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Research Article

On the Global Asymptotic Stability of a Second-Order System of Difference Equations

Ibrahim Yalcinkaya

Department of Mathematics, Faculty of Education, University of Selcuk, 42099 Meram Yeni Yol, Konya, Turkey

Correspondence should be addressed to Ibrahim Yalcinkaya, iyalcinkaya1708@yahoo.com

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Recommended by Guang Zhang

A sufficient condition is obtained for the global asymptotic stability of the following system of difference equations $z_{n+1} = (t_n z_{n-1} + a)/(t_n + z_{n-1})$, $t_{n+1} = (z_n t_{n-1} + a)/(z_n + t_{n-1})$, $n = 0, 1, 2, \ldots$, where the parameter $a\epsilon(0,\infty)$ and the initial values $(z_k, t_k)\epsilon(0,\infty)$ (for $k = -1, 0$).

1. Introduction

Recently, there has been an increasing interest in the study of qualitative analyses of rational difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so forth. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the global behaviors of their solutions (see [1–15] and the references cited therein).

In [9, 10] Papaschinopoluos and Schinas studied the behavior of the positive solutions of the system of two Lyness difference equations

$$x_{n+1} = \frac{by_n + c}{x_{n-1}}, \quad y_{n+1} = \frac{dx_n + e}{y_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where b, c, d, e are positive constants and initial values x_{-1}, x_0, y_{-1}, y_0 are positive.

Iričanin and Stević [6] studied, among others, the following system:

$$x_{n+1}^{(1)} = \frac{1 + x_n^{(k)}}{x_{n-1}^{(k-1)}},\ x_{n+1}^{(2)} = \frac{1 + x_n^{(1)}}{x_{n-1}^{(k)}}, \ldots,\ x_{n+1}^{(k)} = \frac{1 + x_n^{(k-1)}}{x_{n-1}^{(k-2)}}, \tag{1.2}$$

where $k \in \mathbb{N}$.

In [2], Amleh et al. proved that all positive solutions of the difference equations

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \qquad x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \qquad x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-2} + x_{n-1}}, \tag{1.3}$$

where initial values $x_{-2},\ x_{-1},\ x_0$ are positive, converge to 1 as $n \to \infty$.

Moreover, Xianyi and Deming [8] proved that the unique positive equilibrium of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $a \in [0, \infty)$ and $x_{-1},\ x_0$ are positive, is globally asymptotically stable.

In [1], we extended the results obtained in [8] to the following difference equation:

$$x_{n+1} = \frac{x_n x_{n-k} + a}{x_n + x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where k is nonnegative integer, $a \in [0, \infty)$ and $x_{-k}, \ldots, x_0$ are positive and are globally asymptotically stable.

Also in [14], we extended the results obtained in [8] to the following system of difference equations:

$$z_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \quad t_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

where $a \in (0, \infty)$ and the initial values $(z_k, t_k) \in (0, \infty)$ (for $k = -1, 0$) are globally asymptotically stable.

In this paper, we consider the following system of difference equations:

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.7}$$

where $a \in (0, \infty)$ and the initial values $(z_k, t_k) \in (0, \infty)$ (for $k = -1, 0$). Our main aim is to investigate the global asymptotic behavior of its solutions.

It is clear that the change of variables

$$(z_n, t_n) = \left(\sqrt{a}x_n, \sqrt{a}y_n\right) \tag{1.8}$$

reduces the system (1.7) to the system

$$x_{n+1} = \frac{y_n x_{n-1} + 1}{y_n + x_{n-1}}, \quad y_{n+1} = \frac{x_n y_{n-1} + 1}{x_n + y_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.9}$$

where the initial values $(x_k, y_k) \in (0, \infty)$ for $(k = -1, 0)$.

We need the following definitions and theorem.

Let I be some interval of real numbers and let

$$f, g : I \times I \longrightarrow I \tag{1.10}$$

be continuously differentiable functions. Then, for all initial values $(x_k, y_k) \epsilon I$, $k = -1, 0$, the system of difference equations

$$x_{n+1} = f(x_n, y_{n-1}), \quad y_{n+1} = g(y_n, x_{n-1}), \quad n = 0, 1, 2, \ldots, \tag{1.11}$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$.

Definition 1.1. A point $(\overline{x}, \ \overline{y})$ is called an equilibrium point of the system (1.11) if

$$\overline{x} = f(\overline{x}, \overline{y}), \qquad \overline{y} = g(\overline{x}, \overline{y}). \tag{1.12}$$

It is easy to see that the system (1.9) has the unique positive equilibrium $(\overline{x}, \overline{y}) = (1, 1)$ [7].

Definition 1.2. Let $(\overline{x}, \overline{y})$ be an equilibrium point of the system (1.11).

(a) An equilibrium point $(\overline{x}, \overline{y})$ is said to be stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for every initial points (x_{-1}, y_{-1}) and (x_0, y_0) for which $\|(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})\| + \|(x_0, y_0) - (\overline{x}, \overline{y})\| < \delta$, the iterates (x_n, y_n) of (x_{-1}, y_{-1}) and (x_0, y_0) satisfy $\|(x_n, y_n) - (\overline{x}, \overline{y})\| < \varepsilon$ for all $n > 0$. An equilibrium point $(\overline{x}, \overline{y})$ is said to be unstable if it is not stable (the Euclidean norm in $\mathbb{R}^2$ given by $\|(x, y)\| = \sqrt{x^2 + y^2}$) is denoted by $\|\cdot\|$.

(b) An equilibrium point $(\overline{x}, \overline{y})$ is said to be asymptotically stable if there exists $r > 0$ such that $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ for all (x_{-1}, y_{-1}) and (x_0, y_0) that satisfy $\|(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})\| + \|(x_0, y_0) - (\overline{x}, \overline{y})\| < r$ [7].

Definition 1.3. Let $(\overline{x}, \overline{y})$ be an equilibrium point of a map $F = (f, g)$, where f and g are continuously differentiable functions at $(\overline{x}, \overline{y})$. The Jacobian matrix of F at $(\overline{x}, \overline{y})$ is the matrix

$$J_F(\overline{x}, \overline{y}) = \begin{bmatrix} \dfrac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \dfrac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \dfrac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \dfrac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{bmatrix}. \tag{1.13}$$

The linear map $J_F(\overline{x},\overline{y}) : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J_F(p,q)(\overline{x},\overline{y}) = \begin{bmatrix} \dfrac{\partial f}{\partial x}(\overline{x},\overline{y})x + \dfrac{\partial f}{\partial y}(\overline{x},\overline{y})y \\ \dfrac{\partial g}{\partial x}(\overline{x},\overline{y})x + \dfrac{\partial g}{\partial y}(\overline{x},\overline{y})y \end{bmatrix} \tag{1.14}$$

is called the linearization of the map F at $(\overline{x},\overline{y})$ [7].

Theorem 1.4 (linearized stability theorem [7]). *Let $F = (f,g)$ be a continuously differentiable function defined on an open set I in $\mathbb{R}^2$, and let $(\overline{x},\overline{y})$ in I be an equilibrium point of the map $F = (f,g)$.*

(a) *If all the eigenvalues of the Jacobian matrix $J_F(\overline{x},\overline{y})$ have modulus less than one, then the equilibrium point $(\overline{x},\overline{y})$ is asymptotically stable.*

(b) *If at least one of the eigenvalues of the Jacobian matrix $J_F(\overline{x},\overline{y})$ has modulus greater than one, then the equilibrium point $(\overline{x},\overline{y})$ is unstable.*

(c) *An equilibrium point $(\overline{x},\overline{y})$ of the map $F = (f,g)$ is locally asymptotically stable if and only if every solution of the characteristic equation*

$$\lambda^2 - \mathrm{tr}J_F(\overline{x},\overline{y})\lambda + \det J_F(\overline{x},\overline{y}) = 0 \tag{1.15}$$

lies inside the unit circle, that is, if and only if

$$\left|\mathrm{tr}J_F(\overline{x},\overline{y})\right| < 1 + \det J_F(\overline{x},\overline{y}) < 2. \tag{1.16}$$

Definition 1.5. Let $(\overline{x},\overline{y})$ be positive equilibrium point of the system (1.11).

A "string" of consecutive terms $\{x_s,\dots,x_m\}$ (resp., $\{y_s,\dots,y_m\}$), $s \geq -1$, $m \leq \infty$ is said to be a positive semicycle if $x_i \geqslant \overline{x}$ (resp., $y_i \geqslant \overline{y}$), $i\epsilon\{s,\dots,m\}$, $x_{s-1} < \overline{x}$ (resp., $y_{s-1} < \overline{y}$), and $x_{m+1} < \overline{x}$ (resp., $y_{m+1} < \overline{y}$).

A "string" of consecutive terms $\{x_s,\dots,x_m\}$ (resp., $\{y_s,\dots,y_m\}$), $s \geq -1$, $m \leq \infty$ is said to be a negative semicycle if $x_i < \overline{x}$ (resp., $y_i < \overline{y}$), $i\epsilon\{s,\dots,m\}$, $x_{s-1} \geqslant \overline{x}$ (resp., $y_{s-1} \geqslant \overline{y}$), and $x_{m+1} \geqslant \overline{x}$ (resp., $y_{m+1} \geqslant \overline{y}$).

A "string" of consecutive terms $\{(x_s,y_s),\dots,(x_m,y_m)\}$ is said to be a positive (resp., negative) semicycle if $\{x_s,\dots,x_m\}, \{y_s,\dots,y_m\}$ are positive (resp., negative) semicycles. Finally, a "string" of consecutive terms $\{(x_s,y_s),\dots,(x_m,y_m)\}$ is said to be a semicycle positive (resp., negative) with respect to x_n and negative (resp., positive) with respect to y_n if $\{x_s,\dots,x_m\}$ is a positive (resp., negative) semicycle and $\{y_s,\dots,y_m\}$ is a negative (resp., positive) semicycle [9].

We now make new definitions. These definitions can be used for different subsequences of $\{x_n\}$ (resp., $\{y_n\}$).

Definition 1.6. Let $(\overline{x}, \overline{y})$ be positive equilibrium point of the system (1.11).

A "string" of consecutive terms $\{x_{2s}, x_{2s+2}, \ldots, x_{2m}\}$ (resp., $\{y_{2s}, \ldots, y_{2m}\}$), $s \geq 1$, $m \leq \infty$ is said to be a positive sub-semicycle associated with $\{x_{2n}\}$ (resp., $\{y_{2n}\}$) if $x_i \geqslant \overline{x}$ (resp., $y_i \geqslant \overline{y}$), $i \epsilon \{2s, 2s+2, \ldots, 2m\}$, $x_{2s-2} < \overline{x}$ (resp., $y_{2s-2} < \overline{y}$), and $x_{2m+2} < \overline{x}$ (resp., $y_{2m+2} < \overline{y}$).

A "string" of consecutive terms $\{x_{2s}, x_{2s+2}, \ldots, x_{2m}\}$ (resp., $\{y_{2s}, \ldots, y_{2m}\}$), $s \geq 1$, $m \leq \infty$ is said to be a negative sub-semicycle associated with $\{x_{2n}\}$ (resp., $\{y_{2n}\}$) if $x_i < \overline{x}$ (resp., $y_i < \overline{y}$), $i \epsilon \{2s, 2s+2, \ldots, 2m\}$, $x_{2s-2} \geqslant \overline{x}$ (resp., $y_{2s-2} \geqslant \overline{y}$), and $x_{2m+2} \geqslant \overline{x}$ (resp., $y_{2m+2} \geqslant \overline{y}$).

A "string" of consecutive terms $\{(x_{2s}, y_{2s}), (x_{2s+2}, y_{2s+2}), \ldots, (x_{2m}, y_{2m})\}$ is said to be a positive (resp., negative) sub-semicycle if $\{x_{2s}, x_{2s+2}, \ldots, x_{2m}\}, \{y_{2s}, \ldots, y_{2m}\}$ are positive (resp., negative) sub-semicycles. Finally, a "string" of consecutive terms $\{(x_{2s}, y_{2s}), (x_{2s+2}, y_{2s+2}), \ldots, (x_{2m}, y_{2m})\}$ is said to be a sub-semicycle positive (resp., negative) with respect to x_{2n} and negative (resp., positive) with respect to y_{2n} if $\{x_{2s}, x_{2s+2}, \ldots, x_{2m}\}$ is a positive (resp., negative) sub-semicycle and $\{y_{2s}, \ldots, y_{2m}\}$ is a negative (resp., positive) sub-semicycle.

Definition 1.7. Let $(\overline{x}, \overline{y})$ be positive equilibrium point of the system (1.11).

A "string" of consecutive terms $\{x_{2s-1}, x_{2s+1}, \ldots, x_{2m+1}\}$ (resp., $\{y_{2s-1}, \ldots, y_{2m+1}\}$), $s \geq 1$, $m \leq \infty$, is said to be a positive sub-semicycle associated with $\{x_{2n-1}\}$ (resp., $\{y_{2n-1}\}$) if $x_i \geqslant \overline{x}$ (resp., $y_i \geqslant \overline{y}$), $i \epsilon \{2s+1, 2s+1, \ldots, 2m+1\}$, $x_{2s-3} < \overline{x}$ (resp., $y_{2s-3} < \overline{y}$), and $x_{2m+3} < \overline{x}$ (resp., $y_{2m+3} < \overline{y}$).

A "string" of consecutive terms $\{x_{2s-1}, x_{2s+1}, \ldots, x_{2m+1}\}$ (resp., $\{y_{2s-1}, \ldots, y_{2m+1}\}$), $s \geq 1$, $m \leq \infty$, is said to be a negative sub-semicycle associated with $\{x_{2n-1}\}$ (resp., $\{y_{2n-1}\}$) if $x_i < \overline{x}$ (resp., $y_i < \overline{y}$), $i \epsilon \{2s-1, 2s+1, \ldots, 2m+1\}$, $x_{2s-3} \geqslant \overline{x}$ (resp., $y_{2s-3} \geqslant \overline{y}$), and $x_{2m+3} \geqslant \overline{x}$ (resp., $y_{2m+3} \geqslant \overline{y}$).

A "string" of consecutive terms $\{(x_{2s-1}, y_{2s-1}), (x_{2s+1}, y_{2s+1}), \ldots, (x_{2m+1}, y_{2m+1})\}$ is said to be a positive (resp., negative) sub-semicycle if $\{x_{2s-1}, x_{2s+1}, \ldots, x_{2m+1}\}, \{y_{2s-1}, \ldots, y_{2m+1}\}$ are positive (resp., negative) sub-semicycles. Finally, a "string" of consecutive terms $\{(x_{2s-1}, y_{2s-1}), (x_{2s+1}, y_{2s+1}), \ldots, (x_{2m+1}, y_{2m+1})\}$ is said to be a sub-semicycle positive (resp., negative) with respect to x_{2n-1} and negative (resp., positive) with respect to y_{2n-1}, if $\{x_{2s-1}, x_{2s+1}, \ldots, x_{2m+1}\}$ is a positive (resp., negative) sub-semicycle and $\{y_{2s-1}, y_{2s+1}, \ldots, y_{2m+1}\}$ is a negative (resp., positive) sub-semicycle.

2. Some auxiliary results

In this section, we give the following lemmas which show us the behavior of semicycles of positive solutions of system (1.9). The proof of Lemma 2.1 is clear from (1.9). So, it will be omitted.

Lemma 2.1. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of the system* (1.9) *and consider the following cases:*

(Case a) $x_0 = x_{-1} = 1$;

(Case b) $y_0 = y_{-1} = 1$;

(Case c) $x_0 = y_0 = 1$;

(Case d) $x_{-1} = y_{-1} = 1$.

If one of the above cases occurs, then every positive solution of system (1.9) *is equal to* $(1, 1)$.

Lemma 2.2. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a positive solution of the system* (1.9) *which is not eventually equal to* $(1,1)$. *Then the following statements are true:*

(i) $(x_{n+1} - x_{n-1})(x_{n-1} - 1) < 0$ and $(y_{n+1} - y_{n-1})(y_{n-1} - 1) < 0$ for all $n \geq 0$;

(ii) $(x_{n+1} - 1)(x_{n-1} - 1)(y_n - 1) > 0$ and $(y_{n+1} - 1)(y_{n-1} - 1)(x_n - 1) > 0$ for all $n \geq 0$.

Proof. In view of system (1.9), we obtain

$$\begin{aligned} x_{n+1} - x_{n-1} &= \frac{(1 - x_{n-1})(1 + x_{n-1})}{y_n + x_{n-1}}, \\ y_{n+1} - y_{n-1} &= \frac{(1 - y_{n-1})(1 + y_{n-1})}{x_n + y_{n-1}}, \\ x_{n+1} - 1 &= \frac{(x_{n-1} - 1)(y_n - 1)}{y_n + x_{n-1}}, \\ y_{n+1} - 1 &= \frac{(y_{n-1} - 1)(x_n - 1)}{x_n + y_{n-1}} \end{aligned} \tag{2.1}$$

for $n = 0, 1, 2, \ldots$, from which the inequalities in (i) and (ii) follow. □

Lemma 2.3. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1.9) *and suppose that the case,*

(Case 1) $x_k, y_k > 1$ *(for* $k = -1, 0$*), holds.*

Then, (x_{2n-1}, y_{2n-1}) *and* (x_{2n}, y_{2n}) *are positive sub-semicycles of system* (1.9) *with an infinite number of terms and they monotonically tend to the positive equilibrium* $(\overline{x}, \overline{y}) = (1, 1)$.

Proof. If $x_k, y_k > 1$ (for $k = -1, 0$), then by Lemma 2.2(ii), it follows that

$$x_{2n-1}, x_{2n}, y_{2n-1}, y_{2n} > 1 \quad \forall\, n \geq 0, \tag{2.2}$$

that is, these positive sub-semicycles have an infinite number of terms. Furthermore, according to Lemma 2.2(i), we know that (x_{2n-1}, y_{2n-1}) and (x_{2n}, y_{2n}) are strictly decreasing for all $n \geq 0$. So, the limits

$$\begin{aligned} \lim_{n\to\infty} x_{2n-1} &= l_1, \qquad \lim_{n\to\infty} x_{2n} = l_2, \\ \lim_{n\to\infty} y_{2n-1} &= l_3, \qquad \lim_{n\to\infty} y_{2n} = l_4. \end{aligned} \tag{2.3}$$

exist and are finite. From (1.9), we can write

$$x_{2n+1} = \frac{y_{2n}x_{2n-1} + 1}{y_{2n} + x_{2n-1}}, \quad y_{2n+1} = \frac{x_{2n}y_{2n-1} + 1}{x_{2n} + y_{2n-1}}, \quad n = 0, 1, 2, \ldots, \tag{2.4}$$

taking limits on both sides of (2.4), we have

$$l_1 = \frac{l_4 l_1 + 1}{l_4 + l_1}, \qquad l_3 = \frac{l_2 l_3 + 1}{l_2 + l_3}, \tag{2.5}$$

and thus $l_1 = l_3 = 1$. Similarly, one can see that $l_2 = l_4 = 1$. Therefore, the proof is complete. □

Lemma 2.4. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1.9)*, and consider the following cases:*

(Case 2) $x_{-1}, y_{-1} > 1$ *and* $x_0, y_0 < 1$;

(Case 3) $x_{-1}, y_{-1} < 1$ *and* $x_0, y_0 > 1$;

(Case 4) $x_{-1}, x_0 < 1$ *and* $y_{-1}, y_0 > 1$;

(Case 5) $x_{-1}, x_0 > 1$ *and* $y_{-1}, y_0 < 1$;

(Case 6) $x_{-1}, y_{-1}, y_0 < 1$ *and* $x_0 > 1$;

(Case 7) $x_0, y_{-1}, y_0 < 1$ *and* $x_{-1} > 1$;

(Case 8) $x_{-1}, x_0, y_{-1} < 1$ *and* $y_0 > 1$;

(Case 9) $x_{-1}, x_0, y_0 < 1$ *and* $y_{-1} > 1$;

(Case 10) $x_{-1}, x_0, y_{-1}, y_0 < 1$.

If one of the above cases occurs, then the following hold.

(i) *Every positive sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{x_{2n}\}$ *(resp.,* $\{y_{2n-1}\}$ *and* $\{y_{2n}\}$*) of system* (1.9) *consists of one term.*

(ii) *Every negative sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{x_{2n}\}$ *(resp.,* $\{y_{2n-1}\}$ *and* $\{y_{2n}\}$*) of system* (1.9) *consists of two terms.*

(iii) *Every positive sub-semicycle of length one is followed by a negative sub-semicycle of length two.*

(iv) *Every negative sub-semicycle of length two is followed by a positive sub-semicycle of length one.*

Proof. If Case 2 occurs, then in view of inequality (ii) of Lemma 2.2 we have: $x_1, y_1 < 1$; $x_3, y_3 < 1$ and

$$x_{6n+5}, y_{6n+5} > 1; \quad x_{6n+7}, y_{6n+7} < 1, \quad x_{6n+9}, y_{6n+9} < 1 \quad \forall n \geq 0, \tag{2.6}$$

which imply that every positive sub-semicycle associated with $\{x_{2n-1}\}$ and $\{y_{2n-1}\}$ of system (1.9) of length one is followed by a negative sub-semicycle of length two, which in turn is followed by a positive sub-semicycle of length one.

Similarly, if Case 2 occurs, then in view of inequality (ii) of Lemma 2.2 we have

$$x_{6n+2}, y_{6n+2} > 1; \quad x_{6n+4}, y_{6n+4} < 1, \quad x_{6n+6}, y_{6n+6} < 1 \quad \forall n \geq 0, \tag{2.7}$$

which imply that every positive sub-semicycle associated with $\{x_{2n}\}$ and $\{y_{2n}\}$ of system (1.9) of length two is followed by a negative sub-semicycle of length four, which in turn is followed by a positive sub-semicycle of length two.

Proofs of the other cases are similar, so they will be omitted. Therefore, the proof is complete. □

We omit the proofs of the following two results since they can easily be obtained in a way similar to the proof of Lemma 2.4.

Lemma 2.5. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1.9) *and consider the following cases:*

(Case 11) $x_{-1}, y_{-1}, y_0 > 1$ *and* $x_0 < 1$;

(Case 12) $x_{-1}, x_0, y_0 > 1$ *and* $y_{-1} < 1$;

(Case 13) $x_{-1}, y_0 > 1$ *and* $x_0, y_{-1} < 1$.

If one of the above cases occurs, then the following hold.

(i) (x_{2n-1}, y_{2n}) *is a positive sub-semicycle of system* (1.9) *with an infinite number of terms (monotonically tend to the positive equilibrium* $(\overline{x}, \overline{y}) = (1, 1)$*).*

(ii) *Every positive sub-semicycle associated with* $\{x_{2n}\}$ *and* $\{y_{2n-1}\}$ *of system* (1.9) *consists of one term.*

(iii) *Every negative sub-semicycle associated with* $\{x_{2n}\}$ *and* $\{y_{2n-1}\}$ *of system* (1.9) *consists of two terms.*

(iv) *Every positive sub-semicycle of length one is followed by a negative sub-semicycle of length two.*

(v) *Every negative sub-semicycle of length two is followed by a positive sub-semicycle of length one.*

Lemma 2.6. *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1.9) *and consider the following cases:*

(Case 14) $x_{-1}, x_0, y_{-1} > 1$ *and* $y_0 < 1$;

(Case 15) $x_0, y_{-1}, y_0 > 1$ *and* $x_{-1} < 1$;

(Case 16) $x_0, y_{-1} > 1$ *and* $x_{-1}, y_0 < 1$.

If one of the above cases occurs, then the following hold.

(i) (x_{2n}, y_{2n-1}) *is a positive sub-semicycle of system* (1.9) *with an infinite number of terms (monotonically tend to the positive equilibrium* $(\overline{x}, \overline{y}) = (1, 1)$*).*

(ii) *Every positive sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{y_{2n}\}$ *of system* (1.9) *consists of one term.*

(iii) *Every negative sub-semicycle associated with* $\{x_{2n-1}\}$ *and* $\{y_{2n}\}$ *of system* (1.9) *consists of two terms.*

(iv) *Every positive sub-semicycle of length one is followed by a negative sub-semicycle of length two.*

(v) *Every negative sub-semicycle of length two is followed by a positive sub-semicycle of length one.*

3. Main result

Theorem 3.1. *The positive equilibrium point* $(\overline{x}, \overline{y}) = (1, 1)$ *of the system* (1.9) *is globally asymptotically stable.*

Proof. We must show that the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.9) is both locally asymptotically stable and $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ (or equivalently

$(x_{2n-1}, y_{2n-1}) \to (\overline{x}, \overline{y})$ and $(x_{2n}, y_{2n}) \to (\overline{x}, \overline{y})$ as $n \to \infty$). The characteristic equation of the system (1.9) about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ is

$$\lambda^2 - 0.\lambda + 0 = 0, \tag{3.1}$$

and so it is clear from Theorem 1.4 that positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.9) is locally asymptotically stable. It remains to verify that every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (1.9) converges to $(\overline{x}, \overline{y}) = (1, 1)$ as $n \to \infty$. Namely, we want to prove

$$\begin{aligned} \lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} x_{2n-1} = \overline{x} = 1, \\ \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} y_{2n-1} = \overline{y} = 1. \end{aligned} \tag{3.2}$$

If the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of (1.9) is nonoscillatory about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.9), then according to Lemmas 2.1 and 2.3, respectively, we know that the solution is either eventually equal to $(1, 1)$ or an eventually positive one which has an infinite number of terms and monotonically tends the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.9) and so (3.2) holds. Therefore, it suffices to prove that (3.2) holds for strictly oscillatory solutions. Now, let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be strictly oscillatory about the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of the system (1.9). By virtue of Lemmas 2.2(ii) and 2.4, one can see that every positive sub-semicycle associated with $\{x_{2n-1}\}$ (resp. $\{x_{2n}\}, \{y_{2n-1}\}, \{y_{2n}\}$) of this solution has one term, and every negative sub-semicycle associated with $\{x_{2n-1}\}$ (resp., $\{x_{2n}\}, \{y_{2n-1}\}, \{y_{2n}\}$) except perhaps for the first has exactly two terms. Every positive sub-semicycle of length one is followed by a negative sub-semicycle of length two.

We consider the sub-semicycles associated with $\{x_{2n}\}$ and $\{y_{2n}\}$.

For the convenience of statement, without loss of generality, we use the following notation. We denote by x_{2p} and y_{2p} the terms of a positive sub-semicycle of length one, followed by x_{2p+2}, x_{2p+4} and y_{2p+2}, y_{2p+4} which are the terms of a negative sub-semicycle of length two. Afterwards, there are the positive sub-semicycles x_{2p+6} and y_{2p+6} in turn followed by the negative sub-semicycles, and so on.

Therefore, we have the following sequences consisting of positive and negative sub-semicycles (for $n = 0, 1, \ldots$):

$$\{x_{2p+6n}\}_{n=0}^{\infty}, \qquad \{x_{2p+6n+2}, x_{p+6n+4}\}_{n=0}^{\infty}, \qquad \{y_{2p+6n}\}_{n=0}^{\infty}, \qquad \{y_{2p+6n+2}, y_{2p+6n+4}\}_{n=0}^{\infty}. \tag{3.3}$$

We have the following assertions:

(i) $x_{2p+6n+2} < x_{2p+6n+4}$ and $y_{2p+6n+2} < y_{2p+6n+4}$;

(ii) $x_{2p+6n}x_{2p+6n+2} > 1$ and $y_{2p+6n}y_{2p+6n+2} > 1$;

(iii) $x_{2p+6n+4}x_{2p+6n+6} < 1$ and $y_{2p+6n+4}y_{2p+6n+6} < 1$.

In fact, inequality (i) immediately follows from Lemma 2.2(i). From the observations that

$$\begin{aligned} x_{2p+6n+2} &= \frac{y_{2p+6n+1}x_{2p+6n}+1}{y_{2p+6n+1}+x_{2p+6n}} > \frac{y_{2p+6n+1}x_{2p+6n}+1}{y_{2p+6n+1}x_{2p+6n}^2+x_{2p+6n}} = \frac{1}{x_{2p+6n}}, \\ y_{2p+6n+2} &= \frac{x_{2p+6n+1}y_{2p+6n}+1}{x_{2p+6n+1}+y_{2p+6n}} > \frac{x_{2p+6n+1}y_{2p+6n}+1}{x_{2p+6n+1}y_{2p+6n}^2+y_{2p+6n}} = \frac{1}{y_{2p+6n}}, \end{aligned} \tag{3.4}$$

one can see that (ii) is valid.

As for (iii), it is obtained from

$$\begin{aligned} x_{2p+6n+6} &= \frac{y_{2p+6n+5}x_{2p+6n+4}+1}{y_{2p+6n+5}+x_{2p+6n+4}} < \frac{y_{2p+6n+5}x_{2p+6n+4}+1}{y_{2p+6n+5}x_{2p+6n+4}^2+x_{2p+6n+4}} = \frac{1}{x_{2p+6n+4}}, \\ y_{2p+6n+6} &= \frac{x_{2p+6n+5}y_{2p+6n+4}+1}{x_{2p+6n+5}+y_{2p+6n+4}} < \frac{x_{2p+6n+5}y_{2p+6n+4}+1}{x_{2p+6n+5}y_{2p+6n+4}^2+y_{2p+6n+4}} = \frac{1}{y_{2p+6n+4}}, \end{aligned} \tag{3.5}$$

for $n = 0, 1, 2, \ldots$.

Combining the above inequalities, we derive

$$\begin{aligned} \frac{1}{x_{2p+6n}} < x_{2p+6n+2} < x_{2p+6n+4} < \frac{1}{x_{2p+6n+6}}, \\ \frac{1}{y_{2p+6n}} < y_{2p+6n+2} < y_{2p+6n+4} < \frac{1}{y_{2p+6n+6}}. \end{aligned} \tag{3.6}$$

From (3.6), one can see that $\{x_{2p+6n+2}\}_{n=0}^{\infty}$ and $\{y_{2p+6n+2}\}_{n=0}^{\infty}$ are increasing with upper bound 1. So the limits

$$\lim_{n\to\infty} x_{2p+6n+2} = L_1, \qquad \lim_{n\to\infty} y_{2p+6n+2} = L_2 \tag{3.7}$$

exist and are finite. Accordingly, in view of (3.6), we obtain

$$\begin{aligned} \lim_{n\to\infty} x_{2p+6n+4} &= L_1, & \lim_{n\to\infty} x_{2p+6n+6} &= \frac{1}{L_1}, \\ \lim_{n\to\infty} y_{2p+6n+4} &= L_2, & \lim_{n\to\infty} y_{2p+6n+6} &= \frac{1}{L_2}. \end{aligned} \tag{3.8}$$

Now, we consider the sub-semicycles associated with $\{x_{2n-1}\}$ and $\{y_{2n-1}\}$.

Similarly, for the convenience of statement, without loss of generality, we use the following notation. We denote by x_{2p+1} and y_{2p+1} the terms of a positive sub-semicycle of length one, followed by x_{2p+3}, x_{2p+5} and y_{2p+3}, y_{2p+5} which are the terms of a negative sub-semicycle of length two. Afterwards, there are the positive sub-semicycles x_{2p+7} and y_{2p+7} in turn followed by the negative sub-semicycles, and so on.

Therefore, we have the following sequences consisting of positive and negative sub-semicycles (for $n = 0, 1, \ldots$):

$$\{x_{2p+6n+1}\}_{n=0}^{\infty}, \qquad \{x_{2p+6n+3}, x_{p+6n+5}\}_{n=0}^{\infty}, \qquad \{y_{2p+6n+1}\}_{n=0}^{\infty}, \qquad \{y_{2p+6n+3}, y_{2p+6n+5}\}_{n=0}^{\infty}. \tag{3.9}$$

We have the following assertions:

(i) $x_{2p+6n+3} < x_{2p+6n+5}$ and $y_{2p+6n+3} < y_{2p+6n+5}$;

(ii) $x_{2p+6n+1}x_{2p+6n+3} > 1$ and $y_{2p+6n+1}y_{2p+6n+3} > 1$;

(iii) $x_{2p+6n+5}x_{2p+6n+7} < 1$ and $y_{2p+6n+5}y_{2p+6n+7} < 1$.

Combining the above inequalities, we derive

$$\begin{gathered}\frac{1}{x_{2p+6n+1}} < x_{2p+6n+3} < x_{2p+6n+5} < \frac{1}{x_{2p+6n+7}},\\ \frac{1}{y_{2p+6n+1}} < y_{2p+6n+3} < y_{2p+6n+5} < \frac{1}{y_{2p+6n+7}}.\end{gathered} \tag{3.10}$$

From (3.10), one can see that $\{x_{2p+6n+3}\}_{n=0}^{\infty}$ and $\{y_{2p+6n+3}\}_{n=0}^{\infty}$ are increasing with upper bound 1. So the limits

$$\lim_{n\to\infty} x_{2p+6n+3} = L_3, \qquad \lim_{n\to\infty} y_{2p+6n+3} = L_4 \tag{3.11}$$

exist and are finite. Accordingly, in view of (3.10), we obtain

$$\begin{aligned}\lim_{n\to\infty} x_{2p+6n+5} &= L_3, \qquad \lim_{n\to\infty} x_{2p+6n+7} = \frac{1}{L_3},\\ \lim_{n\to\infty} y_{2p+6n+5} &= L_4, \qquad \lim_{n\to\infty} y_{2p+6n+7} = \frac{1}{L_4}.\end{aligned} \tag{3.12}$$

It suffices to verify that

$$L_1 = L_2 = L_3 = L_4 = 1. \tag{3.13}$$

To this end, note that

$$x_{2p+6n+6} = \frac{y_{2p+6n+5}x_{2p+6n+4} + 1}{y_{2p+6n+5} + x_{2p+6n+4}}, \qquad y_{2p+6n+6} = \frac{x_{2p+6n+5}y_{2p+6n+4} + 1}{x_{2p+6n+5} + y_{2p+6n+4}}. \tag{3.14}$$

Take the limits on both sides of the above equality and obtain

$$\frac{1}{L_1} = \frac{L_4 \cdot L_1 + 1}{L_4 + L_1}, \qquad \frac{1}{L_2} = \frac{L_3 \cdot L_2 + 1}{L_3 + L_2}, \tag{3.15}$$

which imply that $L_1 = L_2 = 1$. Similarly, one can see that $L_3 = L_4 = 1$.

Moreover, by virtue of Lemmas 2.2(ii) and 2.5 (resp., 2.6), one can see that (3.2) holds. Therefore, the proof is complete. □

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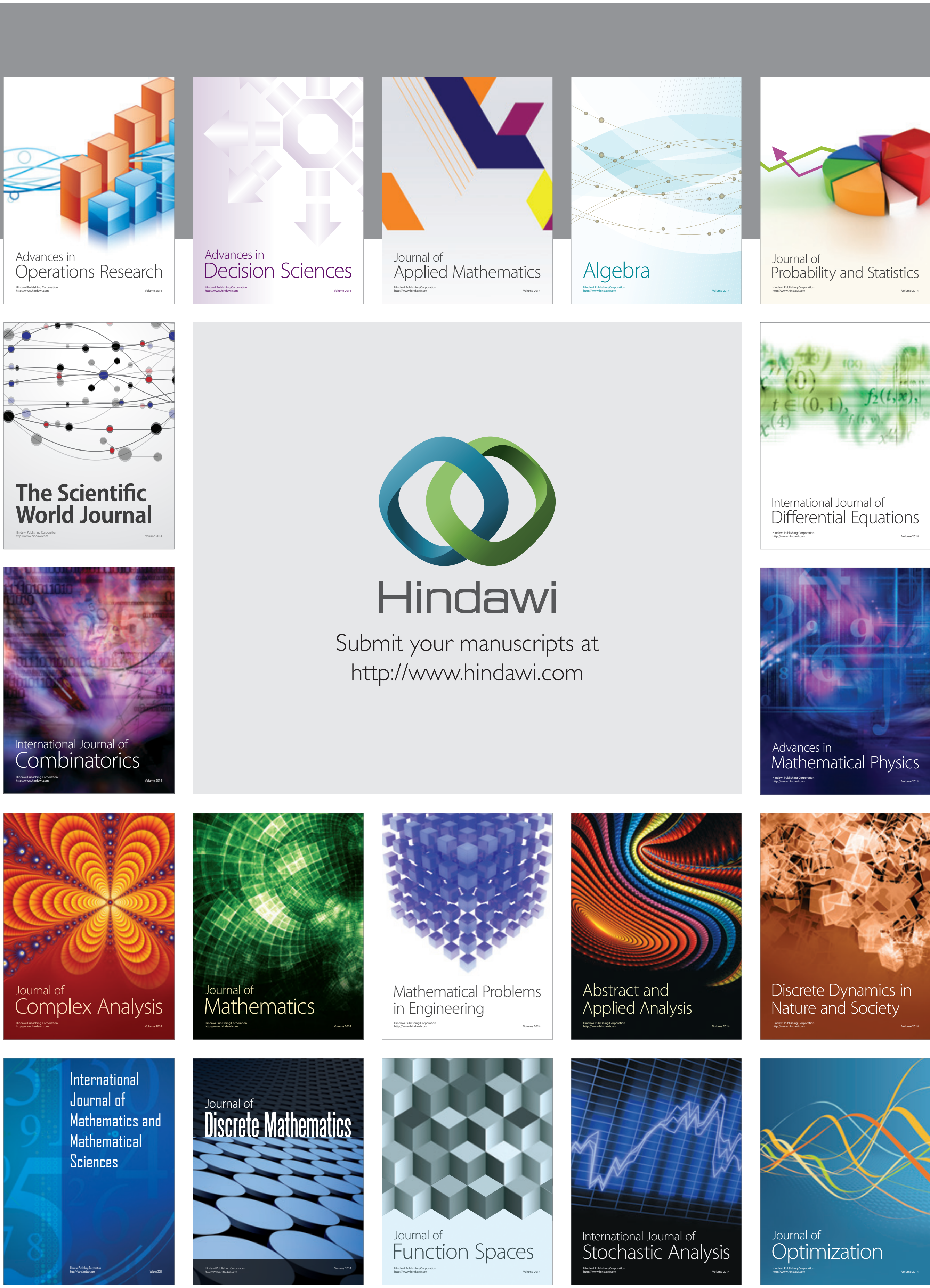
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