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Research Article

Bounds of Double Integral Dynamic Inequalities in Two Independent Variables on Time Scales

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Our aim in this paper is to establish some explicit bounds of the unknown function in a certain class of nonlinear dynamic inequalities in two independent variables on time scales which are unbounded above. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of partial dynamic equations on time scales. Some examples are considered to demonstrate the applications of the results.

1. Introduction

During the past decade, a number of dynamic inequalities have been established by some authors which are motivated by some applications, for example, when studying the behavior of solutions of certain classes of dynamic equations, the bounds provided by earlier inequalities are inadequate in applications and some new and specific type of dynamic inequalities on time scales are required. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. In [1, Theorem 6.1], it is proved that if u, f, and $p \in C_{\mathrm{rd}}$ and $p \in \mathcal{R}^{+}$, then

$$u^{\Delta}(t) \leq f(t) + p(t)u(t), \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

implies

$$u(t) \leq u(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(s))f(s)\Delta s, \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.2}$$

where $\mathcal{R}^+ := \{f \in \mathcal{R} : 1+\mu(t)f(t) > 0,\ t \in \mathbb{T}\}$ and $\mathcal{R}$ is the class of rd-continuous and regressive functions. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided f is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and, for any function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$, where $\sigma(t)$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)f(t) \neq 0,\ t \in \mathbb{T}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by $p \oplus q := p + q + \mu pq$. Throughout this paper, we will assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. The exponential function $e_p(t,s)$ on time scales is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad \text{for } t,s \in \mathbb{T}, \tag{1.3}$$

where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases} \tag{1.4}$$

Alternatively, for $p \in \mathcal{R}$, one can define the exponential function $e_p(\cdot,t_0)$ to be the unique solution of the IVP $x^{\Delta} = p(t)x$, with $x(t_0) = 1$. If $p \in \mathcal{R}$, then $e_p(t,s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathcal{R}^+$, then $e_p(t,t_0)$ is always positive, $e_p(t,t) = 1$, and $e_0(t,s) = 1$. Note that

$$\begin{aligned} e_p(t,t_0) &= \exp\left(\int_{t_0}^t p(s)ds\right), \quad \text{if } \mathbb{T} = \mathbb{R}, \\ e_p(t,t_0) &= \prod_{s=t_0}^{t-1}(1+p(s)), \quad \text{if } \mathbb{T} = \mathbb{N}, \\ e_p(t,t_0) &= \prod_{s=t_0}^{t-1}(1+(q-1)sp(s)), \quad \text{if } \mathbb{T} = q^{\mathbb{N}_0}. \end{aligned} \tag{1.5}$$

The book on the subject of time scales by Bohner and Peterson [1] summarizes and organizes much of time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see [2]), that is, when $\mathbb{T} = \mathbb{R},\ \mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t :\ t \in \mathbb{N}_0\}$, where $q > 1$.

In this paper, we will refer to the (delta) integral which we can define as follows: If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [1]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [3] discusses several possible applications. Also, in [1, Theorem 6.4], it is proved that if u, a, and $p \in C_{\rm rd}$ and $p \in \mathcal{R}^+$, then

$$u(t) \leq a(t) + \int_{t_0}^{t} p(s)u(s)\Delta s, \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.6}$$

implies that

$$u(t) \leq a(t) + \int_{t_0}^{t} e_p(t, \sigma(s))a(s)p(s)\Delta s, \quad \forall t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.7}$$

Since (1.7) provides an explicit bound to the unknown function $u(t)$ and a tool to the study of many qualitative as well as quantitative properties of solutions of dynamic equations, it has become one of the very few classic and most influential results in the theory and applications of dynamic inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (1.7) have been established. Since the discovery of the inequalities (1.1)–(1.7), much work has been done, and many papers which deal with various generalizations and extensions have appeared in the literature, we refer the reader to [4–9] and the references cited therein. On the other hand, a few authors have focused on the theory of partial dynamic equations on time scales [10–15]. However, to the best of author's knowledge, only [16–19] have studied integral inequalities useful in the theory of partial dynamic equations on time scales. Before, we give a brief summary of some of the results of dynamic inequalities in two independent variables, we present some basic definitions about calculus in two variables on time scales (for more details, we refer to [12]).

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with at least two points, and consider the time scale intervals $\Omega_1 = [t_0, \infty) \cap \mathbb{T}_1$ and $\Omega_2 = [s_0, \infty) \cap \mathbb{T}_2$ for $t_0 \in \mathbb{T}_1$ and $s_0 \in \mathbb{T}_2$. Let $\sigma_1,\ \rho_1,\ \Delta_1$ and $\sigma_2,\ \rho_2,\ \Delta_2$ denote the forward jump operators, backward jump operators, and the delta differentiation operator, respectively, on $\mathbb{T}_1$ and $\mathbb{T}_2$. We say that a real valued function f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t, s) \in \Omega \equiv \Omega_1 \times \Omega_2$ has a Δ_1 partial derivative $f^{\Delta_1}(t, s)$ with respect to t if for each $\epsilon > 0$ there exists a neighborhood U_t of t such that

$$\left| f(\sigma_1(t), s) - f(\eta, s) - f^{\Delta_1}(t, s)[\sigma_1(t) - \eta] \right| \leq \epsilon |\sigma(t) - \eta|, \quad \forall \eta \in U_t. \tag{1.8}$$

In this case, we say $f^{\Delta_1}(t, s)$ is the (partial delta) derivative of $f(t, s)$ at t. We say that a real valued function f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t, s) \in \Omega_1 \times \Omega_2$ has a Δ_2 partial derivative $f^{\Delta_2}(t, s)$ with respect to s if for each $\epsilon > 0$ there exists a neighborhood U_s of s such that

$$\left| f(t, \sigma_2(s)) - f(t, \xi) - f^{\Delta_2}(t, s)[\sigma_2(t) - \xi] \right| \leq \epsilon |\sigma(t) - \xi|, \quad \forall \xi \in U_s. \tag{1.9}$$

In this case, we say $f^{\Delta_2}(t, s)$ is the (partial delta) derivative of $f(t, s)$ at s. The function f is called rd-continuous in t if for every $\alpha_2 \in \mathbb{T}_2$ the function $f(t, \alpha_2)$ is rd-continuous on $\mathbb{T}_1$. The function f is called rd-continuous in s if for every $\alpha_1 \in \mathbb{T}_1$ the function $f(\alpha_1, s)$ is rd-continuous on $\mathbb{T}_2$. Now, we are ready to present some results for dynamic inequalities in two independent variables on times scales which are related to the main results in our paper.

In [18], the authors proved that if a, f, and u are positive rd-continuous functions and a is nonnegative and nondecreasing in each of its variables, then

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s,t)u(s,t)\Delta t \Delta s, \tag{1.10}$$

for all $(x,y) \in [x_0,\infty)_{\mathbb{T}} \times [y_0,\infty)_{\mathbb{T}}$, implies

$$u(x,y) \leq a(x,y)e_F(x,x_0), \quad \text{where } F = \int_{y_0}^{y} f(x,t)\Delta t. \tag{1.11}$$

In [19], the author proved that if a, b, g, h, and u are positive continuous real functions defined on $\mathbb{T} \times \mathbb{T}$ and $\gamma > 1$ is a real constant, then

$$u^{\gamma}(x,y) \leq a(x,y) + b(x,y) \int_{x_0}^{x} \int_{y_0}^{y} \left[g(s,t)u^{\gamma}(s,t) + h(s,t)u(s,t)\right]\Delta t \Delta s, \tag{1.12}$$

implies

$$u(x,y) \leq \left[a(x,y) + b(x,y)m(x,y)e_G(t,t_0)\right]^{1/\gamma}, \tag{1.13}$$

for all $(x,y) \in [x_0,\infty)_{\mathbb{T}} \times [y_0,\infty)_{\mathbb{T}}$, where

$$\begin{gathered} m(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} \left[a(s,t)g(s,t) + \left(\frac{\gamma-1}{\gamma} + \frac{a(s,t)}{\gamma}\right)h(s,t)\right]\Delta t \Delta s, \\ G(s,t) = \int_{y_0}^{y} \left[g(x,t) + \frac{h(x,t)}{\gamma}\right]b(x,t)\Delta t. \end{gathered} \tag{1.14}$$

In this paper, we are concerned with bounds of the double integral nonlinear dynamic inequality in two independent variables

$$u^{\gamma}(t,s) \leq a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} \left[f(\tau,\eta)u^{\delta}(\tau,\eta) + g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda} \Delta\eta \Delta\tau, \tag{1.15}$$

for all $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$. For (1.15), we will assume the following hypotheses:

$$(H)\begin{cases} u, a, b, f, \text{ and } g \text{ are rd-continuous positive functions on } \Omega_1 \times \Omega_2, \\ \alpha, \delta, \lambda, \text{ and } \gamma \text{ are positive constants.} \end{cases} \tag{1.16}$$

The main aim in this paper is to establish some explicit bounds of the unknown function $u(t,s)$ of the inequality (1.15). Our results not only complement the results in [18, 19] but also improve the results in [19], in the sense that the results can be applied in the cases when $\gamma \leq 1$. The main results will be proved by employing the Bernoulli inequality [20, Bernoulli's inequality]:

$$(1+x)^{\gamma} \leq 1+\gamma x, \quad \text{for } 0<\gamma \leq 1,\ x>-1, \tag{1.17}$$

the Young inequality [20]:

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}, \quad \text{where } a,b \geq 0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \tag{1.18}$$

and the algebraic inequalities [20]:

$$(a+b)^{\lambda} \leq 2^{\lambda-1}\left(a^{\lambda}+b^{\lambda}\right), \quad \text{for } a,b \geq 0,\ \lambda \geq 1, \tag{1.19}$$

$$(a+b)^{\lambda} \leq a^{\lambda}+b^{\lambda}, \quad \text{for } a,b \geq 0,\ 0 \leq \lambda \leq 1. \tag{1.20}$$

Some examples are considered to illustrate the main results.

2. Main Results

Before, we stated and proved the main results and we proved some Lemmas which play important roles in the proofs of the main results. We will assume that the equations or the inequalities possess such nontrivial solutions.

Lemma 2.1. *Let $\mathbb{T}$ be an unbounded time scale with (t_0,s_0) and $(t,s) \in \mathbb{T}\times\mathbb{T}$. Let $g_i : \mathbb{R} \to \mathbb{R}$ for $i=1,2,\ldots,n$ be functions with $g_i(x_1(t,s)) \leq g_i(x_2(t,s))$ for $i=1,2,\ldots,n$, where $x_i(t,s) : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ for $i=1,2$, whenever $x_1 \leq x_2$. Let $v,w : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ be differentiable with*

$$v^{\Delta_t\Delta_s}(t,s) \leq \sum_{i=1}^{n} a_i(t,s) g_i(v(t,s)), \qquad w^{\Delta_t\Delta_s}(t,s) \geq \sum_{i=1}^{n} a_i(t,s) g_i(w(t,s)), \tag{2.1}$$

for all $(t,s) \in \mathbb{T}\times\mathbb{T}$. Then, $v(t_0,s_0) < w(t_0,s_0)$ implies $v(t,s) \leq w(t,s)$ for all $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$.

Proof. The proof is by induction and similar to the proof of Theorem 6.9 in [1] and hence is omitted. □

Lemma 2.2. *Let $\mathbb{T}$ be an unbounded time scale with (t_0,s_0) and $(t,s) \in \mathbb{T}\times\mathbb{T}$. Suppose that $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $i=1,2,\ldots,n$ and $y : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^+$ is such that $g_i(y)$ is rd-continuous. Let p_i be rd-continuous and positive for $i=1,2,\ldots,n$ and $f : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^+$ differentiable. Then,*

$$y(t,s) \leq f(t,s) + \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s_0}^{s} p_i(\eta,\tau) g_i(y(\eta,s)) \Delta\eta \Delta\tau, \quad \forall (t,s) \in \mathbb{T}\times\mathbb{T}, \tag{2.2}$$

implies $y(t,s) \leq x(t,s)$ *for all* $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$*, where* $x(t,s)$ *solves the initial value problem*

$$x^{\Delta_t\Delta_s}(t,s) = f^{\Delta_t\Delta s}(t,s) + \sum_{i=1}^{n} p_i(t,s) g_i(x(t,s)), \\ x(t_0,s_0) > f(t_0,s_0) > 0. \tag{2.3}$$

Proof. Let

$$v(t,s) := f(t,s) + \sum_{i=1}^{n} \int_{t_0}^{t}\int_{t_0}^{s} p_i(\tau,\eta) g_i(y(\tau,\eta))\Delta\eta\Delta\tau, \tag{2.4}$$

for all $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$. Then,

$$v^{\Delta_t\Delta_s}(t,s) := f^{\Delta_t\Delta s}(t,s) + \sum_{i=1}^{n} p_i(t,s) g_i(y(t,s)), \tag{2.5}$$

for all $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$ and $y(t,s) \leq v(t,s)$ so that

$$v^{\Delta_t\Delta_s}(t,s) \leq f^{\Delta_t\Delta s}(t,s) + \sum_{i=1}^{n} p_i(t,s) g_i(v(t,s)), \tag{2.6}$$

for all $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$. Since $v(t_0,s_0) = f(t_0,s_0) < x_0 = x(t_0,s_0)$, the comparison Lemma 2.1 yields $v(t,s) \leq x(t,s)$ for all $(t,s) \geq (t_0,s_0)$, where $x(t,s)$ solves the initial value problem (2.3). Hence, since $y(t,s) \leq v(t,s)$, we obtain $y(t,s) \leq x(t,s)$. The proof is complete. □

Now, we are ready to state and prove the main results in this paper. First, we consider the case when $\lambda \geq 1$ and $\alpha,\delta \leq \gamma$. For simplicity, we introduce the following notations:

$$F(t,s) := 2^{2(\lambda-1)} \int_{t_0}^{t}\int_{s_0}^{s} \left[f^{\lambda}(\tau,\eta)\left[a^{\delta/\gamma}(\tau,\eta)\right]^{\lambda}\right]\Delta\eta\Delta\tau \\ + 2^{2(\lambda-1)} \int_{t_0}^{t}\int_{s_0}^{s} \left[g^{\lambda}(\tau,\eta)\left[a^{\alpha/\gamma}(\tau,\eta)\right]^{\lambda}\right]\Delta\eta\Delta\tau, \\ G(t,s) := 2^{2(\lambda-1)}\left(f^{\lambda}(t,s)\left[\frac{\delta}{\gamma}a^{(\delta/\gamma)-1}(t,s)\right]^{\lambda} + g^{\lambda}(t,s)\left[\frac{\alpha}{\gamma}a^{(\alpha/\gamma)-1}(t,s)\right]^{\lambda}\right). \tag{2.7}$$

Theorem 2.3. *Let* $\mathbb{T}$ *be an unbounded time scale with* $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$*. Assume that* (H) *holds,* $\lambda,\gamma \geq 1$ *and* $\alpha,\delta \leq \gamma$*. Then,*

$$u^{\gamma}(t,s) \leq a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta) + g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau, \tag{2.8}$$

for $(t,s)\in\Omega$, *implies that*

$$u(t,s)\le a^{1/\gamma}(t,s)+\frac{1}{\gamma}a^{(1/\gamma)-1}(t,s)b(t,s)w(t,s),\quad \forall(t,s)\in\Omega, \tag{2.9}$$

where $w(t)$ *solves the initial value problem*

$$w^{\Delta_t\Delta s}(t,s)=F^{\Delta_t\Delta s}(t,s)+b^{\lambda}(t,s)G(t,s)w^{\lambda}(t,s),\qquad w(t_0,s_0)>0. \tag{2.10}$$

Proof. Define a function $y(t,s)$ by

$$y(t,s):=\int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta)+g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau. \tag{2.11}$$

This reduces (2.8) to

$$u^{\gamma}(t,s)\le a(t,s)+b(t,s)y(t,s),\quad \text{for } (t,s)\in\Omega. \tag{2.12}$$

This implies that

$$u(t,s)\le\left(a(t,s)+b(t,s)y(t,s)\right)^{1/\gamma},\quad \text{for } (t,s)\in\Omega. \tag{2.13}$$

Applying the inequality (1.17) (noting that $1/\gamma\le 1$), we see that

$$u(t,s)\le a^{1/\gamma}(t,s)+\frac{1}{\gamma}a^{(1/\gamma)-1}(t,s)b(t,s)y(t,s),\quad \text{for } (t,s)\in\Omega. \tag{2.14}$$

From (2.13), we obtain

$$u^{\alpha}(t,s)\le a^{\alpha/\gamma}(t,s)\left[1+\frac{b(t,s)y(t,s)}{a(t,s)}\right]^{\alpha/\gamma},\quad \text{for } (t,s)\in\Omega. \tag{2.15}$$

Applying inequality (1.17) on (2.15) (where $\alpha\le\gamma$), we obtain for $(t,s)\in\Omega$ that

$$\begin{aligned} u^{\alpha}(t,s)&\le a^{\alpha/\gamma}(t,s)\left[1+\frac{\alpha}{\gamma}\frac{b(t,s)}{a(t,s)}y(t,s)\right]\\ &=a^{\alpha/\gamma}(t,s)+\frac{\alpha}{\gamma}a^{(\alpha/\gamma)-1}(t,s)b(t,s)y(t,s). \end{aligned} \tag{2.16}$$

Also, from (2.13), we obtain

$$u^{\delta}(t,s) \le a^{\delta/\gamma}(t,s)\left[1+\frac{b(t,s)y(t,s)}{a(t,s)}\right]^{\delta/\gamma}, \quad \text{for } (t,s)\in\Omega. \tag{2.17}$$

Applying inequality (1.17) on (2.17) (where $\delta \le \gamma$), we have for $(t,s)\in\Omega$ that

$$\begin{aligned} u^{\delta}(t,s) &\le a^{\delta/\gamma}(t,s)\left[1+\frac{\delta}{\gamma}\frac{b(t,s)}{a(t,s)}y(t,s)\right] \\ &= a^{\delta/\gamma}(t,s)+\frac{\delta}{\gamma}a^{(\delta/\gamma)-1}(t,s)b(t,s)y(t,s). \end{aligned} \tag{2.18}$$

Combining (2.11), (2.16), and (2.18) and applying the inequality (1.19) (noting that $\lambda \ge 1$), we have

$$\begin{aligned} y(t,s) &= \int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta)+g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\le 2^{\lambda-1}\int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\quad + 2^{\lambda-1}\int_{t_0}^{t}\left[g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\le 2^{\lambda-1}\int_{t_0}^{t}\int_{t_0}^{s}f^{\lambda}(\tau,\eta)\left[a^{\delta/\gamma}(\tau,\eta)+\frac{\delta}{\gamma}a^{(\delta/\gamma)-1}(\tau,\eta)b(\tau,\eta)y(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\quad + 2^{\lambda-1}\int_{t_0}^{t}\int_{t_0}^{s}g^{\lambda}(\tau,\eta)\left[a^{\alpha/\gamma}(\tau,\eta)+\frac{\alpha}{\gamma}a^{(\alpha/\gamma)-1}(\tau,\eta)b(\tau,\eta)y(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau. \end{aligned} \tag{2.19}$$

This implies that

$$\begin{aligned} y(t,s) &\le 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s}f^{\lambda}(\tau,\eta)\left[a^{(\delta/\gamma)}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s}f^{\lambda}(\tau,\eta)\left[\frac{\delta}{\gamma}a^{(\delta/\gamma)-1}(\tau,\eta)b(\tau,\eta)\right]^{\lambda}y^{\lambda}(\tau,\eta)\Delta\eta\Delta\tau \\ &\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s}g^{\lambda}(\tau,\eta)\left[a^{\alpha/\gamma}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s}g^{\lambda}(\tau,\eta)\left[\frac{\alpha}{\gamma}a^{(\alpha/\gamma)-1}(\tau,\eta)b(\tau,\eta)\right]^{\lambda}y^{\lambda}(\tau,\eta)\Delta\eta\Delta\tau \\ &= F(t,s)+\int_{t_0}^{t}\int_{t_0}^{s}G(\tau,\eta)y^{\lambda}(\tau,\eta)\Delta\eta\Delta\tau, \quad \text{for } (t,s)\in\Omega. \end{aligned} \tag{2.20}$$

Now an application of Lemma 2.2 (with $n = 1$ and $g(y) = y^{\lambda}$) gives that

$$y(t,s) < w(t,s), \quad \text{for } (t,s) \in \Omega, \tag{2.21}$$

where $w(t,s)$ solves the initial value problem (2.10). Substituting (2.21) into (2.14), we obtain the desired inequality (2.9). The proof is complete. □

Theorem 2.4. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0, s_0) \in \mathbb{T} \times \mathbb{T}$. Assume that (H) holds, $\lambda, \gamma \geq 1$ and $\alpha, \delta \leq \gamma$. Then,* (2.8) *implies*

$$u(t,s) \leq a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)w^{1/\gamma}(t,s), \quad \forall (t,s) \in \Omega, \tag{2.22}$$

where $w(t)$ solves the initial value problem

$$\begin{gathered} w^{\Delta_t\Delta_s}(t,s) = F^{\Delta_t\Delta_s}(t,s) + G_1(t,s)w^{\lambda(\delta/\gamma)}(t,s) + G_2(t,s)w^{\lambda(\alpha/\gamma)}(t,s), \\ w(t_0,s_0) > 0, \end{gathered} \tag{2.23}$$

where $F(t)$ is defined as in (2.7) *and*

$$G_1(t,s) := 2^{2(\lambda-1)} f^{\lambda}(t,s) b^{\lambda\delta/\gamma}(t,s), \qquad G_2 := 2^{2(\lambda-1)} g^{\lambda}(t,s) b^{\lambda\alpha/\gamma}(t,s). \tag{2.24}$$

Proof. Define a function $y(t,s)$ by (2.11) and proceed as in the proof of Theorem 2.3 to obtain

$$u(t,s) \leq \left(a(t,s) + b(t,s)y(t,s)\right)^{1/\gamma}, \quad \text{for } (t,s) \in \Omega. \tag{2.25}$$

Applying the inequality (1.20), we see that

$$u(t,s) \leq a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s), \quad \text{for } (t,s) \in \Omega. \tag{2.26}$$

From (2.25), we obtain

$$u^{\alpha}(t,s) \leq \left(a(t,s) + b(t,s)y(t,s)\right)^{\alpha/\gamma}, \quad \text{for } (t,s) \in \Omega. \tag{2.27}$$

Applying inequality (1.20) on (2.27) (where $\alpha \leq \gamma$), we obtain for $(t,s) \in \Omega$ that

$$u^{\alpha}(t,s) \leq a^{\alpha/\gamma}(t,s) + b^{\alpha/\gamma}(t,s)y^{\alpha/\gamma}(t,s). \tag{2.28}$$

Also, from (2.25), we have by (1.20) that

$$u^{\delta}(t,s) \leq a^{\delta/\gamma}(t,s) + b^{\delta/\gamma}(t,s)y^{\delta/\gamma}(t,s), \quad \text{for } (t,s) \in \Omega. \tag{2.29}$$

Combining (2.11), (2.28), and (2.29) and applying the inequality (1.19) (noting that $\lambda \geq 1$), we have

$$
\begin{aligned}
y(t,s) &= \int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta)+g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\
&\leq 2^{\lambda-1}\int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\
&\quad + 2^{\lambda-1}\int_{t_0}^{t}\left[g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\
&\leq 2^{\lambda-1}\int_{t_0}^{t}\int_{t_0}^{s} f^{\lambda}(\tau,\eta)\left[a^{\delta/\gamma}(\tau,\eta)+b^{\delta/\gamma}(\tau,\eta)y^{\delta/\gamma}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\
&\quad + 2^{\lambda-1}\int_{t_0}^{t}\int_{t_0}^{s} g^{\lambda}(\tau,\eta)\left[a^{\alpha/\gamma}(\tau,\eta)+b^{\alpha/\gamma}(\tau,\eta)y^{\alpha/\gamma}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau.
\end{aligned}
\tag{2.30}
$$

This implies that

$$
\begin{aligned}
y(t,s) &\leq 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s} f^{\lambda}(\tau,\eta)\left[a^{\delta/\gamma}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\
&\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s} g^{\lambda}(\tau,\eta)\left[a^{\alpha/\gamma}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\
&\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s} f^{\lambda}(\tau,\eta)\left[b^{\delta/\gamma}(\tau,\eta)\right]^{\lambda}y^{\lambda(\delta/\gamma)}(\tau,\eta)\Delta\eta\Delta\tau \\
&\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{t_0}^{s} g^{\lambda}(\tau,\eta)\left[b^{\alpha/\gamma}(\tau,\eta)\right]^{\lambda}y^{\lambda(\alpha/\gamma)}(\tau,\eta)\Delta\eta\Delta\tau \\
&= F(t,s)+\int_{t_0}^{t}\int_{t_0}^{s}\left[G_1(\tau,\eta)y^{\lambda(\delta/\gamma)}(\tau,\eta)+G_2(\tau,\eta)y^{\lambda(\alpha/\gamma)}(\tau,\eta)\right]\Delta\eta\Delta\tau,
\end{aligned}
\tag{2.31}
$$

for $(t,s)\in\Omega$. Now, an application of Lemma 2.2 (with $n=2$, $g_1(y)=y^{\lambda(\delta/\gamma)}$, and $g_2(y)=y^{\lambda(\alpha/\gamma)}$) gives that

$$
y(t,s) < w(t,s), \quad \text{for } (t,s)\in\Omega,
\tag{2.32}
$$

where $w(t,s)$ solves the initial value problem (2.23). Substituting (2.32) into (2.26), we obtain the desired inequality (2.22). The proof is complete. □

As in the proof of Theorem 2.3 by employing the inequality (1.20) instead of the inequality (1.19), we can obtain an explicit bound for $u(t)$ when $0 \le \lambda \le 1$. This will be presented below in Theorem 2.5 without proof since the proof is similar to the proof of Theorem 2.3. For simplicity, we introduce the following notations:

$$F_1(t,s) := \int_{t_0}^{t}\int_{s_0}^{s}\left[f^{\lambda}(\tau,\eta)a^{\lambda(\delta/\gamma)}(\tau,\eta)\right]\Delta\eta\Delta\tau + \int_{t_0}^{t}\int_{s_0}^{s}\left[g^{\lambda}(\tau,\eta)a^{\lambda(\alpha/\gamma)}(\tau,\eta)\right]\Delta\eta\Delta\tau, \tag{2.33}$$

$$G_3(t,s) := \left(f^{\lambda}(t,s)\left[\frac{\delta}{\gamma}a^{(\delta/\gamma)-1}(t,s)\right]^{\lambda} + g^{\lambda}(t,s)\left[\frac{\alpha}{\gamma}a^{(\alpha/\gamma)-1}(t,s)\right]^{\lambda}\right).$$

Theorem 2.5. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0)\in\mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\gamma\ge 1,\ 0<\lambda\le 1,\ \delta\le\gamma$, and $\alpha\le\gamma$. Then,* (2.8) *implies that*

$$u(t,s) \le a^{1/\gamma}(t,s) + \frac{1}{\gamma}a^{(1/\gamma)-1}(t,s)b(t,s)z(t,s), \quad \text{for } (t,s)\in\Omega, \tag{2.34}$$

where $z(t)$ solves the initial value problem

$$z^{\Delta_t\Delta_t}(t) = F_1^{\Delta_t\Delta_s}(t) + G_3(t,s)b^{\lambda}(t,s)z^{\lambda}(t,s), \qquad z(t_0,s_0)>0. \tag{2.35}$$

In the following, we apply the Young inequality (1.18) to find a new explicit upper bound for $u(t)$ of (2.8) when $\lambda\ge 1$ and $0\le\lambda\le 1$. First, we consider the case when $\lambda\ge 1$ and assume that $\lambda(\alpha/\gamma)<1$ and $\lambda(\delta/\gamma)<1$.

Theorem 2.6. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0)\in\mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\gamma,\lambda\ge 1$ and $\alpha,\delta\le\gamma$ such that $(\lambda\alpha/\gamma)<1$ and $(\lambda\delta/\gamma)<1$. Then,* (2.8) *implies that*

$$u(t,s) \le a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)F_3^{1/\gamma}(t,s), \quad \forall (t,s)\in\Omega, \tag{2.36}$$

where

$$F_3(t,s) := F_0(t,s) + e_{\beta(s-s_0)}(t,t_0), \quad \beta = \lambda\left[\frac{\alpha}{\gamma}+\frac{\delta}{\gamma}\right],$$

$$F_0(t,s) := F(t,s) + \frac{(\gamma-\lambda\delta)}{\gamma}\int_{t_0}^{t}(G_1(\tau,\eta))^{\gamma/(\gamma-\lambda\delta)}\Delta\eta\Delta\tau + \frac{(\gamma-\lambda\alpha)}{\gamma}\int_{t_0}^{t}(G_2(\tau,\eta))^{\gamma/(\gamma-\lambda\alpha)}\Delta\eta\Delta\tau, \tag{2.37}$$

and $F,\ G_1$, and G_2 are defined as in (2.7) *and* (2.24).

Proof. Define a function $y(t,s)$ by (2.11) and proceed as in the proof of Theorem 2.4 to obtain

$$u(t,s) \le a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s), \quad \text{for } (t,s) \in \Omega, \tag{2.38}$$

$$y(t,s) \le F(t,s) + \int_{t_0}^{t}\int_{t_0}^{s}\left[G_1(\tau,\eta)y^{\lambda(\delta/\gamma)}(\tau,\eta) + G_2(\tau,\eta)y^{\lambda(\alpha/\gamma)}(\tau,\eta)\right]\Delta\eta\Delta\tau, \tag{2.39}$$

where F, G_1, and G_2 are defined as in (2.7) and (2.24). Applying the Young inequality (1.18) on the term $G_1y^{\lambda(\delta/\gamma)}$ with $q = \gamma/\lambda\delta > 1$ and $p = \gamma/(\gamma-\lambda\delta) > 1$, we see that

$$G_1y^{\lambda(\delta/\gamma)} \le \frac{(\gamma-\lambda\delta)}{\gamma}(G_1)^{\gamma/(\gamma-\lambda\delta)} + \left(\frac{\lambda\delta}{\gamma}\right)y. \tag{2.40}$$

Again applying the Young inequality (1.18) on the term $G_2y^{\lambda(\alpha/\gamma)}$ with $q = \gamma/\lambda\alpha > 1$ and $p = \gamma/(\gamma-\lambda\alpha) > 1$, we see that

$$G_2y^{\lambda(\alpha/\gamma)} \le \frac{(\gamma-\lambda\alpha)}{\gamma}(G_2)^{\gamma/(\gamma-\lambda\alpha)} + \left(\frac{\lambda\alpha}{\gamma}\right)y. \tag{2.41}$$

Substituting (2.40) and (2.41) into (2.39), we have

$$\begin{aligned} y(t,s) \le F(t,s) &+ \frac{(\gamma-\lambda\delta)}{\gamma}\int_{t_0}^{t}\int_{t_0}^{s}(G_1(\tau,\eta))^{\gamma/(\gamma-\lambda\delta)}\Delta\eta\Delta\tau \\ &+ \frac{(\gamma-\lambda\alpha)}{\gamma}\int_{t_0}^{t}\int_{t_0}^{s}(G_2(\tau,\eta))^{\gamma/(\gamma-\lambda\alpha)}\Delta\eta\Delta\tau \\ &+ \left[\frac{\lambda\alpha}{\gamma} + \frac{\lambda\delta}{\gamma}\right]\int_{t_0}^{t}\int_{s_0}^{s}y(\tau,\eta)\Delta\eta\Delta\tau, \quad \forall(t,s) \in \Omega. \end{aligned} \tag{2.42}$$

From the definitions of $F_0(t)$ and β, we get that

$$y(t,s) \le F_0(t,s) + \beta\int_{t_0}^{t}\int_{t_0}^{s}y(\tau,\eta)\Delta\eta\Delta\tau, \quad \text{for } (t,s) \in \Omega. \tag{2.43}$$

Applying the inequality (1.10) with $f(t,s) = \beta$, we have

$$y(t,s) < F_0(t,s) + e_{\beta(s-s_0)}(t,t_0), \quad \forall(t,s) \in \Omega. \tag{2.44}$$

Substituting (2.44) into (2.38), we get the desired inequality (2.36). The proof is complete. □

Theorem 2.7. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\gamma \ge 1$, $0 < \lambda \le 1$, and $\alpha,\delta \le \gamma$. Then,* (2.8) *implies that*

$$u(t,s) \le a^{1/\gamma}(t,s) + \frac{1}{\gamma}a^{(1/\gamma)-1}(t,s)b(t,s)F_4(t,s), \quad \textit{for } (t,s) \in \Omega, \tag{2.45}$$

where

$$F_4(t,s) := F_2(t,s) + e_{\lambda(s-s_0)}(t,t_0),$$
$$F_2(t,s) := F_1(t,s) + (1-\lambda)\int_{t_0}^{t}\int_{s_0}^{s}\left(G_3(\tau,\eta)\right)^{1/(1-\lambda)}\Delta\eta\Delta\tau, \tag{2.46}$$

and F_1 *and* G_3 *are defined as in* (2.33).

Proof. Define a function $y(t,s)$ by (2.11) and proceed as in the proof of Theorem 2.3 to obtain

$$u(t,s) \le a^{1/\gamma}(t,s) + \frac{1}{\gamma}a^{(1/\gamma)-1}(t,s)b(t,s)y(t,s), \quad \text{for } (t,s)\in\Omega, \tag{2.47}$$

$$y(t,s) \le F_1(t,s) + \int_{t_0}^{t}\int_{s_0}^{s} G_3(s)y^{\lambda}(\tau,\eta)\Delta\eta\Delta\tau, \quad \text{for } (t,s)\in\Omega, \tag{2.48}$$

where F_1 and G_3 are defined in (2.33). Applying the Young inequality (1.18) on the term G_3y^{λ} with $q = 1/\lambda > 1$ and $p = 1/(1-\lambda) > 1$, we see that

$$G_3y^{\lambda} \le (1-\lambda)(G_3)^{1/1-\lambda} + \lambda y. \tag{2.49}$$

This and (2.48) imply that

$$y(t,s) \le F_1(t,s) + (1-\lambda)\int_{t_0}^{t}\int_{s_0}^{s}\left(G_3(\tau,\eta)\right)^{1/(1-\lambda)}\Delta\eta\Delta\tau + \int_{t_0}^{t}\int_{s_0}^{s}\lambda y(\tau,\eta)\Delta\eta\Delta\tau. \tag{2.50}$$

Using the definition of $F_2(t,s)$, we get that

$$y(t,s) \le F_2(t,s) + \lambda\int_{t_0}^{t}\int_{s_0}^{s} y(\tau,\eta)\Delta\eta\Delta\tau, \quad \text{for } (t,s)\in\Omega. \tag{2.51}$$

Applying the inequality (1.10) with $f(t,s) = \lambda$, we have

$$y(t,s) < F_2(t,s) + e_{\lambda(s-s_0)}(t,t_0), \quad \forall (t,s)\in\Omega. \tag{2.52}$$

Substituting (2.52) into (2.47), we get the desired inequality (2.45). The proof is complete. □

Next in the following, we consider the case when $\gamma \le 1$ and establish some new explicit bounds of the unknown function $u(t,s)$ of (2.8).

Theorem 2.8. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\lambda \le 1$, $\gamma \le 1$, and $\alpha\lambda, \delta\lambda \le \gamma$. Then,* (2.8) *implies that*

$$u^{\gamma}(t,s) \le a(t,s) + b(t,s)\left[H(t,s) + e_{\beta_1(s-s_0)}(t,t_0)\right], \quad \text{for } (t,s)\in\Omega, \tag{2.53}$$

where $\beta_1 = (\lambda\alpha/\gamma) + (\lambda\delta/\gamma)$ and

$$\begin{aligned} H(t,s) = {} & 2^{\lambda\delta((1/\gamma)-1)} \int_{t_0}^{t}\int_{t_0}^{s} f^{\lambda}(\tau,\eta)a^{\lambda\delta/\gamma}(\tau,\eta)\Delta\eta\Delta\tau \\ & + 2^{\lambda\alpha((1/\gamma)-1)} \int_{t_0}^{t}\int_{t_0}^{s} g^{\lambda}(\tau,\eta)a^{\lambda\alpha/\gamma}(\tau,\eta)\Delta\eta\Delta\tau \\ & + 2^{\lambda\delta((1/\gamma)-1)} \frac{(\gamma-\lambda\delta)}{\gamma}\int_{t_0}^{t}\int_{t_0}^{s} \left(f^{\lambda}(\tau,\eta)b^{\lambda\delta/\gamma}(\tau,\eta)\right)^{\gamma/(\gamma-\lambda\delta)}\Delta\eta\Delta\tau \\ & + 2^{\lambda\alpha((1/\gamma)-1)} \frac{(\gamma-\lambda\alpha)}{\gamma}\int_{t_0}^{t}\int_{t_0}^{s} \left(g^{\lambda}(\tau,\eta)b^{\lambda\alpha/\gamma}(\tau,\eta)\right)^{\gamma/(\gamma-\lambda\alpha)}\Delta\eta\Delta\tau. \end{aligned} \tag{2.54}$$

Proof. Define a function $y(t,s)$ by

$$y(t,s) := \int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta) + g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau. \tag{2.55}$$

This reduces (2.8) to

$$u^{\gamma}(t,s) \le a(t,s) + b(t,s)y(t,s), \quad \text{for } (t,s)\in\Omega. \tag{2.56}$$

This implies that

$$u(t,s) \le \left(a(t,s) + b(t,s)y(t,s)\right)^{1/\gamma}, \quad \text{for } (t,s)\in\Omega. \tag{2.57}$$

Applying the inequality (1.19) (noting that $\gamma \le 1$), we see that

$$u(t,s) \le 2^{(1/\gamma)-1}\left[a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s)\right], \quad \text{for } (t,s)\in\Omega. \tag{2.58}$$

From (2.58), we obtain

$$u^{\alpha}(t,s) \le 2^{\alpha((1/\gamma)-1)}\left[a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s)\right]^{\alpha}, \quad \text{for } (t,s)\in\Omega. \tag{2.59}$$

Also, from (2.58), we obtain

$$u^{\delta}(t,s) \leq 2^{\delta((1/\gamma)-1)}\left[a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s)\right]^{\delta}, \quad \text{for } (t,s) \in \Omega. \tag{2.60}$$

Combining (2.55), (2.59), and (2.60) and applying the inequality (1.19) (noting that $\lambda \leq 1$), we have

$$\begin{aligned} y(t,s) &= \int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta) + g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\leq \int_{t_0}^{t}\int_{t_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\quad + \int_{t_0}^{t}\left[g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau \\ &\leq 2^{\lambda\delta((1/\gamma)-1)}\int_{t_0}^{t}\int_{t_0}^{s}f^{\lambda}(\tau,\eta)\left[a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s)\right]^{\lambda\delta}\Delta\eta\Delta\tau \\ &\quad + 2^{\lambda\alpha((1/\gamma)-1)}\int_{t_0}^{t}\int_{t_0}^{s}g^{\lambda}(\tau,\eta)\left[a^{1/\gamma}(t,s) + b^{1/\gamma}(t,s)y^{1/\gamma}(t,s)\right]^{\alpha\lambda}\Delta\eta\Delta\tau. \end{aligned} \tag{2.61}$$

This implies (noting that $\alpha\lambda \leq 1$ and $\delta\lambda \leq 1$) that

$$\begin{aligned} y(t,s) &\leq 2^{\lambda\delta((1/\gamma)-1)}\int_{t_0}^{t}\int_{t_0}^{s}f^{\lambda}(\tau,\eta)a^{\lambda\delta/\gamma}(\tau,\eta)\Delta\eta\Delta\tau \\ &\quad + 2^{\lambda\delta((1/\gamma)-1)}\int_{t_0}^{t}\int_{t_0}^{s}f^{\lambda}(\tau,\eta)b^{\lambda\delta/\gamma}(\tau,\eta)y^{\lambda\delta/\gamma}(\tau,\eta)\Delta\eta\Delta\tau \\ &\quad + 2^{\lambda\alpha((1/\gamma)-1)}\int_{t_0}^{t}\int_{t_0}^{s}g^{\lambda}(\tau,\eta)a^{\lambda\alpha/\gamma}(\tau,\eta)\Delta\eta\Delta\tau \\ &\quad + 2^{\lambda\alpha((1/\gamma)-1)}\int_{t_0}^{t}\int_{t_0}^{s}g^{\lambda}(\tau,\eta)b^{\lambda\alpha/\gamma}(\tau,\eta)y^{\lambda\alpha/\gamma}(\tau,\eta)\Delta\eta\Delta\tau. \end{aligned} \tag{2.62}$$

Applying the Young inequality (1.18) on the term $H_1 y^{\lambda(\delta/\gamma)}$ with $q = \gamma/\lambda\delta > 1$ and $p = \gamma/(\gamma - \lambda\delta) > 1$, we see that

$$H_1 y^{\lambda(\delta/\gamma)} \leq \frac{(\gamma - \lambda\delta)}{\gamma}(H_1)^{\gamma/(\gamma-\lambda\delta)} + \left(\frac{\lambda\delta}{\gamma}\right)y, \tag{2.63}$$

where $H_1 = f^{\lambda}b^{\lambda\delta/\gamma}$. Again applying the Young inequality (1.18) on the term $H_2 y^{\lambda(\alpha/\gamma)}$ with $q = \gamma/\lambda\alpha > 1$ and $p = \gamma/(\gamma - \lambda\alpha) > 1$, we see that

$$H_2 y^{\lambda(\alpha/\gamma)} \leq \frac{(\gamma - \lambda\alpha)}{\gamma}(H_2)^{\gamma/(\gamma-\lambda\alpha)} + \left(\frac{\lambda\alpha}{\gamma}\right)y, \tag{2.64}$$

where $H_2 = g^{\lambda} b^{\lambda\alpha/\gamma}$. Combining (2.62)–(2.64), we have

$$y(t,s) \le H(t,s) + \beta_1 \int_{t_0}^{t} \int_{t_0}^{s} y(\tau,\eta)\Delta\eta\Delta\tau. \tag{2.65}$$

Applying the inequality (1.10) on (2.65) with $f(t,s) = \beta_1$, we have

$$y(t,s) < H(t,s) + e_{\beta_1(s-s_0)}(t,t_0), \quad \forall (t,s) \in \Omega. \tag{2.66}$$

Substituting (2.66) into (2.56), we get the desired inequality (2.45). The proof is complete. □

Remark 2.9. The above results can be extended to the general inequality

$$u^{\gamma}(t,s) \le a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} L(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau, \tag{2.67}$$

when

$$L(\tau,\eta,u) - L(\tau,\eta,v) \le \left[L_1(\tau,\eta,v)(u-v)^{\delta} + L_2(\tau,\eta,v)(u-v)^{\alpha}\right]^{\lambda}. \tag{2.68}$$

In fact, by using the new substitution

$$y(t,s) = \int_{t_0}^{t} \int_{s_0}^{s} L(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau, \tag{2.69}$$

the inequality (2.67) can be written as

$$u^{\gamma}(t,s) \le a(t,s) + b(t,s)y(t,s), \tag{2.70}$$

and then, since $\gamma \ge 1$, we have

$$u(t,s) \le A(t,s) + B(t,s)y(t,s), \tag{2.71}$$

where $A(t,s) = a^{1/\gamma}(t,s)$ and $B(t,s) = a^{(1/\gamma)-1}(t,s)b(t,s)$. This implies that

$$\begin{aligned} y(t,s) &= \int_{t_0}^{t} \int_{s_0}^{s} L(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau \\ &\le \int_{t_0}^{t} \int_{s_0}^{s} [L(\tau,\eta,A(\tau,\eta) + B(\tau,\eta)y(\tau,\eta)) - L(\tau,\eta,A(\tau,\eta))]\Delta\eta\Delta\tau \\ &\quad + \int_{t_0}^{t} \int_{s_0}^{s} L(\tau,\eta,A(\tau,\eta))\Delta\eta\Delta\tau. \end{aligned} \tag{2.72}$$

Using (2.68) in the last inequality, we get an inequality similar to the inequality (2.8) and then follow the proof of the above results to find some explicit bounds of (2.8). The details are left to the reader.

3. Applications

In this section, we present some applications of our main results.

Example 3.1. Consider the partial dynamic equation on time scales

$$(u^{\gamma}(t,s))^{\Delta_t\Delta_s} = H(t,s,u(t,s)) + h(t,s), \qquad (t,s) \in \Omega \equiv [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}, \tag{3.1}$$

with initial boundary conditions

$$u(t,s_0) = a(t) >, \qquad u(t_0,s) = b(s) >, \qquad a(t_0) = b(s_0) = 0, \tag{3.2}$$

where $\gamma \geq 1$ is a constant and H and h are rd-continuous functions on $\Omega, a : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ and $b : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ are rd-continuous functions. Assume that

$$|H(t,s,u)| \leq f(t,s)|u(t,s)|^{\delta} + g(t,s)|u(t,s)|^{\alpha}, \tag{3.3}$$

where $f(t,s)$ and $g(t,s)$ are nonnegative rd-continuous functions for $(t,s) \in \Omega$ and $\alpha, \delta < \gamma$. If $u(t,s)$ is a solution of (3.1)-(3.2), then $u(t,s)$ satisfies

$$|u(t,s)| \leq a^{1/\gamma}(t,s) + A^{1/\gamma}(t,s), \quad \forall (t,s) \in \Omega, \tag{3.4}$$

where

$$\begin{aligned}
a(t,s) &= a^{\gamma}(t) + b^{\gamma}(s) + \int_{t_0}^{t}\int_{s_0}^{s} |h(\tau,\eta)|\Delta\eta\Delta\tau, \\
A(t,s) &:= H_0(t,s) + e_{\beta(s-s_0)}(t,t_0), \quad \beta = \left[\frac{\alpha}{\gamma} + \frac{\delta}{\gamma}\right], \\
H_0(t,s) &= \int_{t_0}^{t}\int_{s_0}^{s} \left[f(\tau,\eta)a^{\delta/\gamma}(\tau,\eta)\right]\Delta\eta\Delta\tau \\
&\quad + \int_{t_0}^{t}\int_{s_0}^{s} \left[g(\tau,\eta)a^{\alpha/\gamma}(\tau,\eta)\right]\Delta\eta\Delta\tau \\
&\quad + \frac{(\gamma-\delta)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s} (f(\tau,s))^{\gamma/(\gamma-\delta)}\Delta\eta\Delta\tau \\
&\quad + \frac{(\gamma-\alpha)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s} (g(\tau,\eta))^{\gamma/(\gamma-\alpha)}\Delta\eta\Delta\tau.
\end{aligned} \tag{3.5}$$

In fact, the solution of (3.1)-(3.2) satisfies

$$|u(t,s)|^{\gamma} = a^{\gamma}(t) + b^{\gamma}(t) + \int_{t_0}^{t}\int_{s_0}^{s} h(\tau,\eta)\Delta\eta\Delta\tau + \int_{t_0}^{t}\int_{s_0}^{s} H(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau, \tag{3.6}$$

for $(t,s) \in \Omega$. Therefore,

$$|u(t,s)|^{\gamma} \le a(t,s) + \int_{t_0}^{t}\int_{s_0}^{s} \left|H(\tau,\eta,u(\tau,\eta))\right|\Delta\eta\Delta\tau, \quad \text{for } (t,s) \in \Omega. \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$|u(t,s)|^{\gamma} \le a(t,s) + \int_{t_0}^{t}\int_{s_0}^{s} f(\tau,s)\left|u(\tau,\eta)\right|^{\delta} + g(\tau,\eta)\left|u(\tau,\eta)\right|^{\alpha}\Delta\eta\Delta\tau, \tag{3.8}$$

for $(t,s) \in \Omega$. Applying Theorem 2.6 on (3.8) with $\lambda = 1$ and $b(t,s) = 1$, we obtain (3.4).

Example 3.2. Consider the equation

$$u^{\gamma}(t,s) = H(t,s,u(t,s)) + h(t,s), \qquad (t,s) \in \Omega \equiv [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}, \tag{3.9}$$

where $\gamma \ge 1$ is a constant and H and h are rd-continuous on $\Omega, a : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^{+}$ and $b : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^{+}$ are rd-continuous functions. Assume that

$$|H(t,s,u)| \le f(t,s)|u(t,s)|^{\delta} + g(t,s)|u(t,s)|^{\alpha}, \tag{3.10}$$

where $f(t,s)$ and $g(t,s)$ are nonnegative rd-continuous functions for $(t,s) \in \Omega$ and $\alpha, \delta < \gamma$. If $u(t,s)$ is a solution of (3.1)-(3.2), then $u(t,s)$ satisfies

$$|u(t,s)| \le |h(t,s)|^{1/\gamma} + \frac{1}{\gamma}|h(t,s)|^{(1/\gamma)-1}B^{1/\gamma}(t,s), \quad \forall (t,s) \in \Omega, \tag{3.11}$$

where

$$\begin{aligned}
B(t,s) &:= F^{*}(t,s) + (1-\lambda)\int_{t_0}^{t}\int_{s_0}^{t} \left(G^{*}(\tau,\eta)\right)^{1/(1-\lambda)}\Delta\eta\Delta\tau + e_{\lambda(s-s_0)}(t,t_0), \\
F^{*}(t,s) &:= \int_{t_0}^{t}\int_{s_0}^{s} \left[f^{\lambda}(\tau,\eta)a^{\lambda(\delta/\gamma)}(\tau,\eta) + g^{\lambda}(\tau,\eta)a^{\lambda(\alpha/\gamma)}(\tau,\eta)\right]\Delta\eta\Delta\tau, \\
G^{*}(t,s) &:= \left(f^{\lambda}(t,s)\left[\frac{\delta}{\gamma}a^{(\delta/\gamma)-1}(t,s)\right]^{\lambda} + g^{\lambda}(t,s)\left[\frac{\alpha}{\gamma}a^{(\alpha/\gamma)-1}(t,s)\right]^{\lambda}\right).
\end{aligned} \tag{3.12}$$

In fact, the solution of (3.9) satisfies

$$|u(t,s)|^{\gamma} \le |h(t,s)| + \int_{t_0}^{t}\int_{s_0}^{s} \left|H(\tau,\eta,u(\tau,\eta))\right| \Delta\eta\Delta\tau, \quad \text{for } (t,s) \in \Omega. \tag{3.13}$$

It follows from (3.10) and (3.13) that

$$|u(t,s)|^{\gamma} \le |h(t,s)| + \int_{t_0}^{t}\int_{s_0}^{s} \left[f(\tau,s)\left|u(\tau,\eta)\right|^{\delta} + g(\tau,\eta)\left|u(\tau,\eta)\right|^{\alpha}\right]^{\lambda} \Delta\eta\Delta\tau, \tag{3.14}$$

for $(t,s) \in \Omega$. Applying Theorem 2.7 on (3.14) with $b(t,s) = 1$, we obtain (3.11).

Example 3.3. Assume that $\mathbb{T} = \mathbb{R}$ and consider the partial differential equation

$$\frac{\partial}{\partial s}\left(u^{\gamma-1}(t,s)\frac{\partial}{\partial t}u(t,s)\right) + H(t,s,u(t,s)) = h(t,s), \quad (t,s) \in \Omega^{*}, \tag{3.15}$$

where $\Omega^{*} = [0,\infty) \times [0,\infty)$, with initial boundary conditions

$$u(t,0) = a(t) > 0, \qquad u(0,s) = b(s) > 0, \qquad a(0) = b(0) = 0. \tag{3.16}$$

Assume that $\gamma \ge 1$ is a constant and $H : [0,\infty)\times[0,\infty)\times\mathbb{R} \to \mathbb{R}$ and $h : [0,\infty)_{\mathbb{R}} \times [0,\infty)_{\mathbb{R}} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}^{+}$ and $b : \mathbb{R} \to \mathbb{R}^{+}$ are continuous functions. Also, we assume that

$$|H(t,s,u)| \le f(t,s)|u(t,s)|^{\delta} + g(t,s)|u(t,s)|^{\alpha}, \tag{3.17}$$

where $f(t,s)$ and $g(t,s)$ are nonnegative continuous functions for $(t,s) \in \Omega^{*}$ and $\alpha,\delta < \gamma$. If $u(t,s)$ is a solution of (3.1)-(3.2), then $u(t,s)$ satisfies

$$|u(t,s)| \le a^{1/\gamma}(t,s) + \gamma^{1/\gamma}B^{1/\gamma}(t,s), \quad \forall (t,s) \in \Omega^{*}, \tag{3.18}$$

where

$$\begin{aligned}
a(t,s) &= a^{\gamma}(t) + b^{\gamma}(s) + \gamma\int_{t_0}^{t}\int_{s_0}^{s} \left|h(\tau,\eta)\right| \Delta\eta\Delta\tau, \\
B(t,s) &:= H_0(t,s) + e_{\beta(s-s_0)}(t,t_0), \quad \beta = \left[\frac{\alpha}{\gamma} + \frac{\delta}{\gamma}\right], \\
H_0(t,s) &= \int_{t_0}^{t}\int_{s_0}^{s} \left[f(\tau,\eta)a^{\delta/\gamma}(\tau,\eta)\right]\Delta\eta\Delta\tau + \int_{t_0}^{t}\int_{s_0}^{s} \left[g(\tau,\eta)a^{\alpha/\gamma}(\tau,\eta)\right]\Delta\eta\Delta\tau \\
&\quad + \frac{(\gamma-\delta)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s} (f(\tau,s))^{\gamma/(\gamma-\delta)}\Delta\eta\Delta\tau \\
&\quad + \frac{(\gamma-\alpha)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s} (g(\tau,\eta))^{\gamma/(\gamma-\alpha)}\Delta\eta\Delta\tau.
\end{aligned} \tag{3.19}$$

In fact, the solution formula of (3.15)-(3.16), after integration twice, is given by

$$\begin{aligned}|u(t,s)|^{\gamma} - a^{\gamma}(t) - b^{\gamma}(t) + \gamma\int_{t_0}^{t}\int_{s_0}^{s} H(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau \\ = \gamma\int_{t_0}^{t}\int_{s_0}^{s} h(\tau,\eta)\Delta\eta\Delta\tau, \quad \text{for } (t,s)\in\Omega^{*}.\end{aligned} \tag{3.20}$$

Therefore,

$$|u(t,s)|^{\gamma} \le a(t,s) + \gamma\int_{t_0}^{t}\int_{s_0}^{s} \left|H(\tau,\eta,u(\tau,\eta))\right|\Delta\eta\Delta\tau, \quad \text{for } (t,s)\in\Omega^{*}. \tag{3.21}$$

It follows from (3.17) and (3.21) that

$$|u(t,s)|^{\gamma} \le a(t,s) + \gamma\int_{t_0}^{t}\int_{s_0}^{s} \left[f(\tau,s)\left|u(\tau,\eta)\right|^{\delta} + g(\tau,\eta)\left|u(\tau,\eta)\right|^{\alpha}\right]\Delta\eta\Delta\tau, \tag{3.22}$$

for $(t,s)\in\Omega^{*}$. Applying Theorem 2.6 on (3.22) with $\lambda = 1$ and $b(t,s) = \gamma$, we obtain (3.18).

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