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Research Article

Falling d-Ideals in d-Algebras

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Based on the theory of a falling shadow which was first formulated by Wang (1985), a theoretical approach of the ideal structure in d-algebras is established. The notions of a falling d-subalgebra, a falling d-ideal, a falling BCK-ideal, and a falling $d^{\sharp}$-ideal of a d-algebra are introduced. Some fundamental properties are investigated. Relations among a falling d-subalgebra, a falling d-ideal, a falling BCK-ideal, and a falling $d^{\sharp}$-ideal are stated. Characterizations of falling d-ideals and falling $d^{\sharp}$-ideals are discussed. A relation between a fuzzy d-subalgebra and a falling d-subalgebra is provided.

1. Introduction

Iséki and Tanaka introduced two classes of abstract algebras BCK-algebras and BCI-algebras [1, 2]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have several connections with other areas of investigation, such as: lattice ordered groups, MV-algebras, Wajsberg algebras, and implicative commutative semigroups. Font et al. [3] have discussed Wajsberg algebras which are term-equivalent to MV-algebras. Mundici [4] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras. Meng [5] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Neggers and Kim [6] introduced the notion of d-algebras which is another useful generalization of BCK-algebras. They investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. After that, some further aspects were studied in [7, 8]. Neggers et al. [9] introduced the concept of d-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition, they discussed a method of fuzzification of a wide class of algebraic systems onto $[0,1]$ along with some consequences.

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [10] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [11] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [12]. Tan et al. [13, 14] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Kang [15] established a theoretical approach to define a fuzzy positive implicative ideal in a BCK-algebra based on the theory of falling shadows. They provided relations between falling fuzzy positive implicative ideals and falling fuzzy ideals. They also considered relations between fuzzy positive implicative ideals and falling fuzzy positive implicative ideals. Jun and Kang [16] considered the fuzzification of generalized Tarski filters of generalized Tarski algebras and investigated related properties. They established characterizations of a fuzzy-generalized Tarski filter and introduced the notion of falling fuzzy-generalized Tarski filters in generalized Tarski algebras based on the theory of falling shadows. They provided relations between fuzzy-generalized Tarski filters and falling fuzzy-generalized Tarski filters and established a characterization of a falling fuzzy-generalized Tarski filter.

In this paper, we establish a theoretical approach to define a falling d-subalgebra, a falling d-ideal, a falling BCK-ideal, and a falling $d^{\sharp}$-ideal in d-algebras based on the theory of falling shadows which was first formulated by Wang [12]. We provide relations among a falling d-subalgebra, a falling d-ideal, a falling BCK-ideal, and a falling $d^{\sharp}$-ideal. We consider characterizations of falling d-ideals and falling $d^{\sharp}$-ideals and discuss a relation between a fuzzy d-subalgebra and a falling d-subalgebra.

2. Preliminaries

A *d-algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(i) $x * x = 0$,

(ii) $0 * x = 0$,

(iii) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y \in X$.

A *BCK-algebra* is a d-algebra $(X, *, 0)$ satisfying the following additional axioms:

(iv) $((x * y) * (x * z)) * (z * y) = 0$,

(v) $(x * (x * y)) * y = 0$,

for all $x, y, z \in X$.

Any BCK-algebra $(X, *, 0)$ satisfies the following conditions:

(a1) (for all $x, y \in X$) $((x * y) * x = 0)$,

(a2) (for all $x, y, z \in X$) $(((x * z) * (y * z)) * (x * y) = 0)$.

A subset I of a BCK-algebra X is called a *BCK-ideal* of X if it satisfies

(b1) $0 \in I$,

(b2) (for all $x \in X$) (for all $y \in I$) $(x * y \in I \Rightarrow x \in I)$.

We now display the basic theory on falling shadows. We refer the reader to the papers [10–14] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathcal{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.1}$$

and for each $E \in \mathcal{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \tag{2.2}$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hypermeasurable structure on U if $\mathcal{B}$ is a σ-field in $\mathcal{P}(U)$ and $\dot{U} \subseteq \mathcal{B}$. Given a probability space $(\Omega, \mathcal{A}, P)$ and a hypermeasurable structure $(\mathcal{P}(U), \mathcal{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathcal{P}(U)$ which is $\mathcal{A}$-$\mathcal{B}$ measurable, that is,

$$(\forall C \in \mathcal{B}) \quad \left(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathcal{A}\right). \tag{2.3}$$

Suppose that ξ is a random set on U. Let

$$\widetilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U. \tag{2.4}$$

Then $\widetilde{H}$ is a kind of fuzzy set in U. We call $\widetilde{H}$ a falling shadow of the random set ξ, and ξ is called a cloud of $\widetilde{H}$.

For example, $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\widetilde{H}$ be a fuzzy set in U and let $\widetilde{H}_t := \{u \in U \mid \widetilde{H}(u) \geq t\}$ be a t-cut of $\widetilde{H}$. Then

$$\xi : [0,1] \longrightarrow \mathcal{P}(U), \qquad t \longmapsto \widetilde{H}_t \tag{2.5}$$

is a random set and ξ is a cloud of $\widetilde{H}$. We will call ξ defined above as the cut-cloud of $\widetilde{H}$ (see [10]).

3. Falling d-Subalgebras/Ideals

In what follows let X denote a d-algebra unless otherwise specified.

A nonempty subset S of X is called a *d-subalgebra* of X (see [8]) if $x * y \in S$ whenever $x \in S$ and $y \in S$.

A subset I of X is called a *BCK-ideal* of X (see [8]) if it satisfies conditions (b1) and (b2).

A subset I of X is called a *d-ideal* of X (see [8]) if it satisfies conditions (b2) and (b3) (for all $x, y \in X)(x \in I \Rightarrow x * y \in I)$.

Definition 3.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let

$$\xi : \Omega \longrightarrow \mathcal{P}(X) \tag{3.1}$$

be a random set. If $\xi(\omega)$ is a d-subalgebra (resp., BCK-ideal and d-ideal) of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\widetilde{H}$ of the random set ξ, that is,

$$\widetilde{H}(x) = P(\omega \mid x \in \xi(\omega)) \tag{3.2}$$

is called a *falling d-subalgebra* (resp., *falling BCK-ideal* and *falling d-ideal*) of X.

Example 3.2. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$F(X) := \{f \mid f : \Omega \longrightarrow X \text{ is a mapping}\}. \tag{3.3}$$

Define an operation $\circledast$ on $F(X)$ by

$$(\forall \omega \in \Omega) \quad ((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{3.4}$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. It is routine to check that $(F(X); \circledast, \theta)$ is a d-algebra. For any d-subalgebra (resp., BCK-ideal and d-ideal) A of X and $f \in F(X)$, let

$$\begin{gathered} A_f := \{\omega \in \Omega \mid f(\omega) \in A\}, \\ \xi : \Omega \longrightarrow \mathcal{P}(F(X)), \qquad \omega \longmapsto \{f \in F(X) \mid f(\omega) \in A\}. \end{gathered} \tag{3.5}$$

Then $A_f \in \mathcal{A}$ and $\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\}$ is a d-subalgebra (resp., BCK-ideal and d-ideal) of $F(X)$. Since

$$\xi^{-1}(f) = \{\omega \in \Omega \mid f \in \xi(\omega)\} = \{\omega \in \Omega \mid f(\omega) \in A\} = A_f \in \mathcal{A}, \tag{3.6}$$

ξ is a random set of $F(X)$. Hence the falling shadow $\widetilde{H}(f) = P(\omega \mid f(\omega) \in A)$ on $F(X)$ is a falling d-subalgebra (resp., falling BCK-ideal and falling d-ideal) of $F(X)$.

Example 3.3. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

(3.7)

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \emptyset, & \text{if } t \in [0, 0.2), \\ \{0, a, c\}, & \text{if } t \in [0.2, 0.6), \\ X, & \text{if } t \in [0.6, 1]. \end{cases} \tag{3.8}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-subalgebra of X.

Example 3.4. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	0
c	c	c	c	0

(3.9)

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0, a, b\}, & \text{if } t \in [0, 0.9), \\ X, & \text{if } t \in [0.9, 1]. \end{cases} \tag{3.10}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling BCK-ideal of X.

Example 3.5. Let $X := \{0, a, b, c, d\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	0
c	c	c	b	0	c
d	c	c	a	a	0

(3.11)

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set $\xi : [0,1] \rightarrow \mathcal{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0, a\}, & \text{if } t \in [0, 0.3), \\ X, & \text{if } t \in [0.3, 0.8), \\ \emptyset, & \text{if } t \in [0.8, 1]. \end{cases} \tag{3.12}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-ideal of X.

Note that the falling shadow $\widetilde{H}$ of ξ in Example 3.4 is not a falling d-subalgebra of X because if we take $t \in [0, 0.9)$, then $\xi(t) = \{0, a, b\}$ is not a d-subalgebra of X. This shows that, in a d-algebra, a falling BCK-ideal need not be a falling d-subalgebra.

The following example shows that a falling d-subalgebra need not be a falling BCK-ideal in d-algebras.

Example 3.6. Consider the d-algebra X which is given in Example 3.4. Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set

$$\xi : [0,1] \longrightarrow \mathcal{P}(X), \qquad t \longmapsto \begin{cases} \{0, c\}, & \text{if } t \in [0, 0.4), \\ X, & \text{if } t \in [0.4, 1]. \end{cases} \tag{3.13}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-subalgebra of X, but it is not a falling BCK-ideal of X since $\xi(t) = \{0, c\}$ is not a BCK-ideal of X for $t \in [0, 0.4)$.

Theorem 3.7. *Every falling d-ideal is a falling d-subalgebra.*

Proof. It is clear, and we omit the proof. □

The following example shows that the converse of Theorem 3.7 is not true.

Example 3.8. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

(3.14)

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set

$$\xi : [0,1] \longrightarrow \wp(X), \qquad t \longmapsto \begin{cases} \emptyset, & \text{if } t \in [0, 0.2), \\ \{0, a\}, & \text{if } t \in [0.2, 0.5), \\ X, & \text{if } t \in [0.5, 1]. \end{cases} \tag{3.15}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-subalgebra of X, but not a falling d-ideal of X, since $\xi(t) = \{0, a\}$ is not a d-ideal of X for $t \in [0.2, 0.5)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\widetilde{H}$ a falling shadow of a random set $\xi : \Omega \to \wp(X)$. For any $x \in X$, let

$$\Omega(x;\xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}. \tag{3.16}$$

Then $\Omega(x;\xi) \in \mathcal{A}$.

Lemma 3.9. *If $\widetilde{H}$ is a falling d-subalgebra of X, then*

$$(\forall x \in X) \quad (\Omega(x;\xi) \subseteq \Omega(0;\xi)). \tag{3.17}$$

Proof. If $\Omega(x;\xi) = \emptyset$, then it is clear. Assume that $\Omega(x;\xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x;\xi)$. Then $x \in \xi(\omega)$, and so $0 = x * x \in \xi(\omega)$ since $\xi(\omega)$ is a d-subalgebra of X. Hence $\omega \in \Omega(0;\xi)$, and therefore $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ for all $x \in X$. □

Combining Theorem 3.7 and Lemma 3.9, we have the following corollary.

Corollary 3.10. *If $\widetilde{H}$ is a falling d-ideal of X, then* (3.17) *is valid.*

We provide a characterization of a falling d-ideal.

Theorem 3.11. *Let $\widetilde{H}$ be a falling shadow of a random set ξ on X. Then $\widetilde{H}$ is a falling d-ideal of X if and only if the following conditions are valid:*

(a) (*for all* $x, y \in X$) $(\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(b) (*for all* $x, y \in X$) $(\Omega(x; \xi) \subseteq \Omega(x * y; \xi))$.

Proof. Assume that $\widetilde{H}$ is a falling d-ideal of X. For any $x, y \in X$, if

$$\omega \in \Omega(x * y; \xi) \cap \Omega(y; \xi), \tag{3.18}$$

then $x * y \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a d-ideal of X, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence $\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$. Now let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$ and so $x * y \in \xi(\omega)$ by (b3). Thus $\omega \in \Omega(x * y; \xi)$, and therefore $\Omega(x; \xi) \subseteq \Omega(x * y; \xi)$ for all $x, y \in X$.

Conversely, suppose that two conditions (a) and (b) are valid. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x * y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x * y; \xi)$ and $\omega \in \Omega(y; \xi)$. It follows from (a) that $\omega \in \Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ so that $x \in \xi(\omega)$. Now, assume that $x \in \xi(\omega)$ for every $x \in X$ and $\omega \in \Omega$. Then $\omega \in \Omega(x; \xi) \subseteq \Omega(x * y; \xi)$ for all $y \in X$, and so $x * y \in \xi(\omega)$. Therefore $\xi(\omega)$ is a d-ideal of X for all $\omega \in \Omega$. Hence $\widetilde{H}$ is a falling d-ideal of X. □

Proposition 3.12. *For a falling shadow $\widetilde{H}$ of a random set ξ on X, if $\widetilde{H}$ is a falling BCK-ideal of X, then*

(a) (*for all* $x, y \in X$) $(x * y = 0 \Rightarrow \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(b) (*for all* $x, y \in X$) $(\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(c) (*for all* $x \in X$) $(\Omega(x; \xi) \subseteq \Omega(0; \xi))$.

Proof. (a) Let $x, y \in X$ and $\omega \in \Omega$ be such that $x * y = 0$ and $\omega \in \Omega(y; \xi)$. Then $y \in \xi(\omega)$ and $x * y = 0 \in \xi(\omega)$ by (b1). It follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence $\Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$ with $x * y = 0$.

(b) Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x * y; \xi) \cap \Omega(y; \xi)$. Then $x * y \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a BCK-ideal of X, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence $\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$.

(c) It follows from (ii) and (a). □

We give conditions for a falling shadow to be a falling BCK-ideal.

Theorem 3.13. *For a falling shadow $\widetilde{H}$ of a random set ξ on X, assume that the following conditions are satisfied:*

(a) $\Omega = \Omega(0; \xi)$,

(b) (*for all* $x, y \in X$) $(\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$.

Then $\widetilde{H}$ is a falling BCK-ideal of X.

Proof. Using (a), we have $0 \in \xi(\omega)$ for all $\omega \in \Omega$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x * y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ by (b), and so $x \in \xi(\omega)$. Therefore $\xi(\omega)$ is a BCK-ideal of X for all $\omega \in \Omega$. Hence $\widetilde{H}$ is a falling BCK-ideal of X. □

Proposition 3.14. *If $\widetilde{H}$ is a falling d-ideal of X, then*

$$(\forall x, y \in X) \quad (y * x = 0 \Longrightarrow \Omega(x;\xi) \subseteq \Omega(y;\xi)). \tag{3.19}$$

Proof. Let $x, y \in X$ be such that $y * x = 0$. Let $\omega \in \Omega(x;\xi)$. Then $x \in \xi(\omega)$ and $\omega \in \Omega(0;\xi)$ by Corollary 3.10. Hence $y * x = 0 \in \xi(\omega)$. Since $\xi(\omega)$ is a d-ideal of X, it follows from (b2) that $y \in \xi(\omega)$. Therefore (3.19) holds. □

A d-ideal I of X is called a $d^\sharp$*-ideal* of X (see [8]) if, for arbitrary $x, y, z \in X$, (b4) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$.

Definition 3.15. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let

$$\xi : \Omega \longrightarrow \mathcal{P}(X) \tag{3.20}$$

be a random set. If $\xi(\omega)$ is a $d^\sharp$-ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\widetilde{H}$ of the random set ξ is called a *falling* $d^\sharp$*-ideal* of X.

Example 3.16. Let X be a d-algebra as in Example 3.8. Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set

$$\xi : \Omega \longrightarrow \mathcal{P}(X), \qquad \omega \longmapsto \begin{cases} \{0, a, b\}, & \text{if } \omega \in [0, 0.3), \\ X, & \text{if } \omega \in [0.3, 0.8), \\ \emptyset, & \text{if } \omega \in [0.8, 1]. \end{cases} \tag{3.21}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling $d^\sharp$-ideal of X, and it is represented as follows:

$$\widetilde{H}(x) = \begin{cases} 0.8, & \text{if } x \in \{0, a, b\}, \\ 0.5, & \text{if } x = c. \end{cases} \tag{3.22}$$

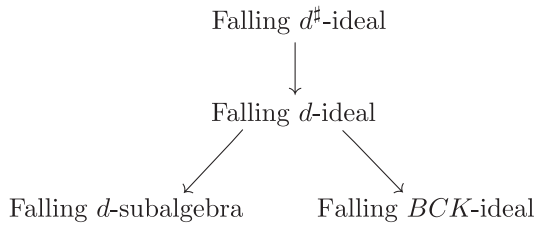
Theorem 3.17. *Every falling $d^\sharp$-ideal is a falling d-ideal.*

Proof. Straightforward. □

We provide an example to show that the converse of Theorem 3.17 is not true.

Example 3.18. Consider the falling d-ideal $\widetilde{H}$ of X which is given in Example 3.5. For $t \in [0, 0.3)$, $\xi(t) = \{0, a\}$ is not a $d^\sharp$-ideal of X since $b * d = 0 \in \xi(t)$, $d * c = a \in \xi(t)$, but $b * c = c \notin \xi(t)$. Hence $\widetilde{H}$ is not a falling $d^\sharp$-ideal of X.

In the above discussion, we can see the following relations:



(3.23)

In this diagram, the reverse implications are not true, and we need additional conditions for considering the reverse implications.

A d-algebra X is called a *d^*-algebra* (see [8]) if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Theorem 3.19. *In a d^*-algebra, every falling BCK-ideal is a falling d-ideal.*

Proof. Let $\widetilde{H}$ be a falling BCK-ideal of a d^*-algebra X. Then $\Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$ by Proposition 3.12. Let $x, y \in X$ and $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$. Since X is a d^*-algebra, we have $(x * y) * x = 0 \in \xi(\omega)$ and so $x * y \in \xi(\omega)$ by (b2). Hence $\omega \in \Omega(x * y; \xi)$, which shows that $\Omega(x; \xi) \subseteq \Omega(x * y; \xi)$ for all $x, y \in X$. Using Theorem 3.11, we conclude that $\widetilde{H}$ is a falling d-ideal of X. □

Corollary 3.20. *In a d^*-algebra, every falling BCK-ideal is a falling d-subalgebra.*

Proof. It follows from Theorems 3.7 and 3.19. □

The following example shows that, in a d^*-algebra, any falling d-subalgebra is neither a falling BCK-ideal nor a falling d-ideal.

Example 3.21. Let $X := \{0, a, b, c\}$ be a d^*-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	a	0

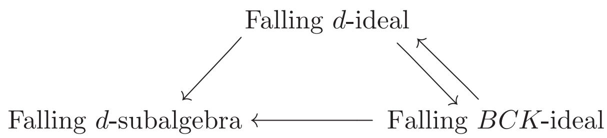
(3.24)

Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set $\xi : [0,1] \to \mathcal{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \emptyset, & \text{if } t \in [0, 0.3), \\ \{0, a, c\}, & \text{if } t \in [0.3, 0.7), \\ X, & \text{if } t \in [0.7, 1]. \end{cases} \tag{3.25}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-subalgebra of X, but it is neither falling BCK-ideal nor a falling d-ideal of X since $\xi(t) = \{0, a, c\}$ is neither a BCK-ideal nor a d-ideal of X for $t \in [0.3, 0.7)$.

Hence, in a d^*-algebra, we have the following relations among falling d-ideals, falling d-subalgebras, and falling BCK-ideals:



(3.26)

We now establish a characterization of a falling $d^\sharp$-ideal.

Theorem 3.22. *For a falling shadow $\widetilde{H}$ of a random set ξ on X, the followings are equivalent.*

(a) *$\widetilde{H}$ is a falling $d^\sharp$-ideal of X.*

(b) *$\widetilde{H}$ is a falling d-ideal of X that satisfies the following inclusion:*

$$(\forall x, y, z \in X) \quad (\Omega(x * y; \xi) \cap \Omega(y * z; \xi) \subseteq \Omega(x * z; \xi)). \tag{3.27}$$

Proof. Assume that $\widetilde{H}$ is a falling $d^\sharp$-ideal of X. Then $\widetilde{H}$ is a falling d-ideal of X. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x * y; \xi) \cap \Omega(y * z; \xi)$. Then $x * y \in \xi(\omega)$ and $y * z \in \xi(\omega)$, and so $x * z \in \xi(\omega)$ since $\xi(\omega)$ is a $d^\sharp$-ideal of X. Hence $\omega \in \Omega(x * z; \xi)$, and therefore $\Omega(x * y; \xi) \cap \Omega(y * z; \xi) \subseteq \Omega(x * z; \xi)$ for all $x, y, z \in X$.

Conversely, let $\widetilde{H}$ be a falling d-ideal of X satisfying the condition (3.27). Then $\xi(\omega)$ is a d-ideal of X. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x * y \in \xi(\omega)$ and $y * z \in \xi(\omega)$. Then $\omega \in \Omega(x * y; \xi) \cap \Omega(y * z; \xi) \subseteq \Omega(x * z; \xi)$ by (3.27), and thus $x * z \in \xi(\omega)$. Hence $\widetilde{H}$ is a falling $d^\sharp$-ideal of X. □

We now discuss relations between a falling d-subalgebra and a fuzzy d-subalgebra. As a result, we can make a statement that the notion of a falling d-subalgebra is a generalization of the notion of a fuzzy d-subalgebra.

A fuzzy set μ on X is called a *fuzzy d-subalgebra* of X (see [7]) if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Lemma 3.23 (see [7]). *A fuzzy set μ of X is a fuzzy d-subalgebra of X if and only if, for every $\lambda \in [0, 1]$, $\mu_\lambda := \{x \in X \mid \mu(x) \geq \lambda\}$ is a d-subalgebra of X when it is nonempty.*

Theorem 3.24. *If one takes the probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure, then every fuzzy d-subalgebra of X is a falling d-subalgebra of X.*

Proof. Let μ be a fuzzy d-subalgebra of X. Then μ_λ is a d-subalgebra of X for all $\lambda \in [0,1]$ by Lemma 3.23. Let

$$\xi : [0,1] \longrightarrow \mathcal{P}(X) \tag{3.28}$$

be a random set and $\xi(\lambda) = \mu_\lambda$ for every $\lambda \in [0,1]$. Then μ is a falling d-subalgebra of X. □

We provide an example to show that the converse of Theorem 3.24 is not true.

Example 3.25. Let X be a d-algebra as in Example 3.4. Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and define a random set

$$\xi : \Omega \longrightarrow \mathcal{P}(X), \qquad \omega \longmapsto \begin{cases} \{0,c\}, & \text{if } t \in [0,0.2), \\ \emptyset, & \text{if } t \in [0.2,0.3), \\ \{0,b\}, & \text{if } t \in [0.3,0.6), \\ \{0,a\}, & \text{if } t \in [0.6,0.85), \\ X, & \text{if } t \in [0.85,1]. \end{cases} \tag{3.29}$$

Then the falling shadow $\widetilde{H}$ of ξ is a falling d-subalgebra of X, and it is represented as follows:

$$\widetilde{H}(x) = \begin{cases} 0.9, & \text{if } x = 0, \\ 0.4, & \text{if } x = a, \\ 0.45, & \text{if } x = b, \\ 0.35, & \text{if } x = c. \end{cases} \tag{3.30}$$

We know that $\widetilde{H}$ is not a fuzzy d-subalgebra of X since

$$\widetilde{H}(b * a) = \widetilde{H}(c) = 0.35 \ngeq 0.4 = \min\left\{\widetilde{H}(b), \widetilde{H}(a)\right\}. \tag{3.31}$$

Theorem 3.26. *Every falling d-subalgebra of X is a T_m-fuzzy d-subalgebra of X; that is, if $\widetilde{H}$ is a falling d-subalgebra of X, then*

$$(\forall x, y \in X) \quad \left(\widetilde{H}(x * y) \geq T_m\left(\widetilde{H}(x), \widetilde{H}(y)\right)\right), \tag{3.32}$$

where $T_m(s,t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0,1]$.

Proof. By Definition 3.1, $\xi(\omega)$ is a d-subalgebra of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence

$$\{\omega \in \Omega \mid x \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x * y \in \xi(\omega)\}, \tag{3.33}$$

which implies that

$$\begin{aligned} \widetilde{H}(x * y) &= P(\omega \mid x * y \in \xi(\omega)) \\ &\geq P(\{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\ &\geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) \\ &\quad - P(\omega \mid x \in \xi(\omega) \text{ or } \omega \mid y \in \xi(\omega)) \\ &\geq \widetilde{H}(x) + \widetilde{H}(y) - 1. \end{aligned} \tag{3.34}$$

Hence

$$\widetilde{H}(x * y) \geq \max\left\{\widetilde{H}(x) + \widetilde{H}(y) - 1, 0\right\} = T_m\left(\widetilde{H}(x), \widetilde{H}(y)\right). \tag{3.35}$$

This completes the proof. □

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