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Research Article

Dynamic Behaviors of a General Discrete Nonautonomous System of Plankton Allelopathy with Delays

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We study the dynamic behaviors of a general discrete nonautonomous system of plankton allelopathy with delays. We first show that under some suitable assumption, the system is permanent. Next, by constructing a suitable Lyapunov functional, we obtain a set of sufficient conditions which guarantee the global attractivity of the two species. After that, by constructing an extinction-type Lyapunov functional, we show that under some suitable assumptions, one species will be driven to extinction. Finally, two examples together with their numerical simulations show the feasibility of the main results.

1. Introduction

The aim of this paper is to investigate the dynamic behaviors of the following general discrete nonautonomous system of plankton allelopathy with delay:

$$\begin{aligned} N_1(k+1) &= N_1(k)\exp\left[r_1(k)-\sum_{l=0}^{m}a_{1l}(k)N_1(k-l)-\sum_{l=0}^{m}b_{1l}(k)N_2(k-l)\right. \\ &\qquad \left.-\sum_{l=0}^{m}c_{1l}(k)N_1(k)N_2(k-l)\right], \\ N_2(k+1) &= N_2(k)\exp\left[r_2(k)-\sum_{l=0}^{m}a_{2l}(k)N_2(k-l)-\sum_{l=0}^{m}b_{2l}(k)N_1(k-l)\right. \\ &\qquad \left.-\sum_{l=0}^{m}c_{2l}(k)N_2(k)N_1(k-l)\right] \end{aligned} \tag{1.1}$$

together with the initial condition

$$N_i(-l) \geq 0, \quad N_i(0) > 0, \quad i = 1, 2;\ l = 0, 1, \ldots, m, \tag{1.2}$$

where m is a positive integer, $N_i(k)$ represent the densities of population i at the kth generation, $r_i(k)$ are the intrinsic growth rate of population i at the kth generation, $a_{il}(k)$ measure the intraspecific influence of the $(k-l)$th generation of population i on the density of own population, $b_{il}(k)$ stand for the interspecific influence of the $(k-l)$th generation of population i on the density of own population, and $c_{il}(k)$ stand for the effect of toxic inhibition of population i by population j at the $(k-l)$th generation, $i, j = 1, 2$ and $i \neq j$. Also, $\{r_i(k)\}$, $\{a_{il}(k)\}$, $\{b_{il}(k)\}$ and $\{c_{il}(k)\}$ are all bounded nonnegative sequences defined for $k \in N$, denoted by the set of all nonnegative integers, and $l \in \{0, 1, \ldots, m\}$ such that

$$\begin{aligned} &0 < r_i^L \leq r_i(k) \leq r_i^M, \qquad 0 < a_{il}^L \leq a_{il}(k) \leq a_{il}^M, \\ &0 < b_{il}^L \leq b_{il}(k) \leq b_{il}^M, \qquad 0 < c_{il}^L \leq c_{il}(k) \leq c_{il}^M, \end{aligned} \tag{1.3}$$

here, for any bounded sequence $\{f(k)\}$, define

$$f^M = \sup_{k \in N} f(k), \qquad f^L = \inf_{k \in N} f(k). \tag{1.4}$$

As was pointed out by Chattopadhyay [1] the effects of toxic substances on ecological communities are an important problem from an environmental point of view. Chattopadhyay [1] and Maynard-Smith [2] proposed the following two species Lotka-Volterra competition system, which describes the changes of size and density of phytoplankton:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - b_1x_1(t)x_2(t)\right], \\ \frac{dx_2(t)}{dt} &= x_2(t)\left[r_2 - a_{21}x_1(t) - a_{22}x_2(t) - b_2x_1(t)x_2(t)\right], \end{aligned} \tag{1.5}$$

where $x_1(t)$ and $x_2(t)$ denote the population density of two competing species at time t for a common pool of resources. The terms $b_1x_1(t)x_2(t)$ and $b_2x_1(t)x_2(t)$ denote the effect of toxic substances. Here, they made the assumption that each species produces a substance toxic to the other, only when the other is present. Noticing that the production of the toxic substance allelopathic to the competing species will not be instantaneous, but delayed by different discrete time lags required for the maturity of both species, thus, Mukhopadhyay et al. [3] also incorporated the discrete time delay into the above system. Tapaswi and Mukhopadhyay [4] also studied a two-dimensional system that arises in plankton allelopathy involving discrete time delays and environmental fluctuations. They assumed that the environmental parameters are assumed to be perturbed by white noise characterized by a Gaussian distribution with mean zero and unit spectral density. They focus on the dynamic behavior of the stochastic system and the fluctuations in population. For more works on system (1.5), one could refer to [1–3, 5–24] and the references cited therein.

Since the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, and discrete time models can also provide efficient computational models of continuous models for numerical simulations, corresponding to system (1.5), Huo and Li [25] argued that it is necessary to study the following discrete two species competition system:

$$x_1(k+1) = x_1(k)\exp\left[r_1(k) - a_{11}(k)x_1(k) - a_{12}(k)x_2(k) - b_1(k)x_1(k)x_2(k)\right],$$
$$x_2(k+1) = x_2(k)\exp\left[r_2(k) - a_{21}(k)x_1(k) - a_{22}(k)x_2(k) - b_2(k)x_1(k)x_2(k)\right], \quad (1.6)$$

where $x_1(k)$ and $x_2(k)$ are the population sizes of the two competitors at generation k, $b_1(k)$ and $b_2(k)$ have respectively, shown that each species produces a toxic substance to the other but the other only is present. In [25], sufficient conditions were obtained to guarantee the permanence of the above system, they also investigated the existence and stability property of the positive periodic solution of system (1.6). Recently, Li and Chen [26] further investigated the dynamic behaviors of the system (1.6). For general nonatonomous case, they obtain a set of sufficient conditions which guarantee the extinction of species x_2 and the global stability of species x_1 when species x_2 is eventually extinct. For periodic case, the other set of sufficient conditions, which concerned with the average condition of the coefficients of he system, were obtained to ensure the eventual extinction of species x_2 and the global stability of positive periodic solution of species x_1 when species x_2 is eventually extinct. For more works on discrete population dynamics, one could refer to [7, 10, 25–45].

Liu and Chen [32] argued that for a more realistic model, both seasonality of the changing environment and some of the past states, that is, the effects of time delays, should be taken into account in a model of multiple species growth. They proposed and studied the system (1.1), which is more general than system (1.6). By applying the coincidence degree theory, they obtained a set of sufficient conditions for the existence of at least one positive periodic solution of system (1.1)-(1.2). Zhang and Fang [46] also investigated the periodic solution of the system (1.1), they showed that under some suitable assumption, system (1.1) could admit at least two positive periodic solution. As we can see, the works [32, 46] are all concerned with the positive periodic solution of the system. However, since few things in the nature are really periodic, it is nature to study the general nonautonomous system (1.1), in this case, it is impossible to study the periodic solution of the system, however, such topics as permanence, extinction, and stability become the most important things. In this paper, we will further investigate the dynamics behaviors of the system (1.1). More precisely, by developing the analysis technique of Liu [31] and Muroya [35, 36], we study the permanence, global attractivity and extinction of system (1.1)-(1.2).

The organization of this paper is as follows. We study the persistence property of the system in Section 2 and the stability property in Section 3. Then in Section 4, by constructing a suitable Lyapunov functional, sufficient conditions which ensure the extinction of species N_2 of system (1.1)-(1.2) are studied. In Section 5, two examples together with their numeric simulations show the feasibility of main results. For more relevant works, one could refer to [2, 3, 5–9, 12, 13, 27–30, 33, 34, 37–45] and the references cited therein.

2. Permanence

In this section, we study the persistent property of system (1.1)-(1.2).

Lemma 2.1. *For any positive solution* $\{(N_1(k), N_2(k))\}$ *of system* (1.1)-(1.2)*,*

$$\limsup_{k\to\infty} N_i(k) \le B_i, \quad i = 1,2, \tag{2.1}$$

where

$$B_i \overset{\text{def}}{=} \frac{\exp\left(r_i^M - 1\right)}{a_{i0}^L}, \quad i = 1,2. \tag{2.2}$$

Proof. Let $\{(N_1(k), N_2(k))\}$ be any positive solution of system (1.1)-(1.2), in view of the system (1.1) for all $k \in N$, we have

$$N_i(k+1) \le N_i(k) \exp\left[r_i(k) - a_{i0}(k) N_i(k)\right], \quad i = 1,2. \tag{2.3}$$

Applying Lemma 2.1 of Yang [44] to (2.3), we can obtain

$$\limsup_{k\to\infty} N_i(k) \le \frac{\exp\left(r_i^M - 1\right)}{a_{i0}^L} \overset{\text{def}}{=} B_i, \quad i = 1,2. \tag{2.4}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Assume that*

$$\begin{aligned}
\Delta_{11} &\overset{\text{def}}{=} r_1^L - \sum_{l=1}^{m} a_{1l}^M B_1 - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M B_1\right] B_2 > 0, \\
\Delta_{21} &\overset{\text{def}}{=} r_2^L - \sum_{l=1}^{m} a_{2l}^M B_2 - \sum_{l=0}^{m} \left[b_{2l}^M + c_{2l}^M B_2\right] B_1 > 0
\end{aligned} \tag{2.5}$$

hold, where B_1 *and* B_2 *are defined in* (2.2)*. Then for any positive solution* $\{(N_1(k), N_2(k))\}$ *of system* (1.1)-(1.2)*,*

$$\liminf_{k\to\infty} N_i(k) \ge A_i, \quad i = 1,2, \tag{2.6}$$

where

$$\begin{aligned}
A_i &= \frac{\Delta_{i1}}{a_{i0}^M} \exp[\Delta_{i2}], \quad i = 1,2, \\
\Delta_{12} &= r_1^L - \sum_{l=0}^{m} a_{1l}^M B_1 - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M B_1\right] B_2, \\
\Delta_{22} &= r_2^L - \sum_{l=0}^{m} a_{2l}^M B_2 - \sum_{l=0}^{m} \left[b_{2l}^M + c_{2l}^M B_2\right] B_1.
\end{aligned} \tag{2.7}$$

Proof. In view of (2.5), we can choose a constant $\varepsilon > 0$ small enough such that

$$r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon) > 0, \tag{2.8}$$

$$r_2^L - \sum_{l=1}^{m} a_{2l}^M (B_2 + \varepsilon) - \sum_{l=0}^{m} \left[b_{2l}^M + c_{2l}^M (B_2 + \varepsilon) \right] (B_1 + \varepsilon) > 0. \tag{2.9}$$

In view of (2.1), for above $\varepsilon > 0$, there exists an integer $k_0 \in N$ such that

$$N_i(k) \le B_i + \varepsilon \quad \forall k \ge k_0,\ i = 1, 2. \tag{2.10}$$

We consider the following two cases.

Case (i). We assume that there exists an integer $l_0 \ge k_0 + m$ such that $N_1(l_0 + 1) \le N_1(l_0)$. Note that

$$\begin{aligned} N_1(l_0 + 1) &= N_1(l_0) \exp \Bigg[r_1(l_0) - \sum_{l=0}^{m} a_{1l}(l_0) N_1(l_0 - l) - \sum_{l=0}^{m} b_{1l}(l_0) N_2(l_0 - l) \\ &\qquad - \sum_{l=0}^{m} c_{1l}(l_0) N_1(l_0) N_2(l_0 - l) \Bigg] \\ &\ge N_1(l_0) \exp \left\{ r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon) - a_{10}^M N_1(l_0) \right\}. \end{aligned} \tag{2.11}$$

So we can obtain

$$r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon) - a_{10}^M N_1(l_0) \le 0. \tag{2.12}$$

It follows from (2.8) that

$$N_1(l_0) \ge \frac{r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon)}{a_{10}^M} > 0. \tag{2.13}$$

Then we have

$$\begin{aligned} N_1(l_0 + 1) \ge{}& \frac{r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon)}{a_{10}^M} \\ & \times \exp \left\{ r_1^L - \sum_{l=0}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon) \right\}. \end{aligned} \tag{2.14}$$

Let

$$N_{1\varepsilon} = \frac{r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon)}{a_{10}^M} \times \exp \left\{ r_1^L - \sum_{l=0}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon) \right\}. \tag{2.15}$$

Note that

$$B_1 = \frac{\exp\left(r_1^M - 1\right)}{a_{10}^L} \geq \frac{r_1^M}{a_{10}^L} \geq \frac{r_1^L}{a_{10}^M}, \tag{2.16}$$

thus $r_1^L - a_{10}^M B_1 \leq 0$, and so, for above $\varepsilon > 0$,

$$r_1^L - \sum_{l=0}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon) < r_1^L - a_{10}^M (B_1 + \varepsilon) < r_1^L - a_{10}^M B_1 \leq 0 \tag{2.17}$$

or

$$\frac{r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon)}{a_{10}^M} \geq N_{1\varepsilon}. \tag{2.18}$$

We can claim that

$$N_1(k) \geq N_{1\varepsilon} \quad \forall k \geq l_0. \tag{2.19}$$

By way of contradiction, assume that there exists an integer $p_0 \geq l_0$ such that $N_1(p_0) < N_{1\varepsilon}$. Then $p_0 \geq l_0 + 2$. Let $\widetilde{p}_0 \geq l_0 + 2$ be the smallest integer such that $N_1(\widetilde{p}_0) < N_{1\varepsilon}$. Then $N_1(\widetilde{p}_0 - 1) > N_1(\widetilde{p}_0)$. The above argument produces that $N_1(\widetilde{p}_0) \geq N_{1\varepsilon}$, a contradiction. Thus (2.19) proved.

Case (ii). We assume that $N_1(k+1) > N_1(k)$ for all $k \geq k_0 + m$, then $\lim_{k \to \infty} N_1(k)$ exists, denoted by $\underline{N}_1$. We can claim that

$$\underline{N}_1 \geq \frac{r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon)}{a_{10}^M}. \tag{2.20}$$

By the way of contradiction, assume that

$$\underline{N}_1 < \frac{r_1^L - \sum_{l=1}^{m} a_{1l}^M (B_1 + \varepsilon) - \sum_{l=0}^{m} \left[b_{1l}^M + c_{1l}^M (B_1 + \varepsilon) \right] (B_2 + \varepsilon)}{a_{10}^M}. \tag{2.21}$$

Taking limit in the first equation of (1.1) gives

$$\lim_{k\to\infty}\left[r_1(k)-\sum_{l=0}^{m}a_{1l}(k)N_1(k-l)-\sum_{l=0}^{m}b_{1l}(k)N_2(k-l)-\sum_{l=0}^{m}c_{1l}(k)N_1(k)N_2(k-l)\right]=0, \tag{2.22}$$

which is a contradiction since

$$\begin{aligned}&\lim_{k\to\infty}\left[r_1(k)-\sum_{l=0}^{m}a_{1l}(k)N_1(k-l)-\sum_{l=0}^{m}b_{1l}(k)N_2(k-l)-\sum_{l=0}^{m}c_{1l}(k)N_1(k)N_2(k-l)\right]\\ &\quad\geq r_1^L-\sum_{l=1}^{m}a_{1l}^M(B_1+\varepsilon)-a_{10}^M\underline{N}_1-\sum_{l=0}^{m}\left[b_{1l}^M+c_{1l}^M(B_1+\varepsilon)\right](B_2+\varepsilon)>0.\end{aligned} \tag{2.23}$$

The claim is thus proved.

From (2.20), we see that

$$\underline{N}_1\geq N_{1\varepsilon}. \tag{2.24}$$

Combining Cases (i) and (ii), we see that

$$\liminf_{k\to\infty}N_1(k)\geq N_{1\varepsilon}. \tag{2.25}$$

Setting $\varepsilon\to 0$, it follows that

$$\lim_{\varepsilon\to 0}N_{1\varepsilon}=\frac{\Delta_{11}}{a_{10}^M}\exp\{\Delta_{12}\}\overset{\text{def}}{=}A_1. \tag{2.26}$$

So we can easily see that

$$\liminf_{k\to\infty}N_1(k)\geq A_1. \tag{2.27}$$

From the second equation of (1.1), similar to above analysis, we have

$$\liminf_{k\to\infty}N_2(k)\geq A_2, \tag{2.28}$$

where A_2 is defined in (2.6). This completes the proof of Lemma 2.2. □

It immediately follows from Lemmas 2.1 and 2.2 that the following theorem holds.

Theorem 2.3. *Assume that* (2.5) *hold, then system* (1.1)-(1.2) *is permanent.*

3. Global attractivity

This section devotes to study the stability property of the positive solution of system (1.1)-(1.2).

Theorem 3.1. *Assume that there exists a constant* $\eta > 0$ *such that*

$$\min\left\{a_{i0}^L, \frac{2}{B_i} - a_{i0}^M\right\} - \sum_{j=1,\, j\neq i}^{2}\left[\sum_{l=1}^{m} a_{il}^M + (m+1)\left(b_j^M + 2B_j c^M\right)\right] > \eta, \quad i = 1,2, \tag{H_0}$$

where, for $i, j = 1,2,\ i \neq j$, B_i *and* B_j *are defined in* (2.2),

$$b_j^M = \max\left\{b_{jl}^M : l = 0,1,\dots,m\right\}, \qquad c_i^M = \max\left\{c_{il}^M : l = 0,1,\dots,m\right\},$$
$$c^M = \max\left\{c_i^M : i = 1,2\right\}, \tag{3.1}$$

then for any two positive solutions $\{(N_1(k), N_2(k))\}$ *and* $\{(N_1^*(k), N_2^*(k))\}$ *of system* (1.1)-(1.2),

$$\lim_{k\to\infty}\left(N_i(k) - N_i^*(k)\right) = 0, \quad i = 1,2. \tag{3.2}$$

Proof. First, let

$$V_{11}(k) = \left|\ln N_1(k) - \ln N_1^*(k)\right|. \tag{3.3}$$

Then from the first equation of (1.1), we have

$$\begin{aligned} V_{11}(k+1) &= \left|\ln N_1(k+1) - \ln N_1^*(k+1)\right| \\ &\leq \left|\ln N_1(k) - \ln N_1^*(k) - a_{10}(k)\left[N_1(k) - N_1^*(k)\right]\right| + \sum_{l=1}^{m} a_{1l}(k)\left|N_1(k-l) - N_1^*(k-l)\right| \\ &\quad + \sum_{l=0}^{m} b_{1l}(k)\left|N_2(k-l) - N_2^*(k-l)\right| + \sum_{l=0}^{m} c_{1l}(k)\left|N_1(k)N_2(k-l) - N_1^*(k)N_2^*(k-l)\right|. \end{aligned} \tag{3.4}$$

Noticing that by mean-value theory

$$\left|\ln N_1(k) - \ln N_1^*(k)\right| = \frac{1}{\theta_1(k)}\left|N_1(k) - N_1^*(k)\right|, \tag{3.5}$$

where $0 < \theta_1(k) \le \max\{N_1(k), N_1^*(k)\}$. Then

$$\begin{aligned}
&\left| \ln N_1(k) - \ln N_1^*(k) - a_{10}(k)\left[N_1(k) - N_1^*(k)\right]\right| \\
&\quad = \left|\ln N_1(k) - \ln N_1^*(k)\right| - \left|\ln N_1(k) - \ln N_1^*(k)\right| \\
&\qquad + \left|\ln N_1(k) - \ln N_1^*(k) - a_{10}(k)\left[N_1(k) - N_1^*(k)\right]\right| \\
&\quad = \left|\ln N_1(k) - \ln N_1^*(k)\right| - \left(\frac{1}{\theta_1(k)} - \left|\frac{1}{\theta_1(k)} - a_{10}(k)\right|\right)\left|N_1(k) - N_1^*(k)\right|.
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.4) leads to

$$\begin{aligned}
V_{11}(k+1) \le{}& \left|\ln N_1(k) - \ln N_1^*(k)\right| - \left(\frac{1}{\theta_1(k)} - \left|\frac{1}{\theta_1(k)} - a_{10}(k)\right|\right)\left|N_1(k) - N_1^*(k)\right| \\
&+ \sum_{l=1}^{m} a_{1l}(k)\left|N_1(k-l) - N_1^*(k-l)\right| + \sum_{l=0}^{m} b_{1l}(k)\left|N_2(k-l) - N_2^*(k-l)\right| \\
&+ \sum_{l=0}^{m} c_{1l}(k)\left|N_1(k)N_2(k-l) - N_1^*(k)N_2^*(k-l)\right|.
\end{aligned} \tag{3.7}$$

So it follows that

$$\begin{aligned}
\Delta V_{11} \le{}& -\left(\frac{1}{\theta_1(k)} - \left|\frac{1}{\theta_1(k)} - a_{10}(k)\right|\right)\left|N_1(k) - N_1^*(k)\right| \\
&+ \sum_{l=1}^{m} a_{1l}(k)\left|N_1(k-l) - N_1^*(k-l)\right| + \sum_{l=0}^{m} b_{1l}(k)\left|N_2(k-l) - N_2^*(k-l)\right| \\
&+ \sum_{l=0}^{m} c_{1l}(k)\left|N_1(k)N_2(k-l) - N_1^*(k)N_2^*(k-l)\right|.
\end{aligned} \tag{3.8}$$

According to (2.1), for any constant $\varepsilon > 0$, there exists an integer $k_0 \in N$ such that

$$N_1(k) \le B_1 + \varepsilon, \quad N_2(k) \le B_2 + \varepsilon \quad \forall k \ge k_0. \tag{3.9}$$

So for all $k \ge k_0 + m$, $l = 0, 1, \dots, m$, it follows that

$$\begin{aligned}
&\left|N_1(k)N_2(k-l) - N_1^*(k)N_2^*(k-l)\right| \\
&\quad = \left|N_1(k)N_2(k-l) - N_1(k)N_2^*(k-l) + N_1(k)N_2^*(k-l) - N_1^*(k)N_2^*(k-l)\right| \\
&\quad = \left|N_1(k)\left[N_2(k-l) - N_2^*(k-l)\right] + N_2^*(k-l)\left[N_1(k) - N_1^*(k)\right]\right| \\
&\quad \le (B_1 + \varepsilon)\left|N_2(k-l) - N_2^*(k-l)\right| + (B_2 + \varepsilon)\left|N_1(k) - N_1^*(k)\right|.
\end{aligned} \tag{3.10}$$

So for all $k \geq k_0 + m$, it follows from (3.4) that

$$\Delta V_{11} \leq -\left(\frac{1}{\theta_1(k)} - \left| \frac{1}{\theta_1(k)} - a_{10}(k) \right| - \sum_{l=0}^{m} (B_2 + \varepsilon) c_{1l}(k) \right) \left| N_1(k) - N_1^*(k) \right| + \sum_{l=1}^{m} a_{1l}(k) \left| N_1(k-l) - N_1^*(k-l) \right| + \sum_{l=0}^{m} \left[b_{1l}(k) + (B_1 + \varepsilon) c_{1l}(k) \right] \left| N_2(k-l) - N_2^*(k-l) \right|. \tag{3.11}$$

Next, let

$$V_{12}(k) = \sum_{l=1}^{m} \sum_{s=k-l}^{k-1} a_{1l}(s+l) \left| N_1(s) - N_1^*(s) \right| + \sum_{l=0}^{m} \sum_{s=k-l}^{k-1} \left[b_{1l}(s+l) + (B_1 + \varepsilon) c_{1l}(s+l) \right] \left| N_2(s) - N_2^*(s) \right|, \tag{3.12}$$

and we can obtain

$$\begin{aligned}
\Delta V_{12} &= V_{12}(k+1) - V_{12}(k) \\
&= \sum_{l=1}^{m} \sum_{s=k+1-l}^{k} a_{1l}(s+l) \left| N_1(s) - N_1^*(s) \right| - \sum_{l=1}^{m} \sum_{s=k-l}^{k-1} a_{1l}(s+l) \left| N_1(s) - N_1^*(s) \right| \\
&\quad + \sum_{l=0}^{m} \sum_{s=k+1-l}^{k} \left[b_{1l}(s+l) + (B_1 + \varepsilon) c_{1l}(s+l) \right] \left| N_2(s) - N_2^*(s) \right| \\
&\quad - \sum_{l=0}^{m} \sum_{s=k-l}^{k-1} \left[b_{1l}(s+l) + (B_1 + \varepsilon) c_{1l}(s+l) \right] \left| N_2(s) - N_2^*(s) \right| \\
&= \sum_{l=1}^{m} a_{1l}(k+l) \left| N_1(k) - N^*(k) \right| - \sum_{l=1}^{m} a_{1l}(k) \left| N_1(k-l) - N_1^*(k-l) \right| \\
&\quad + \sum_{l=0}^{m} \left[b_{1l}(k+l) + (B_1 + \varepsilon) c_{1l}(k+l) \right] \left| N_2(k) - N_2^*(k) \right| \\
&\quad - \sum_{l=0}^{m} \left[b_{1l}(k) + (B_1 + \varepsilon) c_{1l}(k) \right] \left| N_2(k-l) - N_2^*(k-l) \right|.
\end{aligned} \tag{3.13}$$

Now, we define V_1 by

$$V_1(k) = V_{11}(k) + V_{12}(k). \tag{3.14}$$

So for all $k \ge k_0 + m$, it follows from (3.6) and (3.9) that

$$\begin{aligned}\Delta V_1 &= \Delta V_{11} + \Delta V_{12}\\ &\le -\left(\frac{1}{\theta_1(k)} - \left|\frac{1}{\theta_1(k)} - a_{10}(k)\right| - \sum_{l=0}^{m}(B_2+\varepsilon)c_{1l}(k) - \sum_{l=1}^{m} a_{1l}(k+l)\right)\left|N_1(k) - N_1^*(k)\right|\\ &\quad + \sum_{l=0}^{m}\left[b_{1l}(k+l) + (B_1+\varepsilon)c_{1l}(k+l)\right]\left|N_2(k) - N_2^*(k)\right|.\end{aligned} \tag{3.15}$$

Similar to above arguments, we can define

$$V_2(k) = V_{21}(k) + V_{22}(k), \tag{3.16}$$

where

$$\begin{aligned}V_{21}(k) &= \left|\ln N_2(k) - \ln N_2^*(k)\right|,\\ V_{22}(k) &= \sum_{l=1}^{m}\sum_{s=k-l}^{k-1} a_{2l}(s+l)\left|N_2(s) - N_2^*(s)\right|\\ &\quad + \sum_{l=0}^{m}\sum_{s=k-l}^{k-1}\left[b_{2l}(s+l) + (B_2+\varepsilon)c_{2l}(s+l)\right]\left|N_1(s) - N_1^*(s)\right|.\end{aligned} \tag{3.17}$$

Then for all $k \ge k_0 + m$, we can obtain

$$\begin{aligned}\Delta V_2 &= \Delta V_{21} + \Delta V_{22}\\ &\le -\left(\frac{1}{\theta_2(k)} - \left|\frac{1}{\theta_2(k)} - a_{20}(k)\right| - \sum_{l=0}^{m}(B_1+\varepsilon)c_{2l}(k) - \sum_{l=1}^{m} a_{2l}(k+l)\right)\left|N_2(k) - N_2^*(k)\right|\\ &\quad + \sum_{l=0}^{m}\left[b_{2l}(k+l) + (B_2+\varepsilon)c_{2l}(k+l)\right]\left|N_1(k) - N_1^*(k)\right|,\end{aligned} \tag{3.18}$$

where $\theta_2(k)$ lies between $N_2(k)$ and $N_2^*(k)$.

Now, we define V by

$$V(k) = V_1(k) + V_2(k). \tag{3.19}$$

It is easy to see that $V(k) \geq 0$ for all $k \in Z$ and $V(k_0 + m) < +\infty$. For the arbitrariness of $\varepsilon > 0$ and by (H_0), we can choose $\varepsilon > 0$ small enough such that for $i = 1, 2$,

$$\min\left\{a_{i0}^L, \frac{2}{B_i + \varepsilon} - a_{i0}^M\right\} - \sum_{j=1, j\neq i}^{2}\left[\sum_{l=1}^{m} a_{il}(k+l) + (m+1)\left(b_j^M + 2\left(B_j + \varepsilon\right)c^M\right)\right] > \eta. \tag{3.20}$$

So for all $k \geq k_0 + m$, it follows from (3.15) and (3.18) that

$$\begin{aligned}
\Delta V \leq& -\sum_{i=1}^{2}\left\{\frac{1}{\theta_i(k)} - \left|\frac{1}{\theta_i(k)} - a_{i0}(k)\right|\right. \\
&\left. - \sum_{j=1, j\neq i}^{2}\left[\sum_{l=1}^{m} a_{il}(k+l) + \sum_{l=0}^{m}\left[b_{jl}(k+l) + \left(B_j + \varepsilon\right)\left(c_{il}(k) + c_{jl}(k+l)\right)\right]\right]\right\} \\
&\times \left|N_i(k) - N_i^*(k)\right| \\
\leq& -\sum_{i=1}^{2}\left\{\min\left\{a_{i0}^L, \frac{2}{B_i + \varepsilon} - a_{i0}^M\right\} - \sum_{j=1, j\neq i}^{2}\left[\sum_{l=1}^{m} a_{il}(k+l) + (m+1)\left(b_j^M + 2\left(B_j + \varepsilon\right)c^M\right)\right]\right\} \\
&\times \left|N_i(k) - N_i^*(k)\right| \\
\leq& -\eta\sum_{i=1}^{2}\left|N_i(k) - N_i^*(k)\right|,
\end{aligned} \tag{3.21}$$

So we have

$$\sum_{p=k_0+m}^{k}\left[V(p+1) - V(p)\right] \leq -\eta\sum_{p=k_0+m}^{k}\sum_{i=1}^{2}\left|N_i(p) - N_i^*(p)\right|, \tag{3.22}$$

which implies

$$V(k+1) + \eta\sum_{p=k_0+m}^{k}\sum_{i=1}^{2}\left|N_i(p) - N_i^*(p)\right| \leq V\left(k_0 + m\right). \tag{3.23}$$

It follows that

$$\sum_{p=k_0+m}^{k}\sum_{i=1}^{2}\left|N_i(p) - N_i^*(p)\right| \leq \frac{V\left(k_0 + m\right)}{\eta}. \tag{3.24}$$

Then

$$\sum_{k=k_0+m}^{\infty}\sum_{i=1}^{2}\left|N_i(k) - N_i^*(k)\right| \leq \frac{V\left(k_0 + m\right)}{\eta} < +\infty, \tag{3.25}$$

which implies that $\lim_{k\to\infty}\sum_{i=1}^{2}|N_i(k)-N_i^*(k)|=0$, that is,

$$\lim_{k\to\infty}\left(N_i(k)-N_i^*(k)\right)=0,\quad i=1,2. \tag{3.26}$$

This completes the proof of Theorem 3.1. □

4. Extinction of species N_2

This section devotes to study the extinction of the species N_2.

Lemma 4.1. *For any positive solution* $\{(N_1(k),N_2(k))\}$ *of system* (1.1)-(1.2)*, there exists a constant* $\sigma>0$ *such that*

$$\liminf_{k\to+\infty}\left[N_1(k)+N_2(k)\right]>\sigma. \tag{4.1}$$

Proof. By (2.1), there exists a constant $B>0$ such that

$$N_i(k)<B\quad \forall k>k_0,\ i=1,2. \tag{4.2}$$

In view of (1.1) for all $k>k_0+m$, $i=1,2$, it follows that

$$N_i(k)\le N_i(k+1)\exp\left\{-r_i^L+\sum_{l=0}^{m}\left(a_{il}^M+b_{il}^M+c_{il}^M B\right)B\right\}. \tag{4.3}$$

So we have

$$N_i(k-l)\le N_i(k)\exp\left\{l\left[-r_i^L+\sum_{l=0}^{m}\left(a_{il}^M+b_{il}^M+c_{il}^M B\right)B\right]\right\},\quad l=0,1,\ldots,m. \tag{4.4}$$

Let

$$\begin{gathered}C_1=\max\left\{\exp\left\{l\left[-r_i^L+\sum_{l=0}^{m}\left(a_{il}^M+b_{il}^M+c_{il}^M B\right)B\right]\right\}:i=1,2,\ l=0,1,\ldots,m\right\},\\ C_2=\max\left\{d_{il}^M C_1:i=1,2,\ l=0,1,\ldots,m\right\},\end{gathered} \tag{4.5}$$

where $d_{il}^M = \max\{a_{il}^M, b_{il}^M + c_{il}^M B\}$, $i = 1, 2$, $l = 0, 1, \ldots, m$. For all $k > k_0 + m$, $i, j = 1, 2$, $i \neq j$, it follows from (1.1) and (4.4) that

$$\begin{aligned} N_i(k+1) &\geq N_i(k) \exp\left\{ r_i^L - \sum_{l=0}^{m} a_{il}^M C_1 N_i(k) - \sum_{l=0}^{m} \left(b_{il}^M + c_{il}^M B\right) C_1 N_j(k) \right\} \\ &\geq N_i(k) \exp\left\{ r_i^L - \sum_{l=0}^{m} d_{il}^M C_1 \left[N_i(k) + N_j(k)\right] \right\} \\ &\geq N_i(k) \exp\left\{ \min\left\{r_1^L, r_2^L\right\} - (m+1) C_2 \left[N_1(k) + N_2(k)\right] \right\}, \end{aligned} \tag{4.6}$$

so we have

$$N_1(k+1) + N_2(k+1) \geq \left[N_1(k) + N_2(k)\right] \exp\left\{ \min\left\{r_1^L, r_2^L\right\} - (m+1) C_2 \left[N_1(k) + N_2(k)\right] \right\}. \tag{4.7}$$

Let $x(k) = N_1(k) + N_2(k)$, then we have

$$\begin{aligned} x(k+1) &\geq x(k) \exp\left\{ \min\left\{r_1^L, r_2^L\right\} - (m+1) C_2 x(k) \right\} \\ &= x(k) \exp\left\{ \min\left\{r_1^L, r_2^L\right\} \left[1 - \frac{(m+1) C_2}{\min\left\{r_1^L, r_2^L\right\}} x(k)\right] \right\} \\ &\overset{\text{def}}{=} x(k) \exp\left\{ r\left[1 - a x(k)\right] \right\}. \end{aligned} \tag{4.8}$$

Note that for all $k > k_0$, $x(k) = N_1(k) + N_2(k) < 2B$, so similar to the proof of Lemma 2.2 of Chen [27], we have

$$\liminf_{k \to +\infty} x(k) \geq \frac{1}{a} \exp\left\{ r(1 - 2aB) \right\} > 0. \tag{4.9}$$

Then, there is a positive constant $\sigma > 0$ such that

$$\liminf_{k \to +\infty} \left[N_1(k) + N_2(k)\right] = \liminf_{k \to +\infty} x(k) \geq \frac{1}{a} \exp\left\{ r(1 - 2aB) \right\} > \sigma. \tag{4.10}$$

This completes the proof of Lemma 4.1. □

Theorem 4.2. *Assume that*

$$\sum_{l=0}^{m} \left[r_1^L b_{2l}^L - r_2^M a_{1l}^M\right] > 0, \qquad \sum_{l=0}^{m} \left[r_1^L a_{2l}^L - r_2^M \left(b_{1l}^M + c_{1l}^M B_1\right)\right] > 0, \tag{H_1}$$

where B_1 is defined in (2.2). *Let $\{(N_1(k), N_2(k))\}$ be any positive solution of system* (1.1)-(1.2), *then $N_2(k) \to 0$ as $k \to +\infty$.*

Corollary 4.3. *Assume that for all $l = 0, 1, \ldots, m$, the following inequalities*

$$\frac{r_2^M}{r_1^L} - \min\left\{\frac{b_{2l}^L}{a_{1l}^M}, \frac{a_{2l}^L}{b_{1l}^M + c_{1l}^M B_1}\right\} < 0 \tag{H_1^*}$$

hold, where B_1 is defined in (2.2)*. Let $\{(N_1(k), N_2(k))\}$ be any positive solution of system* (1.1)-(1.2)*, then $N_2(k) \to 0$ as $k \to +\infty$.*

Proof of Corollary 4.3. Obviously, if condition (H_1^*) holds, one could easily see that condition (H_1) holds, thus, the conclusion of Corollary 4.3 follows from Theorem 4.2. The proof is complete. □

Proof of Theorem 4.2. It follows from (H_1) that we can choose a constant $\varepsilon > 0$ small enough such that

$$\sum_{l=0}^{m}\left[r_1^L b_{2l}^L - r_2^M a_{1l}^M\right] > 0, \qquad \sum_{l=0}^{m}\left[r_1^L a_{2l}^L - r_2^M\left(b_{1l}^M + c_{1l}^M(B_1 + \varepsilon)\right)\right] > 0. \tag{4.11}$$

Set

$$\Delta^{\varepsilon} = \min\left\{\sum_{l=0}^{m}\left[r_1^L b_{2l}^L - r_2^M a_{1l}^M\right], \sum_{l=0}^{m}\left[r_1^L a_{2l}^L - r_2^M\left(b_{1l}^M + c_{1l}^M(B_1 + \varepsilon)\right)\right]\right\} > 0. \tag{4.12}$$

For above $\varepsilon > 0$ from (2.1), there is an integer $K \in N$ such that for $i = 1, 2$,

$$N_i(k) \le B_i + \varepsilon \quad \forall k > K. \tag{4.13}$$

Lemma 4.1 also implies that there exists $K_1 > K$ such that

$$N_1(k) + N_2(k) \ge \frac{\sigma}{2} \quad \forall k \ge K_1. \tag{4.14}$$

Set

$$\begin{aligned} u(k) = \frac{N_2^{r_1^L}(k)}{N_1^{r_2^M}(k)} \exp\Bigg\{ &-\sum_{l=0}^{m}\sum_{s=k-l}^{k-1} r_1^L\left[a_{2l}^L N_2(s) + b_{2l}^L N_1(s)\right] \\ &+ \sum_{l=0}^{m}\sum_{s=k-l}^{k-1} r_2^M\left[a_{1l}^M N_1(s) + b_{1l}^M N_2(s) + c_{1l}^M(B_1 + \varepsilon)N_2(s)\right]\Bigg\}. \end{aligned} \tag{4.15}$$

So for all $k > K_2 > K_1 + m$, it follows from (1.1), (4.13), and (4.14) that

$$\begin{aligned}
\frac{u(k+1)}{u(k)} = \exp\Bigg\{ & r_1^L r_2(k) - \sum_{l=0}^{m} r_1^L \left[a_{2l}(k)N_2(k-l) + b_{2l}(k)N_1(k-l) + c_{2l}(k)N_2(k)N_1(k-l)\right] \\
& - r_2^M r_1(k) + \sum_{l=0}^{m} r_2^M \left[a_{1l}(k)N_1(k-l) + b_{1l}(k)N_2(k-l) + c_{1l}(k)N_1(k)N_2(k-l)\right] \\
& - \sum_{l=0}^{m}\sum_{s=k+1-l}^{k} r_1^L\left[a_{2l}^L N_2(s) + b_{2l}^L N_1(s)\right] + \sum_{l=0}^{m}\sum_{s=k-l}^{k-1} r_1^L\left[a_{2l}^L N_2(s) + b_{2l}^L N_1(s)\right] \\
& + \sum_{l=0}^{m}\sum_{s=k+1-l}^{k} r_2^M\left[a_{1l}^M N_1(s) + b_{1l}^M N_2(s) + c_{1l}^M(B_1+\varepsilon)N_2(s)\right] \\
& - \sum_{l=0}^{m}\sum_{s=k-l}^{k-1} r_2^M\left[a_{1l}^M N_1(s) + b_{1l}^M N_2(s) + c_{1l}^M(B_1+\varepsilon)N_2(s)\right]\Bigg\} \\
= \exp\Bigg\{ & \left[r_1^L r_2(k) - r_2^M r_1(k)\right] \\
& - \sum_{l=0}^{m} r_1^L\left[\left(a_{2l}(k) - a_{2l}^L\right)N_2(k-l) + \left(b_{2l}(k) - b_{2l}^L\right)N_1(k-l)\right] \\
& - \sum_{l=0}^{m} r_1^L c_{2l}(k)N_2(k)N_1(k-l) \\
& - \sum_{l=0}^{m} r_2^M\Big[\left(a_{1l}^M - a_{1l}(k)\right)N_1(k-l) + \left(b_{1l}^M - b_{1l}(k)\right)N_2(k-l) \\
& \qquad + \left(c_{1l}^M(B_1+\varepsilon) - c_{1l}(k)N_1(k)\right)N_2(k-l)\Big] \\
& - \sum_{l=0}^{m}\left[r_1^L b_{2l}^L - r_2^M a_{1l}^M\right]N_1(k) - \sum_{l=0}^{m}\left[r_1^L a_{2l}^L - r_2^M b_{1l}^M - r_2^M c_{1l}^M(B_1+\varepsilon)\right]N_2(k)\Bigg\} \\
\le \exp\Bigg\{ & -\sum_{l=0}^{m}\left[r_1^L b_{2l}^L - r_2^M a_{1l}^M\right]N_1(k) - \sum_{l=0}^{m}\left[r_1^L a_{2l}^L - r_2^M b_{1l}^M - r_2^M c_{1l}^M(B_1+\varepsilon)\right]N_2(k)\Bigg\}. \\
\le \exp\Big\{ & -\Delta^{\varepsilon}\left(N_1(k) + N_2(k)\right)\Big\} \\
\le \exp\Big\{ & -\Delta^{\varepsilon}\frac{\sigma}{2}\Big\}.
\end{aligned} \tag{4.16}$$

That is, for all $k > K_2$,

$$u(k) \le u(k_2)\exp\left\{-\Delta^{\varepsilon}\frac{\sigma}{2}(k-K_2)\right\}. \tag{4.17}$$

So from the definition of $u(k)$ it follows that

$$\begin{aligned} N_2^{r_1^L}(k) \le\ & N_1^{r_2^M}(k)\exp\left\{\sum_{l=0}^{m}\sum_{s=k-l}^{k-1} r_1^L\left[a_{2l}^L N_2(s)+b_{2l}^L N_1(s)\right]\right.\\ &\left.-\sum_{l=0}^{m}\sum_{s=k-l}^{k-1} r_2^M\left[a_{1l}^M N_1(s)+b_{1l}^M N_2(s)+c_{1l}^M(B_1+\varepsilon)N_2(s)\right]\right\}\\ &\times\exp\left\{-\Delta^{\varepsilon}\frac{\sigma}{2}(k-K_2)\right\}\\ \le\ & (2B_1)^{r_2^M}\exp\left\{\sum_{l=0}^{m}\sum_{s=k-l}^{k-1} 2r_1^L\left[a_{2l}^L B_2+b_{2l}^L B_1\right]\right.\\ &\left.+\sum_{l=0}^{m}\sum_{s=k-l}^{k-1} 2r_2^M\left[a_{1l}^M B_1+b_{1l}^M B_2+c_{1l}^M(B_1+\varepsilon)B_2\right]\right\}\\ &\times\exp\left\{-\Delta^{\varepsilon}\frac{\sigma}{2}(k-K_2)\right\}\longrightarrow 0 \quad \text{as } k\longrightarrow+\infty. \end{aligned} \tag{4.18}$$

The above analysis shows that

$$\lim_{k\to\infty} N_2(k)=0. \tag{4.19}$$

This completes the proof of Theorem 4.2. □

5. Examples

The following two examples show the feasibility of our results.

Example 5.1. Consider the following system

$$\begin{aligned} N_1(k+1) = N_1(k)\exp\big[&1.4-(2.52+0.02\sin(k))N_1(k)-0.5N_1(k-1)\\ &-0.55N_2(k)-0.3N_2(k-1)\\ &-0.1N_1(k)N_2(k)-0.09N_1(k)N_2(k-1)\big],\\ N_2(k+1) = N_2(k)\exp\big[&0.7-(2.62+0.02\sin(k))N_2(k)-1.2N_2(k-1)\\ &-0.01N_1(k)-0.01N_1(k-1)\\ &-0.09N_1(k)N_2(k)-0.1N_2(k)N_1(k-1)\big]. \end{aligned} \tag{5.1}$$

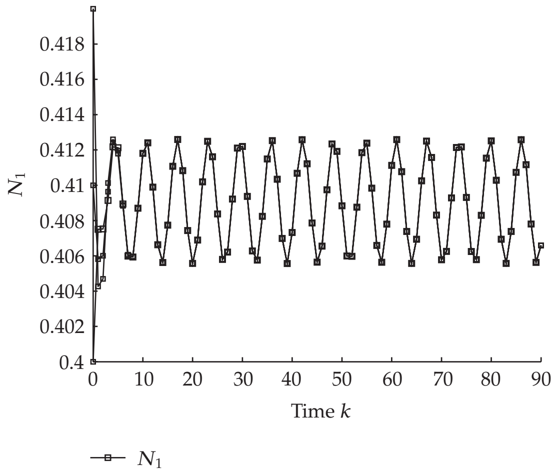

Figure 1: Dynamic behaviors of the species N_1 of system (5.1) with initial conditions $(N_1(p), N_2(p)) = (0.42, 0.175), (0.41, 0.178)$, and $(0.4, 0.18)$, $p = -1, 0$.

One could easily see that

$$\Delta_{11} = r_1^L - a_{11}^M B_1 - \sum_{l=0}^{1} \left[b_{1l}^M + c_{1l}^M B_1 \right] B_2 \approx 0.8271394917 > 0,$$
$$\Delta_{21} = r_2^L - a_{21}^M B_2 - \sum_{l=0}^{1} \left[b_{2l}^M + c_{2l}^M B_2 \right] B_1 \approx 0.3138443044 > 0. \tag{5.2}$$

Clearly, conditions (2.5) are satisfied. From Theorem 2.3, it follows that system (5.1) is permanent. Also, by simple computation, we have

$$\min\left\{ a_{10}^L, \frac{2}{B_1} - a_{10}^M \right\} - \left[a_{11}^M + 2\left(b_2^M + 2B_2 c^M \right) \right] \approx 0.1776281960,$$
$$\min\left\{ a_{20}^L, \frac{2}{B_2} - a_{20}^M \right\} - \left[a_{21}^M + 2\left(b_1^M + 2B_1 c^M \right) \right] \approx 0.613080483. \tag{5.3}$$

The above inequality shows that (H_0) is fulfilled. From Theorem 3.1, it follows that

$$\lim_{k \to \infty} \left(N_i(k) - N_i^*(k) \right) = 0, \quad i = 1, 2. \tag{5.4}$$

Figures 1 and 2 are the numeric simulations of the solution of system (5.1) with initial condition $(N_1(k), N_2(k)) = (0.42, 0.175), (0.41, 0.178)$, and $(0.4, 0.18)$, $k = -1, 0$.

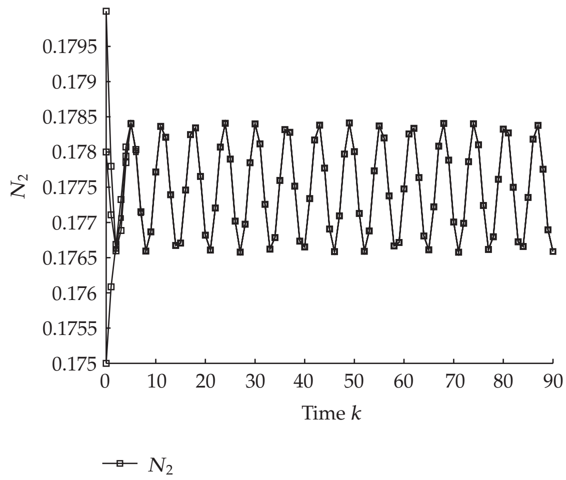

Figure 2: Dynamic behaviors of the species N_2 of system (5.1) with initial conditions $(N_1(k), N_2(k)) = (0.42, 0.175), (0.41, 0.178)$, and $(0.4, 0.18)$, $k = -1, 0$.

Example 5.2. Consider the following system:

$$\begin{aligned} N_1(k+1) = N_1(k)\exp\big[&1.4 - (1.5 + 0.2\sin(k))N_1(k) - 0.9N_1(k-1) \\ &- (0.3 + 0.2\sin(k))N_2(k) - 0.5N_2(k-1) \\ &- (0.4 + 0.1\cos(k))N_1(k)N_2(k) - 0.4N_1(k)N_2(k-1)\big], \\ N_2(k+1) = N_2(k)\exp\big[&0.7 - (1.1 + 0.5\sin(k))N_2(k) - 0.5N_2(k-1) \\ &- (1.1 + 0.2\cos(k))N_1(k) - 0.7N_1(k-1) \\ &- 2.3N_1(k)N_2(k) - 0.4N_2(k)N_1(k-1)\big]. \end{aligned} \tag{5.5}$$

One could easily see that

$$\begin{gathered} \frac{r_2^M}{r_1^L} = \frac{0.7}{1.4} = 0.5, \\ \frac{b_{20}^L}{a_{10}^M} = \frac{0.9}{1.7} \approx 0.5294, \qquad \frac{a_{20}^L}{b_{10}^M + c_{10}^M B_1} \approx \frac{0.6}{0.5 + 0.5 \times 1.1476} \approx 0.5588, \\ \frac{b_{21}^L}{a_{11}^M} = \frac{0.7}{0.9} \approx 0.7778, \qquad \frac{a_{21}^L}{b_{11}^M + c_{11}^M B_1} \approx \frac{0.5}{0.5 + 0.4 \times 1.1476} \approx 0.5214. \end{gathered} \tag{5.6}$$

Then, for $l = 0, 1$,

$$\frac{r_2^M}{r_1^L} - \min\left\{\frac{b_{2l}^L}{a_{1l}^M}, \frac{a_{2l}^L}{b_{1l}^M + c_{1l}^M B_1}\right\} < 0. \tag{5.7}$$

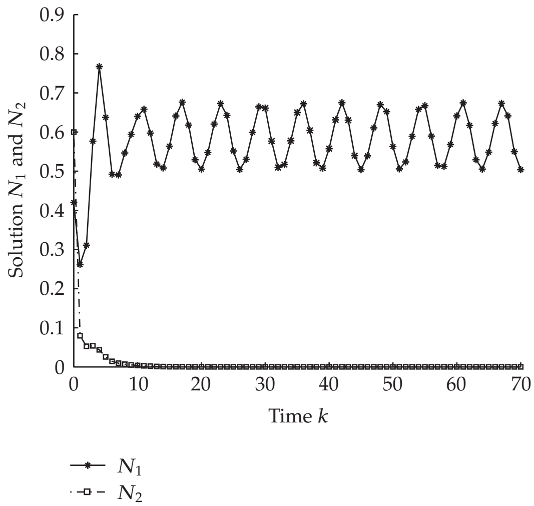

Figure 3: Dynamic behaviors of the species N_2 of system (5.5) with initial conditions $(N_1(k), N_2(k)) = (0.42, 0.6)$, $(k = -1, 0)$, respectively.

The above inequality shows that (H_1^*) is fulfilled. From Theorem 4.2, it follows that $\lim_{k\to\infty} N_2(k) = 0$. Numeric simulation of the dynamic behaviors of system (5.5) with the initial conditions $(N_1(k), N_2(k)) = (0.42, 0.6)$, $(k = -1, 0)$ is presented in Figure 3.

Remark 5.3. In the above two examples, we can take $\sin(k)$, $\cos(k)$ as the perturbation terms. Our numeric simulations show that if the perturbation terms are large enough, then those terms will greatly influence the dynamic behaviors of the system, and in some cases, may lead to the extinction of the species.

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