Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2008, Article ID 179589, 21 pages
doi:10.1155/2008/179589

Research Article

Eventually Periodic Solutions for Difference Equations with Periodic Coefficients and Nonlinear Control Functions

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Received 13 August 2008; Accepted 24 October 2008

Recommended by Guang Zhang

For nonlinear difference equations of the form $x_n = F(n, x_{n-1}, \ldots, x_{n-m})$, it is usually difficult to find periodic solutions. In this paper, we consider a class of difference equations of the form $x_n = a_n x_{n-1} + b_n f(x_{n-k})$, where $\{a_n\}$, $\{b_n\}$ are periodic sequences and f is a nonlinear filtering function, and show how periodic solutions can be constructed. Several examples are also included to illustrate our results.

1. Introduction

There are good reasons to find "eventually periodic solutions" of difference equations of the form

$$x_n = F(n, x_{n-1}, x_{n-2}, \ldots, x_{n-m}), \quad n \in \{0, 1, 2, \ldots\}. \tag{1.1}$$

For instance, the well-known logistic population model

$$x_n = \lambda x_{n-1}(L - x_{n-1}), \quad n \in \{0, 1, 2, \ldots\} \tag{1.2}$$

is of the above form, and the study of the existence of its periodic solutions leads to chaotic solutions. As another example in [1], Chen considers the equation

$$x_n = x_{n-1} + g(x_{n-k-1}), \quad n \in \{0, 1, 2, \ldots\}, \tag{1.3}$$

where k is a nonnegative integer, and $g : \mathbf{R} \to \mathbf{R}$ is a McCulloch-Pitts type function

$$g(\xi) = \begin{cases} -1, & \xi \in (\sigma, \infty), \\ 1, & \xi \in (-\infty, \sigma], \end{cases} \tag{1.4}$$

in which $\sigma \in \mathbf{R}$ is a constant which acts as a threshold. Chen showed that all solutions of (1.3) are eventually periodic and pointed out that such a result may lead to more complicated dynamical behavior of a more general neural network. Recently, Zhu and Huang [2] discussed the periodic solutions of the following difference equation:

$$x_n = ax_{n-1} + (1-a)f(x_{n-k}), \quad n \in \{0, 1, 2, \ldots\}, \tag{1.5}$$

where $a \in (0,1)$, k is a positive integer, and $f : \mathbf{R} \to \mathbf{R}$ is a signal transmission function of the form (1.9). In particular, they obtained the following theorem.

Theorem A. *Let $p, q \in \{0, 1, 2, \ldots\}$. If*

$$\kappa \in \left(a^{p+1}, \frac{a^p(1-a^{k-1})}{(1-a^{k+p-1})} \right) \cap \left(1 - a^q + a^{p+q+k}, 1 - \frac{a^{q+1}(1-a^{k+p})}{1-a^{2k+p+q}} \right), \tag{1.6}$$

then (1.5) *has an eventually* $(2k+p+q)$*-periodic solution* $\{x_n\}_{n=-k}^{\infty}$.

In this paper, we consider the following delay difference equation:

$$x_n = a_n x_{n-1} + b_n f(x_{n-k}), \quad n \in \{0, 1, 2, \ldots\}, \tag{1.7}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are positive ω-periodic sequences such that $a_n + b_n \leq 1$ for $n \geq 0$.

The integer k is assumed to satisfy

$$k = l\omega + 1, \tag{1.8}$$

for some nonnegative integer l. The function f can be chosen in a number of ways. Here, f is a filtering function of the form

$$f(x) = \begin{cases} 1, & x \in (0, \kappa], \\ 0, & x \in (-\infty, 0] \cup (\kappa, \infty), \end{cases} \tag{1.9}$$

where the positive number κ can be regarded as a threshold term. Therefore, if $\omega = 1$, then $a_n = a$, $b_n = b$, and $k = l + 1$ so that (1.7) reduces to

$$x_n = ax_{n-1} + bf(x_{n-l-1}), \tag{1.10}$$

which includes (1.5) as a special case.

When $l = 0$, we have

$$x_n = ax_{n-1} + bf(x_{n-1}), \tag{1.11}$$

which will also be included in the following discussions.

Let Ω denote the set of real finite sequences of the form $\{\phi_{-k}, \phi_{-k+1}, \dots, \phi_{-1}\}$. Given $\phi = \{\phi_{-k}, \dots, \phi_{-1}\} \in \Omega$, if we let $x_{-k} = \phi_{-k}, \dots, x_{-1} = \phi_{-1}$, then we may compute $x_0, x_1, \dots$ successively from (1.7) in a unique manner. Such a sequence $x = \{x_n\}_{n=-k}^{\infty}$ is called a solution of (1.7) determined by $\phi \in \Omega$. Recall that a positive integer η is a period of the sequence $\{x_n\}_{n=-k}^{\infty}$ if $x_{\eta+n} = x_n$ for all $n \geq -k$ and that τ is the least period of $\{x_n\}_{n=-k}^{\infty}$ if τ is the least among all periods of $\{x_n\}_{n=-k}^{\infty}$. The sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be τ-periodic if τ is the least period of $\{x_n\}_{n=-k}^{\infty}$. In case $\{x_n\}_{n=-k}^{\infty}$ is not periodic, it may happen that for some $N \geq -k$, the subsequence $\{x_n\}_{n=N}^{\infty}$ is τ-periodic. Such a sequence is said to be eventually τ-periodic. In other words, let us call $\{y_j\}_{j=-k}^{\infty}$ a translate of $\{x_n\}_{n=-k}^{\infty}$ if $y_j = x_{j+N+k}$ for $j \in \{-k, -k+1, \dots\}$, where N is some integer greater than or equal to $-k$. Then, $\{x_n\}_{n=-k}^{\infty}$ is eventually τ-periodic if one of its translates is τ-periodic.

We will seek eventually periodic solutions of (1.7). This is a rather difficult question since the existence question depends on the sequences $\{a_n\}$, $\{b_n\}$, the "delay" k, and the control term κ.

Throughout this paper, empty sums are taken to be 0 and empty products to be 1. We will also need the following elementary facts. If the real sequence $\{x_n\}_{n=-1}^{\infty}$ satisfies the recurrence relation

$$x_n = a_n x_{n-1} + b_n, \quad n \in \{0, 1, 2, \dots\}, \tag{1.12}$$

then

$$\begin{aligned} x_0 &= a_0 x_{-1} + b_0, \\ x_1 &= a_1 x_0 + b_1 \\ &= a_1(a_0 x_{-1} + b_0) + b_1 \\ &= a_1 a_0 x_{-1} + a_1 b_0 + b_1, \\ x_2 &= a_2 x_1 + b_2 \\ &= a_2(a_1 a_0 x_{-1} + a_1 b_0 + b_1) + b_2 \\ &= a_2 a_1 a_0 x_{-1} + a_2 a_1 b_0 + a_2 b_1 + b_2, \end{aligned} \tag{1.13}$$

and by induction,

$$\begin{aligned} x_n &= \alpha_{0,n} x_{-1} + \frac{\alpha_{0,n}}{\alpha_{0,0}} b_0 + \frac{\alpha_{0,n}}{\alpha_{0,1}} b_1 + \cdots + \frac{\alpha_{0,n}}{\alpha_{0,n}} b_n \\ &= \alpha_{0,n}\left(x_{-1} + \frac{b_0}{\alpha_{0,0}} + \frac{b_1}{\alpha_{0,1}} + \cdots + \frac{b_n}{\alpha_{0,n}} \right), \end{aligned} \tag{1.14}$$

where

$$\alpha_{0,j} = \prod_{n=0}^{j} a_n, \quad j \in \{0,1,2,\ldots\}. \tag{1.15}$$

Since $\{a_n\}$ and $\{b_n\}$ are positive ω-periodic sequences, we see further that

$$\alpha_{0,m\omega+i} = (\alpha_{0,\omega-1})^m \alpha_{0,i}, \quad i \in \{0,\ldots,\omega-1\};\ m \in \{0,1,2,\ldots\}, \tag{1.16}$$

that

$$\begin{aligned}\sum_{j=0}^{m\omega+i} \frac{b_j}{\alpha_{0,j}} &= \left(\frac{b_0}{\alpha_{0,0}} + \cdots + \frac{b_{\omega-1}}{\alpha_{0,\omega-1}}\right) + \left(\frac{b_\omega}{\alpha_{0,\omega}} + \cdots + \frac{b_{2\omega-1}}{\alpha_{0,2\omega-1}}\right) \\ &\quad + \cdots + \left(\frac{b_{(m-1)\omega}}{\alpha_{0,(m-1)\omega}} + \cdots + \frac{b_{m\omega-1}}{\alpha_{0,m\omega-1}}\right) + \left(\frac{b_{m\omega}}{\alpha_{0,m\omega}} + \cdots + \frac{b_{m\omega+i}}{\alpha_{0,m\omega+i}}\right) \\ &= \left(\frac{b_0}{\alpha_{0,0}} + \cdots + \frac{b_{\omega-1}}{\alpha_{0,\omega-1}}\right)\left\{1 + \frac{1}{\alpha_{0,\omega-1}} + \cdots + \frac{1}{(\alpha_{0,\omega-1})^{m-1}}\right\} \\ &\quad + \frac{1}{(\alpha_{0,\omega-1})^m}\left\{\frac{b_0}{\alpha_{0,0}} + \cdots + \frac{b_i}{\alpha_{0,i}}\right\}\end{aligned} \tag{1.17}$$

for $i \in \{0,\ldots,\omega-1\}$ and $m \in \{0,1,2,\ldots\}$, and that

$$\begin{aligned}x_{m\omega+i} &= \alpha_{0,m\omega+i}\left(x_{-1} + \sum_{j=0}^{m\omega+i} \frac{b_j}{\alpha_{0,j}}\right) \\ &= (\alpha_{0,\omega-1})^m \alpha_{0,i} x_{-1} + \alpha_{0,\omega-1} \frac{1-\alpha_{0,\omega-1}^m}{1-\alpha_{0,\omega-1}} \alpha_{0,i}\beta_{0,\omega-1} + \alpha_{0,i}\beta_{0,i}\end{aligned} \tag{1.18}$$

for $i \in \{0,\ldots,\omega-1\}$ and $m \in \{0,1,2,\ldots\}$, where

$$\beta_{0,j} = \sum_{k=0}^{j} \frac{b_k}{\alpha_{0,k}}, \quad j \in \{0,1,\ldots,\omega-1\}. \tag{1.19}$$

2. Convergence of solutions

The filtering function f will return 0 for inputs that fall below 0 or above the threshold constant κ. For this reason, we will single out some subsets of Ω as follows:

$$\begin{aligned}\Omega_- &= \{\{\phi_{-k},\ldots,\phi_{-1}\} \in \Omega \mid \phi_i \le 0,\ -k \le i \le -1\}, \\ \Omega_* &= \{\{\phi_{-k},\ldots,\phi_{-1}\} \in \Omega \mid 0 < \phi_i \le \kappa,\ -k \le i \le -1\}, \\ \Omega_+ &= \{\{\phi_{-k},\ldots,\phi_{-1}\} \in \Omega \mid \phi_i > \kappa,\ -k \le i \le -1\}.\end{aligned} \tag{2.1}$$

Let $x = \{x_n\}_{n=-k}^{\infty}$ be the solution of (1.7) determined by $\phi \in \Omega_-$. By (1.7),

$$\begin{aligned} x_0 &= a_0 x_{-1} + b_0 f(x_{-k}) = a_0 x_{-1} \le 0, \\ x_1 &= a_1 x_0 + b_1 f(x_{-k+1}) = a_1 x_0 = a_1 a_0 x_{-1} \le 0. \end{aligned} \tag{2.2}$$

By induction, we may see that

$$x_n = a_n a_{n-1} \cdots a_1 a_0 x_{-1} \le 0, \quad n \in \{0, 1, 2, \ldots\}. \tag{2.3}$$

Since

$$0 \le \lim_{n\to\infty} a_0 a_1 \cdots a_{n-1} a_n \le \lim_{n\to\infty} \left(\max\{a_0, a_1, \ldots, a_{\omega-1}\}\right)^{n+1} = 0, \tag{2.4}$$

we see that $\lim_{n\to\infty} x_n = 0$.

Next, let $x = \{x_n\}_{n=-k}^{\infty}$ be the solution of (1.7) determined by $\phi \in \Omega_*$. If $\kappa \ge 1$, then by (1.7),

$$\begin{aligned} 0 < x_0 = a_0 x_{-1} + b_0 \le a_0 \kappa + b_0 = a_0 \kappa - a_0 + a_0 + b_0 \le a_0(\kappa - 1) + 1 \le \kappa, \\ 0 < x_1 = a_1 x_0 + b_1 \le a_1 (a_0 \kappa + b_0) + b_1 = a_1 \kappa + b_1 \le \kappa. \end{aligned} \tag{2.5}$$

By induction, we see that

$$0 < x_n = a_n x_{n-1} + b_n \le a_n \kappa + b_n \le \kappa, \quad n \in \{0, 1, 2, \ldots\}. \tag{2.6}$$

By (1.7), we see that

$$x_n = a_n x_{n-1} + b_n f(x_{n-k}) = a_n x_{n-1} + b_n, \quad n \in \{0, 1, 2, \ldots\}. \tag{2.7}$$

In view of (1.18), we see further that

$$\lim_{m\to\infty} x_{m\omega+i} = A_i, \quad i \in \{0, 1, \ldots, \omega - 1\}, \tag{2.8}$$

where

$$A_i = \alpha_{0,i} \left(\frac{\alpha_{0,\omega-1}\beta_{0,\omega-1}}{1 - \alpha_{0,\omega-1}} + \beta_{0,i} \right), \quad i \in \{0, 1, \ldots, \omega - 1\}. \tag{2.9}$$

Next, let $x = \{x_n\}_{n=-k}^{\infty}$ be the solution of (1.7) determined by $\phi \in \Omega_+$. Then, by (1.7),

$$x_0 = a_0 x_{-1} + b_0 f(x_{-k}) = a_0 x_{-1}, \tag{2.10}$$

and by induction,

$$x_n = a_n a_{n-1} \cdots a_0 x_{-1}, \quad n \in \{0, 1, 2, \ldots\}. \tag{2.11}$$

Although $x_{-1} > \kappa$, since (2.4) holds, we see that $\{x_n\}$ is a strictly decreasing sequence tending to 0. Hence, there is a nonnegative integer j such that $x_{j-1} > \kappa$ but $x_j \leq \kappa$. Then, $\kappa \geq x_j > x_{j+1} > x_{j+2} > \cdots > x_{j+k-1}$. If we let $\phi = \{x_j, x_{j+1}, \ldots, x_{j+k-1}\}$, then $\phi \in \Omega_*$. If $\kappa \geq 1$, then by what we have shown above, the solution $\{\tilde{x}_n\}$ of (1.7) determined by ϕ satisfies $\lim_{m\to\infty} \tilde{x}_{m\omega+i} = A_i$ for $i \in \{0, 1, \ldots, \omega-1\}$. By uniqueness, $\tilde{x}_n = x_{n+j+k}$ for $n \geq 0$. In other words, the translate $\{\tilde{x}_n\}$ of the solution $\{x_n\}_{n=-k}^{\infty}$ satisfies $\lim_{m\to\infty} \tilde{x}_{m\omega+i} = A_i$ for $i \in \{0, 1, \ldots, \omega-1\}$.

We summarize the above discussions by means of the following result.

Lemma 2.1. *A solution $x = \{x_n\}_{n=-k}^{\infty}$ determined by $\phi \in \Omega_-$ will tend to 0; and if $\kappa \geq 1$, then a solution $x = \{x_n\}_{n=-k}^{\infty}$ determined by $\phi \in \Omega_* \cup \Omega_+$ will satisfy* (2.8) *or one of its translates will satisfy it.*

Lemma 2.2. *If $0 < \kappa < \min\{1, \max\{A_0, A_1, \ldots, A_{\omega-1}\}\}$, then for any solution $\{x_n\}$ of* (1.7) *determined by a $\phi \in \Omega_* \cup \Omega_+$, there exists an integer $m \in \{0, 1, \ldots\}$ such that $\{x_{m-k}, \ldots, x_{m-1}\} \in \Omega_*$ and $x_m \in (\kappa, 1)$.*

Proof. First let $\{x_n\}_{n=-k}^{\infty}$ be the solution of (1.7) determined by a $\phi \in \Omega_*$. If $x_n \in (0, \kappa]$ for all $n \in \{-k, -k+1, \ldots\}$, then

$$x_n = a_n x_{n-1} + b_n f(x_{n-k}) = a_n x_n + b_n, \quad n \in \{0, 1, 2, \ldots\}, \tag{2.12}$$

so that by (1.18), we see that

$$\lim_{m\to\infty} x_{m\omega+i} = A_i, \quad i \in \{0, 1, \ldots, \omega-1\}. \tag{2.13}$$

But, this is contrary to our assumption that $0 < \kappa < \min\{1, \max\{A_0, A_1, \ldots, A_{\omega-1}\}\}$. Hence, there is some nonnegative integer m such that $x_n \in (0, \kappa]$ for $n \in \{-k, -k+1, \ldots, m-1\}$ but $x_m \in (-\infty, 0] \cup (\kappa, \infty)$. Note that

$$x_m = a_m x_{m-1} + b_m f(x_{m-k}) > 0, \tag{2.14}$$

which implies that $x_m \in (\kappa, \infty)$. Moreover, since $x_{m-1} \in (0, \kappa] \subset (0, 1)$, we then have

$$x_m = a_m x_{m-1} + b_m < a_m + b_m \leq 1, \tag{2.15}$$

so that $x_m \in (\kappa, 1)$.

Next, let $\{x_n\}_{n=-k}^{\infty}$ be the solution of (1.7) determined by a $\phi \in \Omega_+$. As seen in the discussions immediately preceding Lemma 2.1, there is a nonnegative integer j such that $\{x_j, x_{j+1}, \ldots, x_{j+k-1}\} \in \Omega_*$. If $x_n \in (0, \kappa]$ for all $n \in \{j, j+1, \ldots\}$, then as we have just explained, a translate $\{\tilde{x}_n\}$ of $\{x_n\}$ will satisfy

$$\lim_{m\to\infty} \tilde{x}_{m\omega+i} = A_i, \quad i \in \{0, 1, \ldots, \omega-1\}. \tag{2.16}$$

This is again a contradiction. Hence, we may conclude our proof in a manner similar to the above discussions. The proof is complete. □

From the proof of Lemma 2.2, we see that if $\kappa \in (0, \min\{1, \max\{A_0, A_1, \dots, A_{\omega-1}\}\})$, then to study the limiting behavior of a solution $\{x_n\}_{n=-k}^{\infty}$ determined by ϕ in $\Omega_* \cup \Omega_+$, we may assume without loss of generality that $\phi \in \Omega_*$ and $x_0 \in (\kappa, 1)$. As an example, let us consider (1.11), where we recall that $a, b > 0$ and $a + b \le 1$.

Example 2.3. Let $ab/(1-a^2) \le \kappa < b/(1-a^2)$. Then, (1.11) has a 2-periodic solution $\{x_n\}_{k=-1}^{\infty}$ with $x_{-1} \in (0, \kappa]$ and $x_0 \in (\kappa, 1)$. Indeed, let us choose $x_{-1} = ab/(1-a^2)$ (and hence, $x_0 = b/(1-a^2)$). Then,

$$\begin{aligned} 0 < x_{-1} &= \frac{ab}{1-a^2} \le \kappa, \\ \kappa < x_0 = ax_{-1} + b &= \frac{b}{1-a^2} < 1. \end{aligned} \tag{2.17}$$

Furthermore,

$$\begin{aligned} x_1 &= ax_0 = \frac{ab}{1-a^2} \in (0, \kappa], \\ x_2 &= ax_1 + b = a \cdot \frac{ab}{1-a^2} + b = \frac{b}{1-a^2} = x_0, \end{aligned} \tag{2.18}$$

so that $x_1 = x_3 = x_5 = \cdots$ and $x_2 = x_4 = x_6 = \cdots$ and $x_1 \ne x_2$.

3. Existence of eventually periodic solutions

Recall that $G^{[0]}(u) = u$, $G^{[1]}(u) = G(u)$, $G^{[2]}(u) = (G \circ G)(u) = G(G(u)), \dots, G^{[j]}(u) = G(G^{[j-1]}(u))$ are the zeroth, first, second, and so forth and the jth iterate of the function $G(u)$. Also, recall the fact that if $\{u_n\}_{n=0}^{\infty}$ is a sequence that satisfies

$$u_{n+1} = G(u_n), \quad n \in \{0, 1, 2, \dots\}, \tag{3.1}$$

then $\{u_n\}$ is a τ-periodic sequence if and only if

$$\begin{aligned} &u_0 = G^{[\tau]}(u_0), \\ &u_0 \ne G^{[j]}(u_0), \quad j = 1, 2, \dots, \tau - 1. \end{aligned} \tag{3.2}$$

For convenience, denote

$$\alpha_n = \prod_{j=1}^{n} a_j, \quad \beta_n = \sum_{j=1}^{n} \frac{b_j}{\alpha_j}, \quad n \in \{1, 2, \dots\}. \tag{3.3}$$

Since

$$\alpha_n \beta_n + \alpha_n = a_1 \cdots a_n + a_2 \cdots a_n b_1 + a_3 \cdots a_n b_2 + \cdots + b_n \le 1, \tag{3.4}$$

we see that

$$\frac{\alpha_n\beta_n}{1-\alpha_n} \le 1, \quad n \in \{1,2,\ldots\}. \tag{3.5}$$

Theorem 3.1. *Let $k = l\omega + 1$, $p = \tau\omega - 1$, and $q = \sigma\omega - 1$, where $l,\tau,\sigma \in \{1,2,\ldots,k-1\}$. Let*

$$\begin{aligned} I_1(p) &= \left[\alpha_\omega^\tau\left(\alpha_\omega^l + \frac{(1-\alpha_\omega^l)}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right), \frac{\alpha_p(1-\alpha_\omega^l)\alpha_\omega\beta_\omega}{(1-\alpha_p\alpha_\omega^l)(1-\alpha_\omega)}\right), \\ I_2(p,q) &= \left[M, \frac{\alpha_\omega^{\tau+\sigma+l}(1-\alpha_\omega^l) + (1-\alpha_\omega^\sigma)}{(1-\alpha_\omega^{\tau+\sigma+2l})(1-\alpha_\omega)}\alpha_\omega\beta_\omega\right), \end{aligned} \tag{3.6}$$

where

$$M = \max\left\{\alpha_n\alpha_\omega^{\tau+l}\left(\alpha_\omega^l + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_n\beta_n : n \in \{0,1,\ldots,q\}\right\}. \tag{3.7}$$

If $\kappa \in I_1(p) \cap I_2(p,q)$ and

$$0 < \kappa < \min\left\{\frac{\alpha_n\beta_n}{1-\alpha_n} : n \in \{1,2,\ldots,k-1\}\right\}, \tag{3.8}$$

then (1.7) *has an eventually* $(2k+p+q)$*-periodic solution* $\{x_n\}_{n=-k}^{\infty}$ *(which can be explicitly generated).*

Proof. From the condition that $l,\tau,\sigma \in \{1,2,\ldots,k-1\}$, we have $k-1 \ge \omega$. By (3.5), we see that

$$\begin{aligned} 1 &\ge A_0 \\ &= \alpha_{0,0}\left(\frac{\alpha_{0,\omega-1}\beta_{0,\omega-1}}{1-\alpha_{0,\omega-1}} + \beta_{0,0}\right) \\ &= \frac{1}{1-\alpha_\omega}(a_0\alpha_\omega\beta_{0,\omega-1} + b_0 - b_0\alpha_\omega) \\ &= \frac{1}{1-\alpha_\omega}\left(a_0\alpha_\omega\left(\frac{b_0}{a_0} + \frac{b_1}{a_0a_1} + \cdots + \frac{b_{\omega-1}}{a_0\cdots a_{\omega-1}}\right) + b_0 - b_0\alpha_\omega\right) \\ &= \frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega} > \kappa. \end{aligned} \tag{3.9}$$

Hence, $\kappa < \max\{A_0, A_1, \ldots, A_{\omega-1}\}$. Thus, $0 < \kappa < \min\{1, \max\{A_0, A_1, \ldots, A_{\omega-1}\}\}$. By Lemmas 2.1 and 2.2, we may look for our desired eventually periodic solution $\{x_n\}_{n=-k}^{\infty}$ determined by $\phi \in \Omega_*$ such that $x_0 \in (\kappa, 1)$.

Define

$$\begin{aligned} g_n(u) &= \alpha_n u + \alpha_n\beta_n \quad \text{for } n \in \{0,1,2,\ldots\}, \\ h_n(u) &= a_n u \quad \text{for } n \in \{0,1,2,\ldots\}, \end{aligned} \tag{3.10}$$

and the mapping g by

$$g(x) = \left(g_{q+1} \circ (h_\omega \circ \cdots \circ h_1)^{[\tau+l]} \circ g_{k-1}\right)(x). \tag{3.11}$$

We will show that

$$g(x) = \alpha_\omega^{\tau+\sigma+l}\left(\alpha_\omega^l x + \frac{\alpha_\omega \beta_\omega (1-\alpha_\omega^l)}{1-\alpha_\omega}\right) + \frac{\alpha_\omega \beta_\omega (1-\alpha_\omega^\sigma)}{1-\alpha_\omega}, \tag{3.12}$$

and that g maps $D_0 = (\kappa, 1)$ into D_0 with a fixed point $x^* \in D_0$, where

$$x^* = \frac{\beta_\omega \alpha_\omega^{\tau+\sigma+l+1}(1-\alpha_\omega^l) + \beta_\omega \alpha_\omega (1-\alpha_\omega^\sigma)}{(1-\alpha_\omega^{\tau+\sigma+2l})(1-\alpha_\omega)}. \tag{3.13}$$

The first assertion is easy to show. Indeed, since

$$\begin{aligned} g_{k-1}(x) &= \alpha_{k-1}x + \alpha_{k-1}\beta_{k-1}, \\ (h_\omega \circ \cdots \circ h_1)^{[\tau+l]}(x) &= (a_\omega \cdots a_1)^{\tau+l} x = \alpha_\omega^{\tau+l} x, \\ g_{q+1}(x) &= \alpha_{q+1}x + \alpha_{q+1}\beta_{q+1}, \end{aligned} \tag{3.14}$$

we see that

$$\begin{aligned} \left((h_\omega \circ \cdots \circ h_1)^{[\tau+l]} \circ g_{k-1}\right)(x) &= \alpha_\omega^{\tau+l}(\alpha_{k-1}x + \alpha_{k-1}\beta_{k-1}), \\ g(x) &= \alpha_{q+1}\alpha_\omega^{\tau+l}(\alpha_{k-1}x + \alpha_{k-1}\beta_{k-1}) + \alpha_{q+1}\beta_{q+1} \\ &= \alpha_\omega^{\tau+\sigma+l}\left(\alpha_\omega^l x + \frac{\alpha_\omega \beta_\omega (1-\alpha_\omega^l)}{1-\alpha_\omega}\right) + \frac{\alpha_\omega \beta_\omega (1-\alpha_\omega^\sigma)}{1-\alpha_\omega}. \end{aligned} \tag{3.15}$$

We now show the second assertion. Note that the linear maps g_n and h_n satisfy

$$\begin{aligned} g_{m\omega}(u) = \alpha_{m\omega}u + \alpha_{m\omega}\beta_{m\omega} = \alpha_\omega^m u + \frac{(1-\alpha_\omega^m)}{1-\alpha_\omega}\alpha_\omega\beta_\omega, \quad m \in \{0,1,2,\dots\}, \\ h_{m\omega} \circ h_{m\omega-1} \circ \cdots \circ h_1(u) = \alpha_\omega^m u, \quad m \in \{0,1,2,\dots\}. \end{aligned} \tag{3.16}$$

Let $g_n(D_0) = D_n$ for $n \in \{1,\dots,k-1\}$. Since $\phi \in \Omega_*$ and $x_0 \in D_0$, it is clear that the solution $\{x_n\}$ of (1.7) satisfies

$$x_n = g_n(x_0), \quad n \in \{1,\dots,k-1\}. \tag{3.17}$$

Moreover, it is easy to prove that

$$D_n = (g_n(\kappa), g_n(1)), \quad n \in \{1,\dots,k-1\}. \tag{3.18}$$

Indeed, we have

$$\kappa < \alpha_n\kappa + \alpha_n\beta_n = g_n(\kappa) < \alpha_n + \alpha_n\beta_n = g_n(1) < \alpha_n + \beta_n \le 1, \quad n \in \{1, 2, \ldots, k-1\}. \tag{3.19}$$

That is, $D_n \subset D_0$ holds for all $n \in \{0, \ldots, k-1\}$. Let n_1 be the largest integer such that $x_n > \kappa$ for $n \in \{0, 1, \ldots, n_1 + k - 1\}$. Then, from (1.7), we can obtain

$$x_{n+k-1} = a_{n+k-1} \cdots a_k \left(\alpha_\omega^l x_0 + \frac{\alpha_\omega \beta_\omega (1 - \alpha_\omega^l)}{1 - \alpha_\omega} \right), \quad n \in \{1, 2, \ldots, n_1 + k\}, \tag{3.20}$$

which implies that $x_{n+k-1} \in D_{n+k-1}$ for $n \in \{1, 2, \ldots, n_1 + k\}$, where

$$\begin{aligned} D_{n+k-1} &= a_n \cdots a_1 g_{k-1}(D_0) \\ &= \left(\alpha_n \left(\alpha_\omega^l \kappa + \frac{\alpha_\omega \beta_\omega (1 - \alpha_\omega^l)}{1 - \alpha_\omega} \right), \alpha_n \left(\alpha_\omega^l + \frac{\alpha_\omega \beta_\omega (1 - \alpha_\omega^l)}{1 - \alpha_\omega} \right) \right). \end{aligned} \tag{3.21}$$

Since $\kappa \in I_1(p)$, we have

$$\kappa < \frac{\alpha_p (1 - \alpha_\omega^l) \alpha_\omega \beta_\omega}{(1 - \alpha_p \alpha_\omega^l)(1 - \alpha_\omega)}, \tag{3.22}$$

that is,

$$\begin{aligned} \kappa &< \alpha_p \left(\alpha_\omega^l \kappa + \frac{(1 - \alpha_\omega^l) \alpha_\omega \beta_\omega}{(1 - \alpha_\omega)} \right) \\ &< \alpha_{p-1} \left(\alpha_\omega^l \kappa + \frac{(1 - \alpha_\omega^l) \alpha_\omega \beta_\omega}{(1 - \alpha_\omega)} \right) \\ &< \cdots < \alpha_\omega^l \kappa + \frac{(1 - \alpha_\omega^l)}{(1 - \alpha_\omega)} \alpha_\omega \beta_\omega, \end{aligned}$$
$$\alpha_p \left(\alpha_\omega^l + \frac{1 - \alpha_\omega^l}{1 - \alpha_\omega} \alpha_\omega \beta_\omega \right) < \alpha_{p-1} \left(\alpha_\omega^l + \frac{1 - \alpha_\omega^l}{1 - \alpha_\omega} \alpha_\omega \beta_\omega \right) < \cdots < \alpha_\omega^l + \frac{(1 - \alpha_\omega^l)}{(1 - \alpha_\omega)} \alpha_\omega \beta_\omega \le 1, \tag{3.23}$$

which shows that $D_{n+k-1} \subset D_0$ for $n \in \{0, 1, \ldots, p\}$. Thus, $n_1 \ge p$ and

$$x_{n+k-1} \in D_{n+k-1} \subset (0, \kappa] \quad \text{for } n \in \{p + 1, \ldots, p + k\}. \tag{3.24}$$

In fact, from $\kappa \in I_1(p)$, we have

$$\begin{aligned} x_{p+k} &= a_{p+k}x_{p+k-1} + b_{p+k}f\left(x_p\right) \\ &= a_{p+k}x_{p+k-1} \\ &= a_{p+k}a_{p+k-1}x_{p+k-2} \\ &= \cdots = a_{p+k}\cdots a_k\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) \\ &= \alpha_{p+1}\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) \\ &= \alpha_\omega^\tau\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) \\ &\leq \kappa, \\ x_{p+k+1} &= a_{p+k+1}x_{p+k} + b_{p+k+1}f\left(x_{p+1}\right) = a_{p+k+1}x_{p+k} < \kappa, \end{aligned} \tag{3.25}$$

and, by induction,

$$\begin{aligned} x_{p+2k-1} &= a_{p+2k-1}x_{p+2k-2} + b_{p+2k-1}f\left(x_{p+k-1}\right) \\ &= a_{p+2k+1}x_{p+2k-2} < \kappa. \end{aligned} \tag{3.26}$$

Then, it is easy to see that $n_1 = p$.

Taking $n = p + k$ in (3.20), we have

$$\begin{aligned} x_{2k+p-1} &= a_{2k+p-1}\cdots a_k g_{k-1}\left(x_0\right) \\ &= a_{k+p}\cdots a_1 g_{k-1}\left(x_0\right) \\ &= \alpha_\omega^{\tau+l}\left(\alpha_\omega^l x_0 + \frac{\alpha_\omega\beta_\omega\left(1-\alpha_\omega^l\right)}{1-\alpha_\omega}\right). \end{aligned} \tag{3.27}$$

Let n_2 be the largest integer such that $x_{n+2k+p-1} \in (0,\kappa]$ for $n \in \{0,1,\ldots,n_2\}$. Then, it follows from (1.7) that

$$\begin{aligned} x_{n+2k+p-1} &= \prod_{j=2k+p}^{n+2k+p-1} a_j x_{2k+p-1} + \prod_{j=2k+p}^{n+2k+p-1} a_j \sum_{j=2k+p}^{n+2k+p-1} \frac{b_j}{a_{2k+p}\cdots a_j} \\ &= \alpha_n x_{2k+p-1} + \alpha_n\beta_n \\ &= \alpha_n\alpha_\omega^{\tau+l}\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_n\beta_n \\ &= g_n\left(x_{2k+p-1}\right) \end{aligned} \tag{3.28}$$

for $n \in \{1,2,\ldots,n_2+k\}$. This implies that $x_{n+2k+p-1} \in D_{n+2k+p-1}$ for $n \in \{1,2,\ldots,n_2+k\}$, where $D_{n+2k+p-1} = (g_n(h_\omega \circ \cdots \circ h_1)^{[\tau+l]}g_{k-1})(D_0)$.

Substituting (3.21) with $n_1 = p$ into (3.28), we have

$$D_{n+2k+p-1} = \left(g_n \circ (h_\omega \circ \cdots \circ h_1)^{[\tau+l]} g_{k-1}(\kappa),\ g_n(h_\omega \circ \cdots \circ h_1)^{[\tau+l]} g_{k-1}(1)\right)$$
$$= \left(\alpha_n \alpha_\omega^{\tau+l}\left(\alpha_\omega^l \kappa + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_n\beta_n, \alpha_n \alpha_\omega^{\tau+l}\left(\alpha_\omega^l + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_n\beta_n\right) \tag{3.29}$$

for $n \in \{1, 2, \ldots, n_2 + k\}$. Since $\kappa \in I_2(p, q)$, we have

$$\alpha_n \alpha_\omega^{\tau+l}\left(\alpha_\omega^l + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_n\beta_n \le \kappa \quad \text{for } n \in \{0, 1, \ldots, q\}. \tag{3.30}$$

From (3.29), we further have

$$x_{n+2k+p-1} \in D_{n+2k+p-1} \subset (0, \kappa] \quad \text{for } n \in \{0, 1, \ldots, q\}. \tag{3.31}$$

By (3.8), (3.24), (3.28), and (3.31) as well as $\kappa \in I_2(p, q)$, we have

$$\begin{aligned}
x_{2k+p+q} &= a_{2k+p+q} x_{2k+p+q-1} + b_{2k+p+q} \\
&= a_{2k+p+q}\left(\alpha_q \alpha_\omega^{\tau+l}\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_q\beta_q\right) + b_{2k+p+q} \\
&= a_{q+1}\left(\alpha_q \alpha_\omega^{\tau+l}\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + a_{q+1}\alpha_q\beta_q\right) + b_{q+1} \\
&= \alpha_\omega^{\tau+\sigma+l}\left(\alpha_\omega^l x_0 + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \frac{1-\alpha_\omega^\sigma}{1-\alpha_\omega}\alpha_\omega\beta_\omega \\
&> \alpha_\omega^{\tau+\sigma+l}\left(\alpha_\omega^l \kappa + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \frac{1-\alpha_\omega^\sigma}{1-\alpha_\omega}\alpha_\omega\beta_\omega > \kappa, \\
x_{2k+p+q+1} &= a_{2k+p+q+1} x_{2k+p+q} + b_{2k+p+q+1} \\
&> a_1\kappa + b_1 > \kappa, \\
x_{2k+p+q+2} &= a_{2k+p+q+2} x_{2k+p+q+1} + b_{2k+p+q+2} \\
&> a_2(a_1\kappa + b_1) + b_2 > \kappa, \\
&\ \ \vdots \\
x_{2k+p+q+k-1} &= a_{2k+p+q+k-1} x_{2k+p+q+k-2} + b_{2k+p+q+k-1} \\
&= \prod_{j=2k+p+q+1}^{2k+p+q+k-1} a_j x_{2k+p+q} + \prod_{j=2k+p+q+1}^{2k+p+q+k-1} a_j \sum_{j=2k+p+q+1}^{2k+p+q+k-1} \frac{b_j}{a_{2k+p+q+1}\cdots a_j} \\
&= \alpha_{k-1} x_{2k+p+q} + \alpha_{k-1}\beta_{k-1} \\
&> \alpha_{k-1}\kappa + \alpha_{k-1}\beta_{k-1} \\
&> \kappa.
\end{aligned} \tag{3.32}$$

Hence,

$$x_{n+2k+p-1} \in D_{n+2k+p-1} \subset D_0 \quad \text{for } n \in \{q+1,\dots,q+k\}, \tag{3.33}$$

which implies that $n_2 = q$. In particular, taking $n = q+1$ in (3.33) and (3.28), we have, respectively,

$$\begin{gathered} x_{2k+p+q} \in D_{2k+p+q} \subset D_0, \\ x_{2k+p+q} = g(x_0) = \alpha_\omega^{\tau+\sigma+l}\left(\alpha_\omega^l x_0 + \frac{\alpha_\omega \beta_\omega (1-\alpha_\omega^l)}{1-\alpha_\omega}\right) + \frac{\alpha_\omega \beta_\omega (1-\alpha_\omega^\sigma)}{1-\alpha_\omega}. \end{gathered} \tag{3.34}$$

Since g is a linear map sending D_0 into D_0, then it is easy to see that it has a unique fixed point x^* in D_0 which satisfies (3.13).

Next, we assert that there is a $\phi^* \in \Omega_*$ such that the solution $\{x_n\}$ determined by ϕ^* satisfies $x_0 = x^*$, and that $\{x_n\}$ is a periodic solution of (1.7) with minimal period $2k+p+q$. To see this, we choose $\phi_{-1} = (x^* - b_0)/a_0$ and arbitrary $\phi_{-2},\dots,\phi_{-k} \in (0,\kappa]$. Then, clearly, the solution $\{x_n\}$ of (1.7) determined by $\phi_{-k},\dots,\phi_{-1}$ will satisfy $x_0 = x^*$. Furthermore, we may show that $x_{-1} = \phi_{-1} \in (0,\kappa]$. Indeed, from

$$\alpha_\omega^{\tau+\sigma+l} + \alpha_\omega > \alpha_\omega^\sigma + \alpha_\omega^{\tau+\sigma+2l+1}, \tag{3.35}$$

we have

$$\begin{gathered} \alpha_\omega^{\tau+\sigma+l}(1-\alpha_\omega^l) + (1-\alpha_\omega^\sigma) > 1 - \alpha_\omega - \alpha_\omega^{\tau+\sigma+2l} + \alpha_\omega^{\tau+\sigma+2l+1}, \\ \frac{\alpha_\omega^{\tau+\sigma+l}(1-\alpha_\omega^l) + (1-\alpha_\omega^\sigma)}{1-\alpha_\omega^{\tau+\sigma+2l}} > (1-\alpha_\omega) = \frac{\alpha_\omega\beta_\omega(1-\alpha_\omega)}{\alpha_\omega\beta_\omega} > \frac{b_0(1-\alpha_\omega)}{\alpha_\omega\beta_\omega}, \end{gathered} \tag{3.36}$$

hence,

$$x^* = \frac{\beta_\omega \alpha_\omega^{\tau+\sigma+l+1}(1-\alpha_\omega^l) + \beta_\omega\alpha_\omega(1-\alpha_\omega^\sigma)}{(1-\alpha_\omega^{\tau+\sigma+2l})(1-\alpha_\omega)} > b_0. \tag{3.37}$$

Thus, $\phi_{-1} = (x^* - b_0)/a_0 > 0$. Next, from

$$0 \le \alpha_\omega^{\tau+\sigma+2l} - \alpha_\omega^{\tau+\sigma+3l}, \tag{3.38}$$

we get

$$\alpha_\omega^{\tau+l} - 1 \le \alpha_\omega^{\tau+2l} + \alpha_\omega^{\tau+l}\left(1-\alpha_\omega^l\right) - 1 - \alpha_\omega^{2\tau+\sigma+4l} - \alpha_\omega^{2\tau+\sigma+3l}\left(1-\alpha_\omega^l\right) + \alpha_\omega^{\tau+\sigma+2l}, \tag{3.39}$$

so that

$$\begin{aligned}
\frac{\alpha_\omega^{\tau+l}-1}{1-\alpha_\omega^{\tau+\sigma+2l}} &\le \alpha_\omega^{\tau+2l} + \alpha_\omega^{\tau+l}\left(1-\alpha_\omega^l\right) - 1,\\
1 + \frac{\alpha_\omega^{\tau+\sigma+l}-\alpha_\omega^{\sigma}}{1-\alpha_\omega^{\tau+\sigma+2l}} &\le \alpha_\omega^{\sigma+\tau+2l} + \alpha_\omega^{\sigma+\tau+l}\left(1-\alpha_\omega^l\right) + 1 - \alpha_\omega^{\sigma},\\
\frac{\alpha_\omega^{\tau+\sigma+l+1}\left(1-\alpha_\omega^l\right) + \alpha_\omega\left(1-\alpha_\omega^{\sigma}\right)}{1-\alpha_\omega^{\tau+\sigma+2l}} &\le \alpha_\omega^{\sigma+\tau+2l+1} + \alpha_\omega^{\sigma+\tau+l+1}\left(1-\alpha_\omega^l\right) + \alpha_\omega\left(1-\alpha_\omega^{\sigma}\right),\\
\frac{\alpha_\omega^{\tau+\sigma+l+1}\left(1-\alpha_\omega^l\right) + \alpha_\omega\left(1-\alpha_\omega^{\sigma}\right)}{\left(1-\alpha_\omega^{\tau+\sigma+2l}\right)\left(1-\alpha_\omega\right)} &\le \frac{\alpha_\omega^{\sigma+\tau+2l+1}}{1-\alpha_\omega} + \frac{\alpha_\omega^{\sigma+\tau+l+1}\left(1-\alpha_\omega^l\right)}{1-\alpha_\omega} + \frac{\alpha_\omega\left(1-\alpha_\omega^{\sigma}\right)}{1-\alpha_\omega}.
\end{aligned} \tag{3.40}$$

On the other hand, by (3.5), we have

$$\frac{\alpha_\omega}{1-\alpha_\omega} \le \frac{1}{\beta_\omega}, \tag{3.41}$$

so that

$$\begin{aligned}
\frac{\alpha_\omega^{\tau+\sigma+l+1}\beta_\omega\left(1-\alpha_\omega^l\right) + \alpha_\omega\beta_\omega\left(1-\alpha_\omega^{\sigma}\right)}{\left(1-\alpha_\omega^{\tau+\sigma+2l}\right)\left(1-\alpha_\omega\right)} &\le \alpha_\omega^{\sigma+\tau+2l} + \frac{\alpha_\omega^{\sigma+\tau+l+1}\beta_\omega\left(1-\alpha_\omega^l\right)}{1-\alpha_\omega} + \frac{\alpha_\omega\beta_\omega\left(1-\alpha_\omega^{\sigma}\right)}{1-\alpha_\omega}\\
&= \alpha_\omega^{\sigma+\tau+l}\left(\alpha_\omega^l + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \frac{\alpha_\omega\beta_\omega\left(1-\alpha_\omega^{\sigma}\right)}{1-\alpha_\omega}\\
&= \alpha_\omega^{\sigma+\tau+l}\left(\alpha_\omega^l + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + \alpha_{\sigma\omega}\beta_{\sigma\omega}\\
&= a_0\alpha_q\alpha_\omega^{\tau+l}\left(\alpha_\omega^l + \frac{1-\alpha_\omega^l}{1-\alpha_\omega}\alpha_\omega\beta_\omega\right) + a_0\alpha_q\beta_q + b_0.
\end{aligned} \tag{3.42}$$

In view of our assumption that $\kappa \in I_2(p,q)$, we may now see that $x_{-1} \le \kappa$.

In view of the above discussions, we see that $0 < x_n \le \kappa$ for $n \in \{-k,\dots,-1\}$, $x_n > \kappa$ for $n \in \{0,\dots,p+k-1\}$, and $0 < x_n \le \kappa$ for $n \in \{p+k,\dots,p+2k+q-1\}$. Since x^* is the unique fixed point of $g(x)$ in D_0, we have $g(x^*) = x^*$, $g^{[2]}(x^*) = x^*, \dots, g^{[n]}(x^*) = x^*$, and so forth, and hence,

$$\begin{aligned}
x_{2k+p+q} &= g(x^*) = x^*, \\
x_{2k+p+q+1} &= a_{2k+p+q+1}x_{2k+p+q} + b_{2k+p+q+1}f(x_{k+p+q+1}) \\
&= a_1x^* + b_1 = x_1 > \kappa, \\
&\ \ \vdots \\
x_{p+2k+q+k-1} &= a_{p+2k+q+k-1}x_{p+2k+q+k-2} + b_{p+2k+q+k-1}f(x_{p+q+2k-1}) \\
&= a_{k-1}x_{p+2k+q+k-2} + b_{k-1} \\
&= a_{k-1}x_{k-2} + b_{k-1} = x_{k-1} > \kappa, \\
x_{p+2k+q+k} &= a_{p+2k+q+k}x_{p+2k+q+k-1} + b_{p+2k+q+k}f(x_{p+q+2k}) \\
&= a_kx_{k-1} = x_k > \kappa, \\
x_{p+2k+q+k+1} &= a_{p+2k+q+k+1}x_{p+2k+q+k} + b_{p+2k+q+k+1}f(x_{p+q+2k+1}) \\
&= a_{p+2k+q+k+1}x_{p+2k+q+k} \\
&= a_{k+1}x_k = x_{k+1} > \kappa, \\
&\ \ \vdots \\
x_{p+2k+q+k+p-1} &= a_{p+2k+q+k+p-1}x_{p+2k+q+k+p-2} + b_{p+2k+q+k+p-1}f(x_{p+q+2k+p-1}) \\
&= a_{p+2k+q+k+p-1}x_{p+2k+q+k+p-2} \\
&= a_{k+p-1}x_{k+p-2} = x_{k+p-1} > \kappa, \\
x_{p+2k+q+k+p} &= a_{zp+2k+q+k+p}x_{p+2k+q+k+p-1} + b_{p+2k+q+k+p}f(x_{p+q+2k+p}) \\
&= a_{p+k}x_{p+k-1} + b_{p+k}f(x_p) \\
&= a_{k+p}x_{k+p-1} = x_{k+p} \le \kappa, \\
&\ \ \vdots \\
x_{2k+p+q+2k+p+q-1} &= a_{2k+p+q-1}x_{2k+p+q+2k+p+q-2} + b_{p+2k+q-1}f(x_{2k+p+q+k+p+q-1}) \\
&= a_{2k+p+q-1}x_{2k+p+q-2} + b_{2k+p+q-1} \\
&= x_{2k+p+q-1} \le \kappa, \\
x_{2(2k+p+q)} &= g^{[2]}(x^*) = g(x_{2k+p+q}) = g(x^*) = x^*,
\end{aligned} \tag{3.43}$$

and so forth. Thus,

$$\begin{aligned}
x_n > \kappa &\quad \text{for } n \in \{0, \dots, p+k-1\}, \\
0 < x_n \le \kappa &\quad \text{for } n \in \{p+k, \dots, p+2k+q-1\}, \\
x_n > \kappa &\quad \text{for } n \in \{p+2k+q, \dots, p+2k+q+p+k-1\}, \\
0 < x_n \le \kappa &\quad \text{for } n \in \{p+2k+q+p+k, \dots, 2(p+2k+q)-1\},
\end{aligned} \tag{3.44}$$

and so forth.

By induction, we may see that $x_n > \kappa$ for $n \in \{m(p+2k+q), \dots, m(p+2k+q)+p+k-1\}$, $0 < x_n \le \kappa$ for $n \in \{m(p+2k+q)+p+k, \dots, (m+1)(p+2k+q)-1\}$, where $m \in \{0, 1, 2, \dots\}$, and $x_{n(2k+p+q)} = x^*$, $x_{n(2k+p+q)+1} = x_1, \dots, x_{n(2k+p+q)+2k+p+q-1} = x_{2k+p+q-1}$. This shows that $\{x_n\}$

is an eventually periodic solution of (1.7), whose minimal period is $2k + p + q$. The proof is complete. □

We remark that in the above result, l cannot be 0. We may, however, show the following by similar considerations.

Theorem 3.2. *Let* $k = 1$,

$$I_1 = \left[\frac{\alpha_\omega^3 \beta_\omega}{(1-\alpha_\omega^3)}, \frac{\alpha_{2\omega-1}\alpha_\omega\beta_\omega}{1-\alpha_\omega^3}\right), \quad I_2 = \left[M, \frac{\alpha_{2\omega-1}\alpha_\omega\beta_\omega}{1-\alpha_\omega^3}\right), \tag{3.45}$$

where

$$M = \max\left\{\alpha_n \frac{\alpha_\omega^3\beta_\omega}{1-\alpha_\omega^3} + \alpha_n\beta_n : n \in \{0, 1, \ldots, \omega - 1\}\right\}. \tag{3.46}$$

If $\kappa \in I_1 \cap I_2$ *and*

$$0 < \kappa < \min\left\{\frac{\alpha_n\beta_n}{1-\alpha_n} : n \in \{1, 2, \ldots, \omega\}\right\}, \tag{3.47}$$

then (1.7) *has an eventually* 3ω*-periodic solution* $\{x_n\}_{n=-k}^{\infty}$ *(which can be generated explicitly).*

Proof. Similar to the proof of the Theorem 3.1, set (3.10) and define the mapping g by

$$g(x) = g_\omega \circ (h_\omega \circ h_{\omega-1} \circ \cdots \circ h_1)^{[2]}(x). \tag{3.48}$$

We may show that

$$g(x) = \alpha_\omega(\alpha_\omega^2 x) + \alpha_\omega\beta_\omega, \tag{3.49}$$

and that g maps $D_0 = (\kappa, 1)$ into D_0 with a unique fixed point $x^* \in D_0$, where

$$x^* = \frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3}. \tag{3.50}$$

Let us choose

$$x_{-1} = \frac{1}{a_0}\left(\frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} - b_0\right). \tag{3.51}$$

By

$$\alpha_\omega > \alpha_\omega(1-\alpha_\omega^3) > \frac{b_0}{\alpha_\omega\beta_\omega}\alpha_\omega(1-\alpha_\omega^3) = \frac{b_0}{\beta_\omega}(1-\alpha_\omega^3), \tag{3.52}$$

we have

$$\frac{\alpha_\omega\beta_\omega}{(1-\alpha_\omega^3)} > b_0, \tag{3.53}$$

and hence,

$$x_{-1} = \frac{1}{a_0}\left(\frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} - b_0\right) > 0. \tag{3.54}$$

Since

$$\alpha_\omega^3(b_0 + \alpha_\omega\beta_{\omega-1}) = \alpha_\omega^4\beta_\omega, \tag{3.55}$$

then

$$\frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} - b_0 = \frac{\alpha_\omega^4\beta_\omega}{1-\alpha_\omega^3} + \alpha_\omega\beta_{\omega-1}, \tag{3.56}$$

and hence,

$$\begin{aligned}
x_{-1} &= \frac{1}{a_0}\left(\frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} - b_0\right) = \alpha_{\omega-1}\left(\frac{\alpha_\omega^3\beta_\omega}{1-\alpha_\omega^3}\right) + \alpha_{\omega-1}\beta_{\omega-1} \le \kappa, \\
x_0 &= \frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} > \kappa, \\
x_1 &= a_1 x_0 = \frac{a_1\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} > \kappa, \\
&\vdots \\
x_{2\omega-1} &= a_{2\omega-1}\cdots a_1\frac{\alpha_\omega\beta_\omega}{1-\alpha_\omega^3} > \kappa, \\
x_{2\omega} &= \frac{\alpha_\omega^3\beta_\omega}{1-\alpha_\omega^3} \le \kappa, \\
&\vdots \\
x_{3\omega-1} &= \alpha_{\omega-1}\frac{\alpha_\omega^3\beta_\omega}{1-\alpha_\omega^3} + \alpha_{\omega-1}\beta_{\omega-1} \le \kappa, \\
x_{3\omega} &= \frac{\alpha_\omega^4\beta_\omega}{1-\alpha_\omega^3} + \alpha_\omega\beta_\omega = x_0, \\
&\vdots
\end{aligned} \tag{3.57}$$

so that $\{x_n\}$ is an eventually 3ω-periodic solution of the system (1.7). □

4. Examples and remarks

Let $\{a_n\}$, $\{b_n\}$ be 2-periodic sequences, $k=3$, $p=1$, $q=1$, and

$$I_1 = \left[\alpha_2(\alpha_2+\alpha_2\beta_2), \frac{a_1\alpha_2\beta_2}{1-a_1\alpha_2}\right),$$
$$I_2 = \left[\alpha_2^2(\alpha_2+\alpha_2\beta_2), \frac{\alpha_2^3+1}{1-\alpha_2^4}\alpha_2\beta_2\right) \cap \left[\alpha_1\alpha_2^2(\alpha_2+\alpha_2\beta_2)+\alpha_1\beta_1, \frac{\alpha_2^3+1}{1-\alpha_2^4}\alpha_2\beta_2\right). \tag{4.1}$$

Suppose $\kappa \in I_1 \cap I_2$ and

$$0 < \kappa < \min\left\{\frac{\alpha_1}{1-\alpha_1}\beta_1, \frac{\alpha_2}{1-\alpha_2}\beta_2\right\}. \tag{4.2}$$

Consider the following "delay" difference equation:

$$x_n = a_n x_{n-1} + b_n f(x_{n-3}), \quad n \in \{0,1,2,\dots\}. \tag{4.3}$$

We can check that (4.3) has an eventually 8-periodic solution $\{x_n\}_{n=-3}^{\infty}$ with $x_0 \in (\kappa, 1)$.

In fact, as in the proof of Theorem 3.1, let

$$x^* = \frac{\alpha_2^4\beta_2 + \alpha_2\beta_2}{1-\alpha_2^4},$$
$$\phi_{-1} = \frac{x^* - b_0}{a_0}, \tag{4.4}$$

and ϕ_{-2}, ϕ_{-3} be arbitrary numbers in $(0,\kappa]$. Then, as shown in the proof of Theorem 3.1, the solution of (4.3) determined by ϕ_{-3}, ϕ_{-2}, ϕ_{-1} satisfies $x_{-1} = \phi_{-1} \in (0,\kappa]$ and $x_0 = x^*$.

Since $\kappa \in I_2$, we have $x_0 > \kappa$. On the other hand, by (3.3), and $(\alpha_2^3+1)/(1+\alpha_2^2)(1+\alpha_2) < 1$, hence

$$\begin{aligned}
&\kappa < x_0 < 1,\\
&x_1 = a_1x_0 + b_1 f(x_{-2}) = a_1x_0 + b_1 > a_1\kappa + b_1 > \kappa,\\
&x_2 = a_2x_1 + b_2 f(x_{-1}) = a_2x_1 + b_2 = a_2(a_1x_0 + b_1) + b_2\\
&\quad = a_1a_2x_0 + a_2b_1 + b_2 = \alpha_2x_0 + \alpha_2\beta_2 > \alpha_2\kappa + \alpha_2\beta_2 > \kappa,\\
&x_3 = a_3x_2 + b_3 f(x_0) = a_3x_2 > a_1(\alpha_2\kappa + \alpha_2\beta_2) > \kappa,\\
&x_4 = a_4x_3 + b_4 f(x_1) = a_4x_3 = a_2a_1(\alpha_2x_0 + \alpha_2\beta_2)\\
&\quad = \alpha_2(\alpha_2x_0 + \alpha_2\beta_2) < \alpha_2(\alpha_2 + \alpha_2\beta_2) \le \kappa,\\
&x_5 = a_5x_4 + b_5 f(x_2) = a_5\alpha_2(\alpha_2x_0 + \alpha_2\beta_2)\\
&\quad = a_1\alpha_2(\alpha_2x_0 + \alpha_2\beta_2) < \alpha_2(\alpha_2 + \alpha_2\beta_2) \le \kappa,\\
&x_6 = a_6x_5 + b_5 f(x_3) = a_6x_5 = a_2a_1\alpha_2(\alpha_2x_0 + \alpha_2\beta_2)\\
&\quad = \alpha_2^2(\alpha_2x_0 + \alpha_2\beta_2) < \alpha_2^2(\alpha_2 + \alpha_2\beta_2) \le \kappa,
\end{aligned}$$

$$\begin{aligned}
x_7 &= a_7x_6 + b_7f(x_4) = a_7\alpha_2^2(\alpha_2x_0 + \alpha_2\beta_2) + b_7\\
&= \alpha_1\alpha_2^2(\alpha_2x_0 + \alpha_2\beta_2) + b_1 = \alpha_1\alpha_2^2(\alpha_2x_0 + \alpha_2\beta_2) + \alpha_1\beta_1\\
&< \alpha_1\alpha_2^2(\alpha_2 + \alpha_2\beta_2) + \alpha_1\beta_1 \le \kappa,\\
x_8 &= a_8x_7 + b_8f(x_5)\\
&= a_8x_7 + b_8 = a_8(\alpha_1\alpha_2^2(\alpha_2x_0 + \alpha_2\beta_2) + \alpha_1\beta_1) + b_8\\
&= \alpha_2^3(\alpha_2x_0 + \alpha_2\beta_2) + \alpha_2\beta_2\\
&= \alpha_2^3\left(\alpha_2\frac{\alpha_2^4\beta_2 + \alpha_2\beta_2}{1-\alpha_2^4} + \alpha_2\beta_2\right) + \alpha_2\beta_2\\
&= \frac{\alpha_2^4\beta_2 + \alpha_2\beta_2}{1-\alpha_2^4} = x_0,\\
x_9 &= a_9x_8 + b_9f(x_6) = a_9x_8 + b_9 = a_9x_0 + b_9 = a_1x_0 + b_1 = x_1,\\
x_{10} &= a_{10}x_9 + b_{10}f(x_7) = a_{10}x_9 + b_{10} = a_2x_9 + b_2 = a_2x_1 + b_2 = x_2,\\
x_{11} &= a_{11}x_{10} + b_{11}f(x_8) = a_{11}x_{10} = a_9x_2 = x_3,\\
&\ \vdots
\end{aligned} \tag{4.5}$$

so that $\{x_n\}$ is an eventually 8-periodic solution of the system (4.3).

Next, let $a_n \equiv a$ and $b_n \equiv 1-a$ in (1.7). We have

$$\begin{aligned}
&\frac{\alpha_n}{1-\alpha_n}\beta_n = \frac{a^n}{1-a^n}\left(\frac{b}{a} + \frac{b}{a^2} + \cdots + \frac{b}{a^n}\right) = 1,\\
&\alpha_\omega^\tau\left(\alpha_\omega^l + \frac{\alpha_\omega\beta_\omega(1-\alpha_\omega^l)}{1-\alpha_\omega}\right) = a^{p+1}(\alpha_\omega^l + 1 - \alpha_\omega^l) = a^{p+1},\\
&\frac{\alpha_p\alpha_\omega\beta_\omega(1-\alpha_\omega^l)}{(1-\alpha_p\alpha_\omega^l)(1-\alpha_\omega)} = \frac{a^p(1-a^{k-1})}{1-a^{p+k-1}},\\
&\alpha_i\alpha_\omega^{\tau+l}\left(\alpha_\omega^l + \frac{\alpha_\omega\beta_\omega(1-\alpha_\omega^l)}{1-\alpha_\omega}\right) + \alpha_i\beta_i = a^{p+k+i} + 1 - a^i \le a^{p+q+k} + 1 - a^q, \quad i \in \{0,\dots,q-1\},\\
&\frac{\alpha_\omega^{\tau+\sigma+l}(1-\alpha_\omega^l) + (1-\alpha_\omega^\sigma)}{(1-\alpha_\omega^{\tau+\sigma+2l})(1-\alpha_\omega)}\alpha_\omega\beta_\omega = 1 - \frac{a^{q+1}(1-a^{k+p})}{1-a^{p+q+2k}}.
\end{aligned} \tag{4.6}$$

Hence,

$$I_1(p) = \left(a^{p+1}, \frac{a^p(1-a^{k-1})}{(1-a^{k+p-1})}\right), \qquad I_2(p,q) \supset \left(1-a^q+a^{p+q+k}, 1-\frac{a^{q+1}(1-a^{k+p})}{1-a^{2k+p+q}}\right). \tag{4.7}$$

Form the above, we can see that Theorem A is just a special case of Theorem 3.1, hence Theorem 3.1 is an extension of Theorem A.

Further, if $k = 1$ in (1.7), then the intervals I_1 and I_2 in Theorem 3.2 are, respectively,

$$I_1 = I_2 = \left[\frac{a^2}{1+a+a^2}, \frac{a}{1+a+a^2}\right). \tag{4.8}$$

Corollary 4.1. *Let* $a_n \equiv a$, $b_n \equiv 1-a$, *and* $k = 1$. *If*

$$\kappa \in (0,1) \cap \left[\frac{a^2}{1+a+a^2}, \frac{a}{1+a+a^2}\right), \tag{4.9}$$

then (1.7) *has an eventually* 3*-periodic solution* $\{x_n\}_{n=-k}^{\infty}$ *(which can be generated explicitly).*

As our final remark, note that under the conditions of Theorems 3.1 or 3.2 if $\{x_n'\}$ is an arbitrary solution of (1.7) with $x_{-k}', \dots, x_{-2}', x_{-1}' \in (0,\kappa]$ such that $x_0' \in (\kappa, 1)$, then in view of the proofs of Theorems 3.1 or 3.2,

$$\lim_{j\to\infty} g^{[j]}(x_0') = x^* = x_0. \tag{4.10}$$

This shows, by means of the continuity properties of the maps g_n and h_n, that $\lim_{n\to\infty}|x_n' - x_n| = 0$. Note that the requirement $x_{-k}', \dots, x_{-2}', x_{-1}' \in (0,\kappa]$ with $x_0' \in (\kappa, 1)$ is the same as requiring

$$x_{-1}' = \frac{1}{a_0}(x_0' - b_0) \in \left(\frac{1}{a_0}(\kappa - b_0), \frac{1}{a_0}(1-b_0)\right) \cap (0,\kappa]. \tag{4.11}$$

In other words, let $\{x_n'\}$ be a solution determined by $\phi_{-k}, \dots, \phi_{-1} \in (0,\kappa]$ such that

$$\phi_{-1} \in \left(\frac{1}{a_0}(\kappa - b_0), \frac{1}{a_0}(1-b_0)\right) \cap (0,\kappa], \tag{4.12}$$

then $\{x_n'\}$ will be "attracted" to the periodic solution $\{x_n\}$ in the proofs of Theorems 3.1 or 3.2. We remark that $(1-b_0)/a_0 > 0$. Thus, if

$$\frac{1}{a_0}(\kappa - b_0) \le \kappa, \tag{4.13}$$

then the above intersection is nonempty. And, if

$$\kappa - b_0 \le 0, \qquad \frac{1}{a_0}(1-b_0) > \kappa, \tag{4.14}$$

then

$$\left(\frac{1}{a_0}(\kappa - b_0), \frac{1}{a_0}(1-b_0)\right) \cap (0,\kappa] = (0,\kappa]. \tag{4.15}$$

Since a_0 and b_0 can be chosen in arbitrary manners in Theorems 3.1 and 3.2, such additional conditions can easily be achieved once κ is determined.

We may illustrate the above discussions by the following example. Let $k = 1$ and $a_n = 1/2 = b_n$ for all $n \in \{0, 1, 2, \ldots\}$. According to Corollary 4.1, if

$$\kappa \in (0,1) \cap \left[\frac{1}{7}, \frac{2}{7}\right) = \left[\frac{1}{7}, \frac{2}{7}\right), \tag{4.16}$$

then the solution $\{x_n\}$ of (1.7) determined by $x_0 = x^*$ in (3.50), that is, $x_{-1} = 1/7$, is eventually 3-periodic. Furthermore, let $\{x_n'\}$ be the solution determined by $x_{-1}' = \phi_{-1}$. If $\phi_{-1} \leq 0$, then by Lemma 2.1, $\lim_{n\to\infty} x_n' = 0$. If

$$\phi_{-1} \in (2\kappa - 1, 1) \cap (0, \kappa] = (0, \kappa], \tag{4.17}$$

then the solution $\{x_n'\}$ will satisfy $\lim_{n\to\infty} |x_n' - x_n| = 0$. If $\phi_{-1} > \kappa$, then by Lemma 2.2, a translate $\{y_n\}$ of $\{x_n'\}$ will satisfy $\lim_{n\to\infty} |y_n - x_n| = 0$.

Acknowledgment

Project supported by the National Natural Science Foundation of China (10661011).

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