n \in \mathbb{Z}$, $f(A^{-1}u(n-$

$\tau(n))) \le c_2 A^{-1}u(n-\tau(n)) \le (c_2/(1-|c|))\|u\|_X$ and $Qu(n)=u(n)$. Therefore, by Lemma 2.8, for $\lambda \in (0,\lambda_3)$, we have

$$\|u\|_X = \|Qu\|_X \le \frac{\lambda B c_2 \Sigma_{s=0}^{T-1} b(s) + BL|c|\Sigma_{s=0}^{T-1} a(s)}{1-|c|}\|u\|_X < \|u\|_X, \tag{3.13}$$

which is a contradiction. So, When $i_\infty = 0$, (1.2) has no positive T-periodic solution for any $0 < \lambda < \lambda_3$. □

Proof of Theorem 3.2. Following the proof of part (c) of Theorem 3.1, we can obtain this result immediately. □

Proof of Theorem 3.3
Case 1. If $f_0 \le f_\infty$, then

$$\frac{1-c^2}{f_\infty A_1(\alpha-|c|)\Sigma_{s=0}^{T-1} b(s)} < \lambda < \frac{1-|c|-BL|c|\Sigma_{s=0}^{T-1} a(s)}{f_0 B \Sigma_{s=0}^{T-1} b(s)}. \tag{3.14}$$

We can choose $0 < \varepsilon < f_\infty$ such that

$$\frac{1-c^2}{(f_\infty-\varepsilon)A_1(\alpha-|c|)\Sigma_{s=0}^{T-1} b(s)} < \lambda < \frac{1-|c|-BL|c|\Sigma_{s=0}^{T-1} a(s)}{(f_0+\varepsilon) B \Sigma_{s=0}^{T-1} b(s)}. \tag{3.15}$$

From the definition of f_0, there exists $\overline{r}_1 > 0$ such that $f(u) \le (f_0+\varepsilon)u$ for any $0 \le u \le \overline{r}_1$. Take $r_1 = (1-|c|)\overline{r}_1$. For $y \in \partial\Omega_{r_1}$, since $0 \le (A^{-1}y)(n-\tau(n)) \le (1/(1-|c|))\|y\|_X \le \overline{r}_1$, then $f((A^{-1}y)(n-\tau(n))) \le (f_0+\varepsilon)(A^{-1}y)(n-\tau(n))$. By Lemma 2.8, for any $y \in \partial\Omega_{r_1}$, we get

$$\|Qy\|_X \le \frac{B\lambda(f_0+\varepsilon)\Sigma_{s=0}^{T-1} b(s) + BL|c|\Sigma_{s=0}^{T-1} a(s)}{1-|c|}\|y\|_X < \|y\|_X. \tag{3.16}$$

On the other hand, we can choose $\widetilde{H} > 0$ such that $f(u) \ge (f_\infty - \varepsilon)u$ for $u \ge \widetilde{H}$. Let $r_2 = \max\{2r_1, ((1-c^2)/(\alpha-|c|))\widetilde{H}\}$. If $y \in \partial\Omega_{r_2}$, then $(A^{-1}y)(n-\tau(n)) \ge ((\alpha-|c|)/(1-c^2))\|y\|_X \ge \widetilde{H}$. So $f((A^{-1}y)(n-\tau(n))) \ge (f_\infty-\varepsilon)(A^{-1}y)(n-\tau(n))$ for any $y \in \partial\Omega_{r_2}$. From Lemma 2.7, for $y \in \partial\Omega_{r_2}$, we have

$$\|Qy\|_X \ge \lambda(f_\infty-\varepsilon)A_1\frac{\alpha-|c|}{1-c^2}\Sigma_{s=0}^{T-1} b(s)\|y\|_X > \|y\|_X. \tag{3.17}$$

It follows from Lemma 2.1 that

$$i(Q,\Omega_{r_1},K) = 1, \qquad i(Q,\Omega_{r_2},K) = 0, \qquad i(Q,\Omega_{r_2}\setminus\overline{\Omega}_{r_1},K) = -1. \tag{3.18}$$

Then Q has at least one fixed point $y(n)$ in $\Omega_{r_2}\setminus\overline{\Omega}_{r_1}$. By Lemma 2.6, $(A^{-1}y)(n)$ is the positive T-periodic solution of (1.2).
Case 2. If $f_0 > f_\infty$, then

$$\frac{1-c^2}{f_0 A_1(\alpha-|c|)\Sigma_{s=0}^{T-1} b(s)} < \lambda < \frac{1-|c|-BL|c|\Sigma_{s=0}^{T-1} a(s)}{f_\infty B \Sigma_{s=0}^{T-1} b(s)}. \tag{3.19}$$

So we can take a constant $0 < \varepsilon < f_0$ satisfying

$$\frac{1-c^2}{(f_0-\varepsilon)A_1(\alpha-|c|)\Sigma_{s=0}^{T-1}b(s)} < \lambda < \frac{1-|c|-BL|c|\Sigma_{s=0}^{T-1}a(s)}{(f_\infty+\varepsilon)B\Sigma_{s=0}^{T-1}b(s)}. \tag{3.20}$$

$0 < f_0 < \infty$ implies that there exists $\overline{r}_1 > 0$ such that for any $0 \le u \le \overline{r}_1$, $f(u) \ge (f_0-\varepsilon)u$.

Let $r_1 = (1-|c|)\overline{r}_1$. If $y \in \partial\Omega_{r_1}$, then $0 \le (A^{-1}y)(n-\tau(n)) \le (1/(1-|c|))\|y\|_X \le \overline{r}_1$. So we have $f((A^{-1}y)(n-\tau(n))) \ge (f_0-\varepsilon)(A^{-1}y)(n-\tau(n))$ for $y \in \partial\Omega_{r_1}$. From Lemma 2.7, for any $y \in \partial\Omega_{r_1}$, we obtain

$$\|Qy\|_X \ge \lambda(f_0-\varepsilon)A_1\Sigma_{s=0}^{T-1}b(s)\frac{\alpha-|c|}{1-c^2}\|y\|_X > \|y\|_X. \tag{3.21}$$

If $0 < f_\infty < \infty$, then there exists $\widetilde{H} > 0$ satisfying for any $u \ge \widetilde{H}$, $f(u) \le (f_\infty+\varepsilon)u$. Take $r_2 = \max\{2r_1, ((1-c^2)/(\alpha-|c|))\widetilde{H}\}$. $y \in \partial\Omega_{r_2}$ tells that $(A^{-1}y)(n-\tau(n)) \ge ((\alpha-|c|)/(1-c^2))\|y\|_x \ge \widetilde{H}$. So $f((A^{-1}y)(n-\tau(n))) \le (f_\infty+\varepsilon)(A^{-1}y)(n-\tau(n))$ for $y \in \partial\Omega_{r_2}$. Thus, by Lemma 2.8, for $y \in \partial\Omega_{r_2}$, we have

$$\|Qy\|_X \le \frac{\lambda B(f_\infty+\varepsilon)\Sigma_{s=0}^{T-1}b(s)+BL|c|\Sigma_{s=0}^{T-1}a(s)}{1-|c|}\|y\|_X < \|y\|_X. \tag{3.22}$$

It follows from Lemma 2.1 that

$$i(Q,\Omega_{r_1},K) = 0, \qquad i(Q,\Omega_{r_2},K) = 1. \tag{3.23}$$

Therefore, $i(Q,\Omega_{r_2}\setminus\overline{\Omega}_{r_1},K) = 1$ and Q has at least one fixed point $y(n)$ in $\Omega_{r_2}\setminus\overline{\Omega}_{r_1}$. Lemma 2.6 shows that $(A^{-1}y)(n)$ is a positive T-periodic solution of (1.2).

Our results are applicable to consider multiplicity of periodic solutions for many neutral difference equations. □

Example 3.4. We consider the following neutral difference equation:

$$\Delta\left[u(n)+\frac{1}{3}u(n-1)\right] = \frac{1}{4}u(n)-\lambda[1-\sin\pi n]u^a(n-\tau(n))e^{-u(n-\tau(n))}, \quad n \in \mathbb{Z}, \tag{3.24}$$

where λ and a are two positive parameters, $\tau(n+2) \equiv \tau(n)$. Take $\tau = 1$, $c = -1/3$, $a(n) \equiv 1/4$, $b(n) = 1-\sin\pi n$, $g(u) \equiv 1$, $f(u) = u^a e^{-u}$, $L = l = 1$. Then assumptions (E_1) and (E_2) hold, $f_\infty = 0$, and $\max_{u\in[0,\infty)} f(u) = f(a)$.

By direct computations, we have $k = \alpha = 2/5$, $f_0 = +\infty$ if $a \in (0,1)$, $f_0 = 1$ when $a = 1$, and $f_0 = 0$ as $a > 1$. Furthermore, let $t_0 = \min\{a, (3/2)\}$, we have

$$\begin{gathered} M(1) = \max\left\{f(t) : 0 \le t \le \frac{3}{2}\right\} = f(t_0), \\ m(1) = \min\left\{f(t) : \frac{3}{40} \le t \le \frac{3}{2}\right\} = \min\left\{f\left(\frac{3}{2}\right), f\left(\frac{3}{40}\right)\right\} = r_0. \end{gathered} \tag{3.25}$$

Thus

$$\lambda_0 = \frac{1}{A_1 m(1)\Sigma_{s=0}^{T-1} b(s)} = \frac{3}{4r_0}, \qquad \lambda_1 = \frac{1-|c|-BL|c|\Sigma_{s=0}^{T-1} a(s)}{BM(1)\Sigma_{s=0}^{T-1} b(s)(1-|c|)} = \frac{7}{40f(t_0)}. \tag{3.26}$$

Applying Theorem 3.1 to (3.24), we obtain the following results.

4. Conclusion

(a) If $a \in (0,1)$, then (3.24) has one positive two-periodic solution for $\lambda > 3/4r_0 > 0$ or $0 < \lambda < 7/40f(a)$.
(b) If $a = 1$, then (3.24) has one positive two-periodic solution for $\lambda > 3/4r_0 > 0$.
(c) If $a > 1$, then (3.24) has two positive two-periodic solutions for $\lambda > 3/4r_0 > 0$.

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