

## Research Article

# Stability Analysis of a System of Exponential Difference Equations

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We study the boundedness character and persistence, existence and uniqueness of positive equilibrium, local and global behavior, and rate of convergence of positive solutions of the following system of exponential difference equations:  $x_{n+1} = (\alpha_1 + \beta_1 e^{-x_n} + \gamma_1 e^{-x_{n-1}})/(a_1 + b_1 y_n + c_1 y_{n-1})$ ,  $y_{n+1} = (\alpha_2 + \beta_2 e^{-y_n} + \gamma_2 e^{-y_{n-1}})/(a_2 + b_2 x_n + c_2 x_{n-1})$ , where the parameters  $\alpha_i$ ,  $\beta_i$ ,  $\gamma_i$ ,  $a_i$ ,  $b_i$ , and  $c_i$  for  $i \in \{1, 2\}$  and initial conditions  $x_0$ ,  $x_{-1}$ ,  $y_0$ , and  $y_{-1}$  are positive real numbers. Furthermore, by constructing a discrete Lyapunov function, we obtain the global asymptotic stability of the positive equilibrium. Some numerical examples are given to verify our theoretical results.

## 1. Introduction

Many population models are governed by exponential difference equations. We refer to [1–6] and the references therein. Systems of nonlinear difference equations of higher-order are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of systems differential and delay differential equations which model diverse phenomena in biology, ecology, physiology, physics, engineering, and economics. For applications and basic theory of rational difference equations we refer to [7–9]. In [10–17] applications of difference equations in mathematical biology are given. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points.

El-Metwally et al. [1] investigated boundedness character, asymptotic behavior, periodicity nature of the positive solutions, and stability of equilibrium point of the following population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}. \quad (1)$$

Papaschinopoulos et al. [2] studied the boundedness, the persistence, and the asymptotic behavior of positive solutions

of the following two directional interactive and invasive species models:

$$x_{n+1} = a + b x_{n-1} e^{-y_n}, \quad y_{n+1} = c + d y_{n-1} e^{-x_n}. \quad (2)$$

Papaschinopoulos et al. [3] investigated the asymptotic behavior of the solutions of the following three systems of difference equations of exponential form:

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, & y_{n+1} &= \frac{\delta + \epsilon e^{-x_n}}{\zeta + x_{n-1}}, \\ x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, & y_{n+1} &= \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}}, \\ x_{n+1} &= \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, & y_{n+1} &= \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}}. \end{aligned} \quad (3)$$

Recently, Papaschinopoulos and Schinas [4] studied the asymptotic behavior of the positive solutions of the systems of the two difference equations:

$$\begin{aligned} x_{n+1} &= a + b y_{n-1} e^{-y_n}, & y_{n+1} &= c + d x_{n-1} e^{-x_n}, \\ x_{n+1} &= a + b y_{n-1} e^{-x_n}, & y_{n+1} &= c + d x_{n-1} e^{-y_n}. \end{aligned} \quad (4)$$

Motivated by the above study, our aim in this paper is to investigate the qualitative behavior of positive solutions of the following system of exponential difference equations:

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 e^{-x_n} + \gamma_1 e^{-x_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}}, \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 e^{-y_n} + \gamma_2 e^{-y_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}}, \end{aligned} \quad (5)$$

where the parameters  $\alpha_i, \beta_i, \gamma_i, a_i, b_i,$  and  $c_i$  for  $i \in \{1, 2\}$  and initial conditions  $x_0, x_{-1}, y_0,$  and  $y_{-1}$  are positive real numbers.

More precisely, we investigate the boundedness character and persistence, existence and uniqueness of positive steady state, local asymptotic stability and global behavior of unique positive equilibrium point, and rate of convergence of positive solutions of system (5) which converge to its unique positive equilibrium point. Some special cases of system (5) can be treated as population models of two species [3].

## 2. Main Results

The following theorem shows that every solution of (5) is bounded and persists.

**Theorem 1.** *Every positive solution  $\{(x_n, y_n)\}$  of system (5) is bounded and persists.*

*Proof.* For any positive solution  $\{(x_n, y_n)\}$  of system (5), one has

$$\begin{aligned} x_{n+1} &\leq \frac{\alpha_1 + \beta_1 + \gamma_1}{a_1} = U_1, \\ y_{n+1} &\leq \frac{\alpha_2 + \beta_2 + \gamma_2}{a_2} = U_2, \\ n &= 0, 1, 2, \dots \end{aligned} \quad (6)$$

Furthermore, from systems (5) and (6) we obtain that

$$\begin{aligned} x_{n+1} &\geq \frac{\alpha_1 + \beta_1 e^{-U_1} + \gamma_1 e^{-U_1}}{a_1 + b_1 U_2 + c_1 U_2} = L_1, \\ y_{n+1} &\geq \frac{\alpha_2 + \beta_2 e^{-U_2} + \gamma_2 e^{-U_2}}{a_2 + b_2 U_1 + c_2 U_1} = L_2, \\ n &= 2, 3, \dots \end{aligned} \quad (7)$$

From (6) and (7), it follows that

$$L_1 \leq x_n \leq U_1, \quad L_2 \leq y_n \leq U_2, \quad n = 3, 4, \dots \quad (8)$$

Hence, the theorem is proved.  $\square$

**Lemma 2.** *Let  $\{(x_n, y_n)\}$  be a positive solution of system (5). Then,  $[L_1, U_1] \times [L_2, U_2]$  is invariant set for system (5).*

*Proof.* The proof follows by induction.  $\square$

**2.1. Stability Analysis.** Let us consider four-dimensional discrete dynamical system of the form:

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, y_n, y_{n-1}) \\ y_{n+1} &= g(x_n, x_{n-1}, y_n, y_{n-1}), \quad n = 0, 1, \dots, \end{aligned} \quad (9)$$

where  $f: I^2 \times J^2 \rightarrow I$  and  $g: I^2 \times J^2 \rightarrow J$  are continuously differentiable functions and  $I, J$  are some intervals of real numbers. Furthermore, a solution  $\{(x_n, y_n)\}_{n=-1}^{\infty}$  of system (9) is uniquely determined by initial conditions  $(x_i, y_i) \in I \times J$  for  $i \in \{-1, 0\}$ . Along with system (9) we consider the corresponding vector map  $F = (f, x_n, g, y_n)$ . An equilibrium point of (9) is a point  $(\bar{x}, \bar{y})$  that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \bar{y}, \bar{y}), \\ \bar{y} &= g(\bar{x}, \bar{x}, \bar{y}, \bar{y}). \end{aligned} \quad (10)$$

The point  $(\bar{x}, \bar{y})$  is also called a fixed point of the vector map  $F$ .

**Definition 3.** Let  $(\bar{x}, \bar{y})$  be an equilibrium point of the system (9).

- (i) An equilibrium point  $(\bar{x}, \bar{y})$  is said to be stable if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every initial condition  $(x_i, y_i), i \in \{-1, 0\}$ , if  $\|\sum_{i=-1}^0 (x_i, y_i) - (\bar{x}, \bar{y})\| < \delta$  implies  $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$  for all  $n > 0$ , where  $\|\cdot\|$  is usual Euclidian norm in  $\mathbb{R}^2$ .
- (ii) An equilibrium point  $(\bar{x}, \bar{y})$  is said to be unstable if it is not stable.
- (iii) An equilibrium point  $(\bar{x}, \bar{y})$  is said to be asymptotically stable if there exists  $\eta > 0$  such that  $\|\sum_{i=-1}^0 (x_i, y_i) - (\bar{x}, \bar{y})\| < \eta$  and  $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$  as  $n \rightarrow \infty$ .
- (iv) An equilibrium point  $(\bar{x}, \bar{y})$  is called global attractor if  $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$  as  $n \rightarrow \infty$ .
- (v) An equilibrium point  $(\bar{x}, \bar{y})$  is called asymptotic global attractor if it is a global attractor and stable.

**Definition 4.** Let  $(\bar{x}, \bar{y})$  be an equilibrium point of a map  $F = (f, x_n, g, y_n)$ , where  $f$  and  $g$  are continuously differentiable functions at  $(\bar{x}, \bar{y})$ . The linearized system of (9) about the equilibrium point  $(\bar{x}, \bar{y})$  is

$$X_{n+1} = F(X_n) = F_J X_n, \quad (11)$$

where  $X_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$  and  $F_J$  is Jacobian matrix of system (9) about the equilibrium point  $(\bar{x}, \bar{y})$ .

To construct corresponding linearized form of system (5) we consider the following transformation:

$$(x_n, y_n, x_{n-1}, y_{n-1}) \mapsto (f, g, f_1, g_1), \quad (12)$$

where  $f = x_{n+1}, g = y_{n+1}, f_1 = x_n,$  and  $g_1 = y_n$ . The linearized system of (5) about  $(\bar{x}, \bar{y})$  is given by

$$Z_{n+1} = F_J(\bar{x}, \bar{y}) Z_n, \quad (13)$$

where  $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$  and the Jacobian matrix about the fixed point  $(\bar{x}, \bar{y})$  under transformation (12) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (14)$$

where

$$\begin{aligned} A_1 &= -\frac{\beta_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1) \bar{y}}, \\ A_2 &= -\frac{b_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}})}{(a_1 + (b_1 + c_1) \bar{y})^2}, \\ A_3 &= -\frac{\gamma_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1) \bar{y}}, \\ A_4 &= -\frac{c_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}})}{(a_1 + (b_1 + c_1) \bar{y})^2}, \\ B_1 &= -\frac{b_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x})^2}, \\ B_2 &= -\frac{\beta_2 e^{-\bar{y}}}{a_2 + (b_2 + c_2) \bar{x}}, \\ B_3 &= -\frac{c_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x})^2}, \\ B_4 &= -\frac{\gamma_2 e^{-\bar{y}}}{a_2 + (b_2 + c_2) \bar{x}}. \end{aligned} \quad (15)$$

**Lemma 5** (see [9]). Assume that  $X_{n+1} = F(X_n)$ ,  $n = 0, 1, \dots$ , is a system of difference equations such that  $\bar{X}$  is a fixed point of  $F$ . If all eigenvalues of the Jacobian matrix  $J_F$  about  $\bar{X}$  lie inside the open unit disk  $|\lambda| < 1$ , then  $\bar{X}$  is locally asymptotically stable. If one of them has a modulus greater than one, then  $\bar{X}$  is unstable.

**Theorem 6.** System (5) has a unique positive equilibrium point  $(\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$ , if the following condition is satisfied:

$$a_2 + L_1 (b_2 + c_2) < \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-K}}{K}, \quad (16)$$

where

$$K = \frac{\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1) - a_1 L_1}{L_1 (b_1 + c_1)}. \quad (17)$$

*Proof.* Consider the following system of equations:

$$x = \frac{\alpha_1 + (\beta_1 + \gamma_1) e^{-x}}{a_1 + (b_1 + c_1) y}, \quad y = \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-y}}{a_2 + (b_2 + c_2) x}. \quad (18)$$

From (18), it follows that

$$\begin{aligned} y &= \frac{\alpha_1 + (\beta_1 + \gamma_1) e^{-x} - a_1 x}{(b_1 + c_1) x}, \\ x &= \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-y} - a_2 y}{(b_2 + c_2) y}. \end{aligned} \quad (19)$$

Set

$$F(x) = \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-f(x)} - a_2 f(x)}{(b_2 + c_2) f(x)} - x, \quad (20)$$

where  $f(x) = (\alpha_1 + (\beta_1 + \gamma_1) e^{-x} - a_1 x) / ((b_1 + c_1) x)$  and  $x \in [L_1, U_1]$ . Then, it follows that

$$\begin{aligned} F(L_1) &= L_1 (b_1 + c_1) \\ &\times \left( \alpha_2 + (\beta_2 + \gamma_2) e^{-(\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1) - a_1 L_1) / L_1 (b_1 + c_1)} \right. \\ &\quad \left. - \frac{a_2 (\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1) - a_1 L_1)}{L_1 (b_1 + c_1)} \right) \\ &\times ((b_2 + c_2) (\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1) - a_1 L_1))^{-1} - L_1. \end{aligned} \quad (21)$$

Furthermore, it is easy to see that

$$\begin{aligned} (a_1 + (b_1 + c_1) U_2) (\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1)) \\ > a_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-U_1}). \end{aligned} \quad (22)$$

From (22) it follows that

$$\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1) - a_1 L_1 > 0. \quad (23)$$

Let  $K = (\alpha_1 + e^{-L_1} (\beta_1 + \gamma_1) - a_1 L_1) / (L_1 (b_1 + c_1)) > 0$ . Then  $F(L_1)$  can be expressed as

$$F(L_1) = \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-K} - a_2 K}{(b_2 + c_2) K} - L_1. \quad (24)$$

Suppose that  $a_2 + L_1 (b_2 + c_2) < (\alpha_2 + (\beta_2 + \gamma_2) e^{-K}) / K$ ; then, it follows that  $F(L_1) > 0$ . Furthermore, we have

$$\begin{aligned} F(U_1) &= U_1 (b_1 + c_1) \left( (\beta_2 + \gamma_2) e^{(\beta_1 + \gamma_1)(1 - e^{-U_1}) / U_1 (b_1 + c_1)} \right. \\ &\quad \left. + \alpha_2 + \frac{a_2 (\beta_1 + \gamma_1) (1 - e^{-U_1})}{U_1 (b_1 + c_1)} \right) \\ &\times ((b_2 + c_2) (\alpha_1 + e^{-U_1} (\beta_1 + \gamma_1) - a_1 U_1))^{-1} - U_1. \end{aligned} \quad (25)$$

Then it is easy to see that

$$\alpha_1 + e^{-U_1} (\beta_1 + \gamma_1) - a_1 U_1 = (\beta_1 + \gamma_1) (e^{-U_1} - 1) < 0, \quad (26)$$

which gives that  $F(U_1) < 0$ . Hence,  $F(x) = 0$  has at least one positive solution in  $[L_1, U_1]$ . Moreover, we obtain that

$$F'(x) = -1 - \frac{f'(x)}{(b_2 + c_2)f(x)} \times [a_2 + (b_2 + c_2)x + (\beta_2 + \gamma_2)e^{-f(x)}], \quad (27)$$

where

$$f'(x) = -\frac{\alpha_1 + (\beta_1 + \gamma_1)(1+x)e^{-x}}{(b_1 + c_1)x^2}. \quad (28)$$

Then, from (28) it follows that  $f(U_1) < f(x) < f(L_1)$  and using (26) we obtain

$$\begin{aligned} F'(x) &= -1 + \frac{\alpha_1 + (\beta_1 + \gamma_1)(1+x)e^{-x}}{(b_1 + c_1)(b_2 + c_2)x^2 f(x)} \\ &\quad \times [a_2 + (b_2 + c_2)x + (\beta_2 + \gamma_2)e^{-f(x)}] \\ &\leq -1 + \frac{\alpha_1 + (\beta_1 + \gamma_1)(1+x)e^{-x}}{(b_1 + c_1)(b_2 + c_2)x^2 f(U_1)} \\ &\quad \times [a_2 + (b_2 + c_2)x + (\beta_2 + \gamma_2)e^{-f(x)}] < 0. \end{aligned} \quad (29)$$

Hence,  $F(x) = 0$  has a unique positive solution in  $[L_1, U_1]$ . The proof is therefore completed.  $\square$

**Theorem 7.** *The unique positive equilibrium point of system (5) is locally asymptotically stable under the following condition:*

$$\begin{aligned} &\frac{\beta_1 e^{-L_1}}{a_1 + (b_1 + c_1)L_2} + \left(1 + \frac{\beta_1 e^{-L_1}}{a_1 + (b_1 + c_1)L_2}\right) \\ &\quad \times \left(\frac{c_2(\alpha_2 + (\beta_2 + \gamma_2)e^{-L_2})}{(a_2 + (b_2 + c_2)L_1)^2} + \frac{(\beta_2 + \gamma_2)e^{-L_2}}{a_2 + (b_2 + c_2)L_1}\right) \\ &\quad + \frac{b_2(\alpha_2 + (\beta_2 + \gamma_2)e^{-L_2})}{(a_2 + (b_2 + c_2)L_1)^2} \\ &\quad \times \left(\frac{(b_1 + c_1)(\alpha_1 + (\beta_1 + \gamma_1)e^{-L_1})}{(a_1 + (b_1 + c_1)L_2)^2} + \frac{\gamma_1 e^{-L_1}}{a_1 + (b_1 + c_1)L_2}\right) < 1. \end{aligned} \quad (30)$$

*Proof.* The characteristic polynomial of Jacobian matrix  $F_J(\bar{x}, \bar{y})$  about  $(\bar{x}, \bar{y})$  is given by

$$\begin{aligned} P(\lambda) &= \lambda^4 - (A_1 + B_2)\lambda^3 - (A_2B_1 - A_1B_2 + B_3 + B_4)\lambda^2 \\ &\quad - ((A_3 + A_4)B_1 - (B_3 + B_4)A_1)\lambda. \end{aligned} \quad (31)$$

Clearly, one root of  $P(\lambda)$  is 0. To check the behavior of the other three roots of  $P(\lambda)$ , we let  $\Phi(\lambda) = \lambda^3$  and  $\Psi(\lambda) = (A_1 + B_2)\lambda^2 + (A_2B_1 - A_1B_2 + B_3 + B_4)\lambda + (A_3 + A_4)B_1 - (B_3 + B_4)A_1$ . Assume that (30) holds and  $|\lambda| = 1$ ; then, one has

$$\begin{aligned} |\Psi(\lambda)| &< |A_1 + B_2| + |A_2B_1 - A_1B_2 + B_3 + B_4| \\ &\quad + |(A_3 + A_4)B_1 - (B_3 + B_4)A_1| \\ &< \frac{\beta_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1)\bar{y}} + \left(1 + \frac{\beta_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1)\bar{y}}\right) \\ &\quad \times \left(\frac{c_2(\alpha_2 + (\beta_2 + \gamma_2)e^{-\bar{y}})}{(a_2 + (b_2 + c_2)\bar{x})^2} + \frac{(\beta_2 + \gamma_2)e^{-\bar{y}}}{a_2 + (b_2 + c_2)\bar{x}}\right) \\ &\quad + \frac{b_2(\alpha_2 + (\beta_2 + \gamma_2)e^{-\bar{y}})}{(a_2 + (b_2 + c_2)\bar{x})^2} \\ &\quad \times \left(\frac{(b_1 + c_1)(\alpha_1 + (\beta_1 + \gamma_1)e^{-\bar{x}})}{(a_1 + (b_1 + c_1)\bar{y})^2} + \frac{\gamma_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1)\bar{y}}\right) \\ &< \frac{\beta_1 e^{-L_1}}{a_1 + (b_1 + c_1)L_2} + \left(1 + \frac{\beta_1 e^{-L_1}}{a_1 + (b_1 + c_1)L_2}\right) \\ &\quad \times \left(\frac{c_2(\alpha_2 + (\beta_2 + \gamma_2)e^{-L_2})}{(a_2 + (b_2 + c_2)L_1)^2} + \frac{(\beta_2 + \gamma_2)e^{-L_2}}{a_2 + (b_2 + c_2)L_1}\right) \\ &\quad + \frac{b_2(\alpha_2 + (\beta_2 + \gamma_2)e^{-L_2})}{(a_2 + (b_2 + c_2)L_1)^2} \\ &\quad \times \left(\frac{(b_1 + c_1)(\alpha_1 + (\beta_1 + \gamma_1)e^{-L_1})}{(a_1 + (b_1 + c_1)L_2)^2} + \frac{\gamma_1 e^{-L_1}}{a_1 + (b_1 + c_1)L_2}\right) \\ &< 1. \end{aligned} \quad (32)$$

Then, by Rouché's Theorem,  $\Phi(\lambda)$  and  $\Phi(\lambda) - \Psi(\lambda)$  have the same number of zeroes in an open unit disk  $|\lambda| < 1$ . Hence, all the roots of (31) satisfy  $|\lambda| < 1$ , and it follows from Lemma 5 that the unique positive equilibrium point  $(\bar{x}, \bar{y})$  of the system (5) is locally asymptotically stable.  $\square$

**Theorem 8.** *The unique positive equilibrium point  $(\bar{x}, \bar{y})$  of system (5) is globally asymptotically stable, if the following condition is satisfied:*

$$\begin{aligned} \alpha_1 + (\beta_1 + \gamma_1)e^{-L_1} &< \bar{x}(a_1 + (b_1 + c_1)L_2), \\ \alpha_2 + (\beta_2 + \gamma_2)e^{-L_2} &< \bar{y}(a_2 + (b_2 + c_2)L_1). \end{aligned} \quad (33)$$

*Proof.* Arguing as in [18], we consider the following discrete time analogue of Lyapunov function:

$$V_n = \bar{x} \left( \frac{x_n}{\bar{x}} - 1 - \ln \frac{x_n}{\bar{x}} \right) + \bar{y} \left( \frac{y_n}{\bar{y}} - 1 - \ln \frac{y_n}{\bar{y}} \right). \quad (34)$$

Then nonnegativity of  $V_n$  follows from the following inequality:

$$x - 1 - \ln x \geq 0 \quad \forall x > 0. \quad (35)$$

Furthermore, we have

$$\begin{aligned} -\ln \left( \frac{x_{n+1}}{x_n} \right) &= \ln \left( 1 - \left( 1 - \frac{x_n}{x_{n+1}} \right) \right) \leq -\frac{x_{n+1} - x_n}{x_{n+1}}, \\ -\ln \left( \frac{y_{n+1}}{y_n} \right) &= \ln \left( 1 - \left( 1 - \frac{y_n}{y_{n+1}} \right) \right) \leq -\frac{y_{n+1} - y_n}{y_{n+1}}. \end{aligned} \quad (36)$$

Assume that (33) holds true; then, it follows that

$$\begin{aligned} V_{n+1} - V_n &= \bar{x} \left( \frac{x_{n+1}}{\bar{x}} - 1 - \ln \frac{x_{n+1}}{\bar{x}} \right) \\ &\quad + \bar{y} \left( \frac{y_{n+1}}{\bar{y}} - 1 - \ln \frac{y_{n+1}}{\bar{y}} \right) \\ &\quad - \bar{x} \left( \frac{x_n}{\bar{x}} - 1 - \ln \frac{x_n}{\bar{x}} \right) - \bar{y} \left( \frac{y_n}{\bar{y}} - 1 - \ln \frac{y_n}{\bar{y}} \right) \\ &\leq (x_{n+1} - x_n) + (y_{n+1} - y_n) \\ &\quad - \frac{\bar{x}}{x_{n+1}} (x_{n+1} - x_n) - \frac{\bar{y}}{y_{n+1}} (y_{n+1} - y_n) \\ &= (x_{n+1} - x_n) \left( 1 - \frac{\bar{x}}{x_{n+1}} \right) \\ &\quad + (y_{n+1} - y_n) \left( 1 - \frac{\bar{y}}{y_{n+1}} \right) \\ &\leq (U_1 - L_1) \\ &\quad \times \left( \frac{\alpha_1 + (\beta_1 + \gamma_1) e^{-L_1} - \bar{x} (a_1 + (b_1 + c_1) L_2)}{\alpha_1 + (\beta_1 + \gamma_1) e^{-L_1}} \right) \\ &\quad + (U_2 - L_2) \\ &\quad \times \left( \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-L_2} - \bar{y} (a_2 + (b_2 + c_2) L_1)}{\alpha_2 + (\beta_2 + \gamma_2) e^{-L_2}} \right) \\ &\leq 0 \end{aligned} \quad (37)$$

for all  $n \geq 0$  so that  $V_n \geq 0$  is monotonically decreasing sequence. It follows that  $\lim_{n \rightarrow \infty} V_n \geq 0$ . Hence, we obtain that

$$\lim_{n \rightarrow \infty} (V_{n+1} - V_n) = 0. \quad (38)$$

Then it follows that  $\lim_{n \rightarrow \infty} x_{n+1} = \bar{x}$  and  $\lim_{n \rightarrow \infty} y_{n+1} = \bar{y}$ . Furthermore,  $V_n \leq V_0$  for all  $n \geq 0$ , which gives that  $(\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$  is uniformly stable. Hence, unique positive equilibrium point  $(\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$  of system (5) is globally asymptotically stable.  $\square$

**2.2. Rate of Convergence.** In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of the system (5).

The following result gives the rate of convergence of solutions of a system of difference equations:

$$X_{n+1} = (A + B(n)) X_n, \quad (39)$$

where  $X_n$  is an  $m$ -dimensional vector,  $A \in \mathbb{C}^{m \times m}$  is a constant matrix, and  $B : \mathbb{Z}^+ \rightarrow \mathbb{C}^{m \times m}$  is a matrix function satisfying

$$\|B(n)\| \rightarrow 0 \quad (40)$$

as  $n \rightarrow \infty$ , where  $\|\cdot\|$  denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}. \quad (41)$$

**Proposition 9** (Perron's Theorem) [19]. *Suppose that condition (40) holds. If  $X_n$  is a solution of (39), then either  $X_n = 0$  for all large  $n$  or*

$$\rho = \lim_{n \rightarrow \infty} (\|X_n\|)^{1/n} \quad (42)$$

*exists and is equal to the modulus of one of the eigenvalues of matrix  $A$ .*

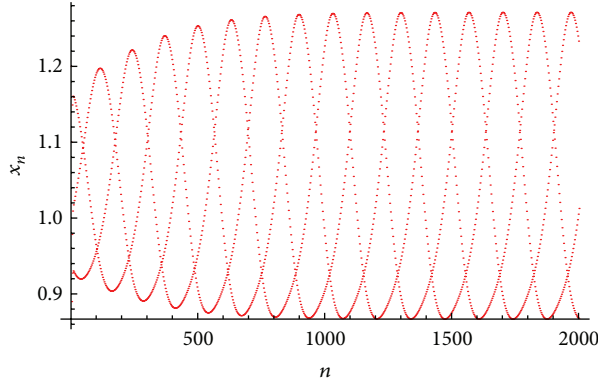
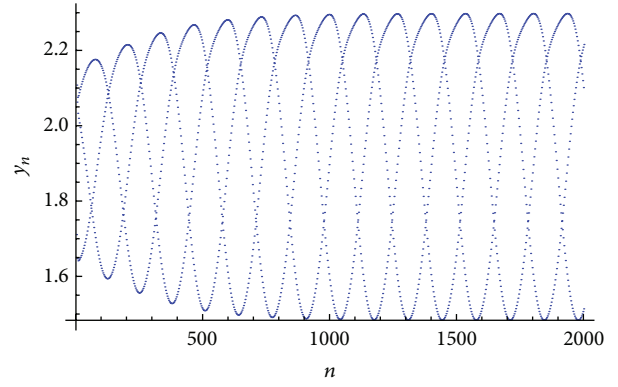
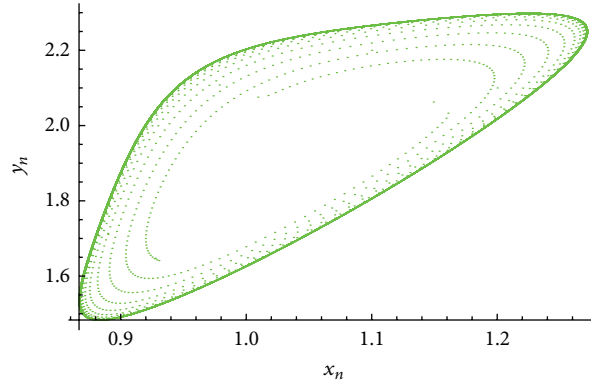
**Proposition 10** (see [19]). *Suppose that condition (40) holds. If  $X_n$  is a solution of (39), then either  $X_n = 0$  for all large  $n$  or*

$$\rho = \lim_{n \rightarrow \infty} \frac{\|X_{n+1}\|}{\|X_n\|} \quad (43)$$

*exists and is equal to the modulus of one of the eigenvalues of matrix  $A$ .*

Let  $\{(x_n, y_n)\}$  be any solution of the system (5) such that  $\lim_{n \rightarrow \infty} x_n = \bar{x}$  and  $\lim_{n \rightarrow \infty} y_n = \bar{y}$ , where  $\bar{x} \in [L_1, U_1]$  and  $\bar{y} \in [L_2, U_2]$ . To find the error terms, one has from the system (5)

$$\begin{aligned} &x_{n+1} - \bar{x} \\ &= \frac{\alpha_1 + \beta_1 e^{-x_n} + \gamma_1 e^{-x_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}} - \frac{\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}}}{a_1 + (b_1 + c_1) \bar{y}} \\ &= \frac{\beta_1 (e^{-x_n} - e^{-\bar{x}})}{(a_1 + b_1 y_n + c_1 y_{n-1}) (x_n - \bar{x})} (x_n - \bar{x}) \\ &\quad - \frac{b_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}})}{(a_1 + (b_1 + c_1) \bar{y}) (a_1 + b_1 y_n + c_1 y_{n-1})} (y_n - \bar{y}) \\ &\quad + \frac{\gamma_1 (e^{-x_{n-1}} - e^{-\bar{x}})}{(a_1 + b_1 y_n + c_1 y_{n-1}) (x_{n-1} - \bar{x})} (x_{n-1} - \bar{x}) \\ &\quad - \frac{c_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}})}{(a_1 + (b_1 + c_1) \bar{y}) (a_1 + b_1 y_n + c_1 y_{n-1})} (y_{n-1} - \bar{y}), \end{aligned}$$

(a) Plot of  $x_n$  for the system (50)(b) Plot of  $y_n$  for the system (50)

(c) Phase portrait of system (50)

FIGURE 1: Plots for the system (50).

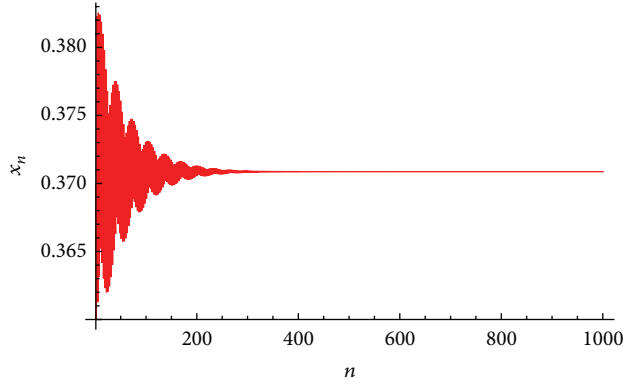
$$\begin{aligned}
& y_{n+1} - \bar{y} \\
&= \frac{\alpha_2 + \beta_2 e^{-y_n} + \gamma_2 e^{-y_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}} - \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}}}{a_2 + (b_2 + c_2) \bar{x}} \\
&= \frac{\beta_2 (e^{-y_n} - e^{-\bar{y}})}{(a_2 + b_2 x_n + c_2 x_{n-1}) (y_n - \bar{y})} (y_n - \bar{y}) \\
&\quad - \frac{b_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x}) (a_2 + b_2 x_n + c_2 x_{n-1})} (x_n - \bar{x}) \\
&\quad + \frac{\gamma_2 (e^{-y_{n-1}} - e^{-\bar{y}})}{(a_2 + b_2 x_n + c_2 x_{n-1}) (y_{n-1} - \bar{y})} (y_{n-1} - \bar{y}) \\
&\quad - \frac{c_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x}) (a_2 + b_2 x_n + c_2 x_{n-1})} (x_{n-1} - \bar{x}). \tag{44}
\end{aligned}$$

Let  $e_n^1 = x_n - \bar{x}$  and  $e_n^2 = y_n - \bar{y}$ ; then, one has

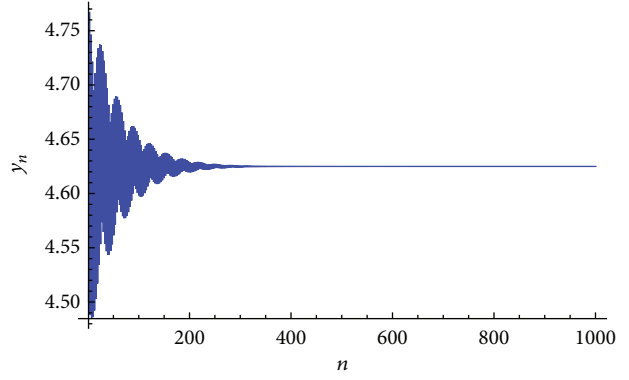
$$\begin{aligned}
e_{n+1}^1 &= a_n e_n^1 + b_n e_n^2 + c_n e_{n-1}^1 + d_n e_{n-1}^2, \\
e_{n+1}^2 &= f_n e_n^1 + g_n e_n^2 + h_n e_{n-1}^1 + k_n e_{n-1}^2, \tag{45}
\end{aligned}$$

where

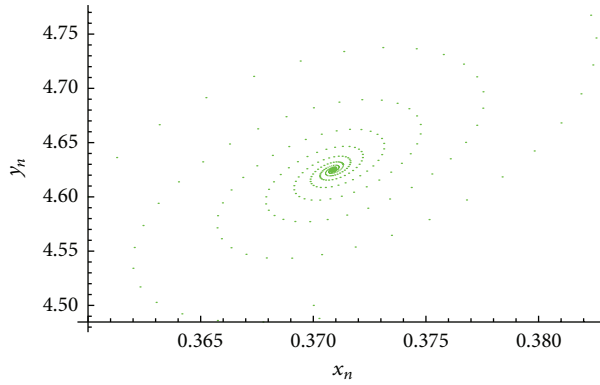
$$\begin{aligned}
a_n &= \frac{\beta_1 (e^{-x_n} - e^{-\bar{x}})}{(a_1 + b_1 y_n + c_1 y_{n-1}) (x_n - \bar{x})}, \\
b_n &= -\frac{b_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}})}{(a_1 + (b_1 + c_1) \bar{y}) (a_1 + b_1 y_n + c_1 y_{n-1})}, \\
c_n &= \frac{\gamma_1 (e^{-x_{n-1}} - e^{-\bar{x}})}{(a_1 + b_1 y_n + c_1 y_{n-1}) (x_{n-1} - \bar{x})}, \\
d_n &= -\frac{c_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}})}{(a_1 + (b_1 + c_1) \bar{y}) (a_1 + b_1 y_n + c_1 y_{n-1})}, \\
f_n &= -\frac{b_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x}) (a_2 + b_2 x_n + c_2 x_{n-1})}, \\
g_n &= \frac{\beta_2 (e^{-y_n} - e^{-\bar{y}})}{(a_2 + b_2 x_n + c_2 x_{n-1}) (y_n - \bar{y})}, \\
h_n &= -\frac{c_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x}) (a_2 + b_2 x_n + c_2 x_{n-1})}, \\
k_n &= \frac{\gamma_2 (e^{-y_{n-1}} - e^{-\bar{y}})}{(a_2 + b_2 x_n + c_2 x_{n-1}) (y_{n-1} - \bar{y})}. \tag{46}
\end{aligned}$$



(a) Plot of  $x_n$  for the system (51)



(b) Plot of  $y_n$  for the system (51)



(c) An attractor of the system (51)

FIGURE 2: Plots for the system (51).

Moreover,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n &= -\frac{\beta_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1) \bar{y}} = A_1, \\
 \lim_{n \rightarrow \infty} b_n &= -\frac{b_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}})}{(a_1 + (b_1 + c_1) \bar{y})^2} = A_2, \\
 \lim_{n \rightarrow \infty} c_n &= -\frac{\gamma_1 e^{-\bar{x}}}{a_1 + (b_1 + c_1) \bar{y}} = A_3, \\
 \lim_{n \rightarrow \infty} d_n &= -\frac{c_1 (\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{x}})}{(a_1 + (b_1 + c_1) \bar{y})^2} = A_4, \\
 \lim_{n \rightarrow \infty} f_n &= -\frac{b_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x})^2} = B_1, \\
 \lim_{n \rightarrow \infty} g_n &= -\frac{\beta_2 e^{-\bar{y}}}{a_2 + (b_2 + c_2) \bar{x}} = B_2, \\
 \lim_{n \rightarrow \infty} h_n &= -\frac{c_2 (\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{y}})}{(a_2 + (b_2 + c_2) \bar{x})^2} = B_3, \\
 \lim_{n \rightarrow \infty} k_n &= -\frac{\gamma_2 e^{-\bar{y}}}{a_2 + (b_2 + c_2) \bar{x}} = B_4.
 \end{aligned}$$

(47)

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e_{n+1}^1 \\ e_{n+1}^2 \\ e_n^1 \\ e_n^2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{bmatrix}, \quad (48)$$

which is similar to linearized system of (5) about the equilibrium point  $(\bar{x}, \bar{y})$ .

Using Proposition 9, one has the following result.

**Theorem 11.** Assume that  $\{(x_n, y_n)\}$  is a positive solution of the system (5) such that  $\lim_{n \rightarrow \infty} x_n = \bar{x}$ , and  $\lim_{n \rightarrow \infty} y_n = \bar{y}$ , where  $\bar{x} \in [L_1, U_1]$  and  $\bar{y} \in [L_2, U_2]$ . Then, the error vector

$e_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix}$  of every solution of (5) satisfies both of the

following asymptotic relations:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} (\|e_n\|)^{1/n} &= |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|, \\
 \lim_{n \rightarrow \infty} \frac{\|e_{n+1}\|}{\|e_n\|} &= |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|,
 \end{aligned} \quad (49)$$

where  $\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})$  are the characteristic roots of Jacobian matrix  $F_J(\bar{x}, \bar{y})$ .

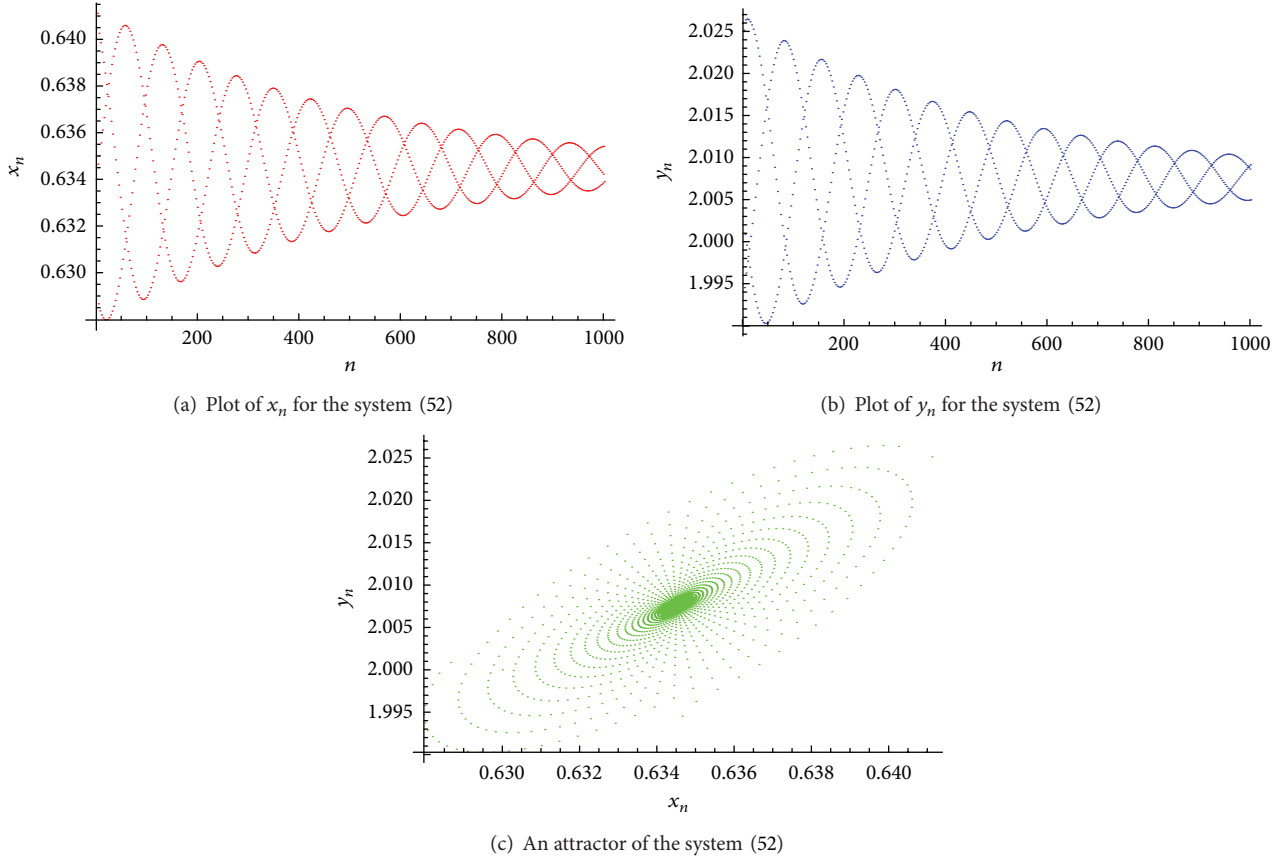


FIGURE 3: Plots for the system (52).

**2.3. Examples.** In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations (5). The first and last examples show that positive equilibrium of system (5) is unstable with suitable parametric choices. Moreover, from the remaining examples it is clear that unique positive equilibrium point of system (5) is globally asymptotically stable with different parametric values.

*Example 1.* Let  $\alpha_1 = 6.5$ ,  $\beta_1 = 8.5$ ,  $\gamma_1 = 12.3$ ,  $a_1 = 1.6$ ,  $b_1 = 5.02$ ,  $c_1 = 1.2$ ,  $\alpha_2 = 3.5$ ,  $\beta_2 = 5.5$ ,  $\gamma_2 = 1.3$ ,  $a_2 = 0.1$ ,  $b_2 = 0.003$ , and  $c_2 = 2.2$ . Then, system (5) can be written as

$$\begin{aligned} x_{n+1} &= \frac{6.5 + 12.3e^{-x_n} + 8.5e^{-x_{n-1}}}{1.6 + 1.2y_n + 5.02y_{n-1}}, \\ y_{n+1} &= \frac{3.5 + 1.3e^{-y_n} + 5.5e^{-y_{n-1}}}{0.1 + 2.2x_n + 0.003x_{n-1}}, \end{aligned} \quad (50)$$

with initial conditions  $x_{-1} = 1.2$ ,  $x_0 = 1.1$ ,  $y_{-1} = 2.1$ , and  $y_0 = 2.2$ .

In this case the positive equilibrium point of the system (50) is unstable. Moreover, in Figure 1 the plot of  $x_n$  is shown in Figure 1(a), the plot of  $y_n$  is shown in Figure 1(b), and a phase portrait of the system (50) is shown in Figure 1(c).

*Example 2.* Let  $\alpha_1 = 1.5$ ,  $\beta_1 = 7.5$ ,  $\gamma_1 = 13.5$ ,  $a_1 = 2.1$ ,  $b_1 = 7.92$ ,  $c_1 = 0.95$ ,  $\alpha_2 = 4.1$ ,  $\beta_2 = 7.2$ ,  $\gamma_2 = 16.7$ ,  $a_2 = 0.01$ ,  $b_2 = 0.2$ , and  $c_2 = 2.3$ . Then, system (5) can be written as

$$\begin{aligned} x_{n+1} &= \frac{1.5 + 7.5e^{-x_n} + 13.5e^{-x_{n-1}}}{2.1 + 7.92y_n + 0.95y_{n-1}}, \\ y_{n+1} &= \frac{4.1 + 7.2e^{-y_n} + 16.7e^{-y_{n-1}}}{0.01 + 0.2x_n + 2.3x_{n-1}}, \end{aligned} \quad (51)$$

with initial conditions  $x_{-1} = 0.37$ ,  $x_0 = 0.36$ ,  $y_{-1} = 4.5$ , and  $y_0 = 4.6$ .

In this case the unique positive equilibrium point of the system (51) is given by  $(\bar{x}, \bar{y}) = (0.370864, 4.62495)$ . Moreover, in Figure 2 the plot of  $x_n$  is shown in Figure 2(a), the plot of  $y_n$  is shown in Figure 2(b), and an attractor of the system (51) is shown in Figure 2(c).

*Example 3.* Let  $\alpha_1 = 8.98$ ,  $\beta_1 = 75$ ,  $\gamma_1 = 135$ ,  $a_1 = 21$ ,  $b_1 = 79$ ,  $c_1 = 5$ ,  $\alpha_2 = 41$ ,  $\beta_2 = 71.9$ ,  $\gamma_2 = 16$ ,  $a_2 = 6$ ,  $b_2 = 2$ , and  $c_2 = 30$ . Then, system (5) can be written as

$$\begin{aligned} x_{n+1} &= \frac{8.98 + 75e^{-x_n} + 135e^{-x_{n-1}}}{21 + 79y_n + 5y_{n-1}}, \\ y_{n+1} &= \frac{41 + 71.9e^{-y_n} + 16e^{-y_{n-1}}}{6 + 2x_n + 30x_{n-1}}, \end{aligned} \quad (52)$$



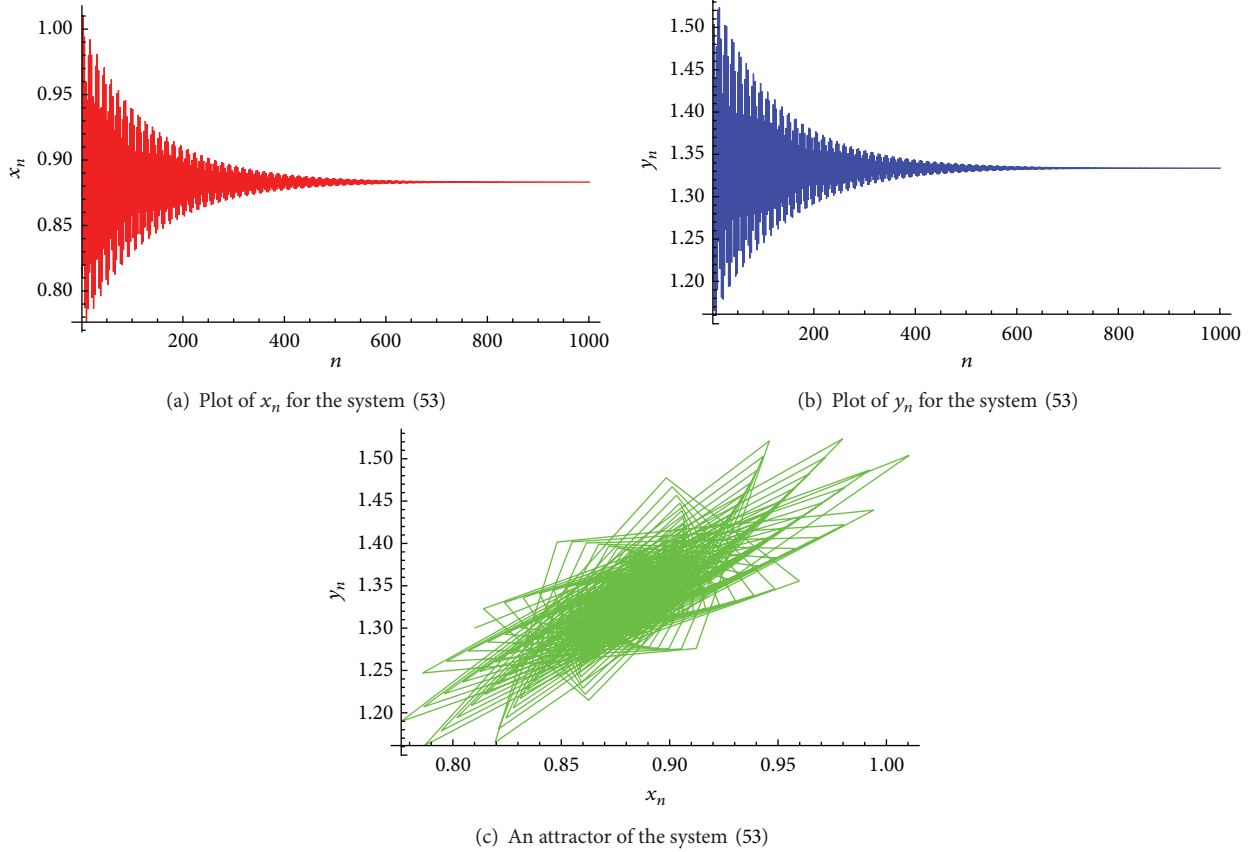


FIGURE 4: Plots for the system (53).

with initial conditions  $x_{-1} = 0.634$ ,  $x_0 = 0.63$ ,  $y_{-1} = 1.99$ , and  $y_0 = 2$ .

In this case the unique positive equilibrium point of the system (52) is given by  $(\bar{x}, \bar{y}) = (0.634497, 2.00755)$ . Moreover, in Figure 3 the plot of  $x_n$  is shown in Figure 3(a), the plot of  $y_n$  is shown in Figure 3(b), and an attractor of the system (52) is shown in Figure 3(c).

*Example 4.* Let  $\alpha_1 = 23.8$ ,  $\beta_1 = 350$ ,  $\gamma_1 = 66$ ,  $a_1 = 55$ ,  $b_1 = 14$ ,  $c_1 = 111$ ,  $\alpha_2 = 66$ ,  $\beta_2 = 35$ ,  $\gamma_2 = 87$ ,  $a_2 = 1.7$ ,  $b_2 = 80$ , and  $c_2 = 1.4$ . Then, system (5) can be written as

$$\begin{aligned} x_{n+1} &= \frac{23.8 + 350e^{-x_n} + 66e^{-x_{n-1}}}{55 + 14y_n + 111y_{n-1}}, \\ y_{n+1} &= \frac{66 + 35e^{-y_n} + 87e^{-y_{n-1}}}{1.7 + 80x_n + 1.4x_{n-1}}, \end{aligned} \tag{53}$$

with initial conditions  $x_{-1} = 0.82$ ,  $x_0 = 0.81$ ,  $y_{-1} = 1.2$ , and  $y_0 = 1.3$ .

In this case the unique positive equilibrium point of the system (53) is given by  $(\bar{x}, \bar{y}) = (0.883148, 1.33372)$ . Moreover, in Figure 4 the plot of  $x_n$  is shown in Figure 4(a), the plot of  $y_n$  is shown in Figure 4(b), and an attractor of system (53) is shown in Figure 4(c).

*Example 5.* Let  $\alpha_1 = 8$ ,  $\beta_1 = 3$ ,  $\gamma_1 = 6$ ,  $a_1 = 0.05$ ,  $b_1 = 4$ ,  $c_1 = 16$ ,  $\alpha_2 = 9$ ,  $\beta_2 = 17$ ,  $\gamma_2 = 7$ ,  $a_2 = 0.07$ ,  $b_2 = 22$ , and  $c_2 = 0.2$ . Then, system (5) can be written as

$$\begin{aligned} x_{n+1} &= \frac{8 + 3e^{-x_n} + 6e^{-x_{n-1}}}{0.05 + 4y_n + 16y_{n-1}}, \\ y_{n+1} &= \frac{9 + 17e^{-y_n} + 7e^{-y_{n-1}}}{0.07 + 22x_n + 0.2x_{n-1}}, \end{aligned} \tag{54}$$

with initial conditions  $x_{-1} = 0.5$ ,  $x_0 = 0.4$ ,  $y_{-1} = 1.2$ , and  $y_0 = 1.3$ .

In this case the positive equilibrium point of the system (54) is unstable. Moreover, in Figure 5 the plot of  $x_n$  is shown in Figure 5(a), the plot of  $y_n$  is shown in Figure 5(b), and a phase portrait of system (54) is shown in Figure 5(c).

### 3. Concluding Remarks

In literature several articles are related to qualitative behavior of exponential systems of rational difference equations. It is a very interesting mathematical problem to study the dynamics of such systems because these are closely related to models in population dynamics and biological sciences. This work is related to qualitative behavior of an exponential system of second-order rational difference equations. We have investigated the existence and uniqueness of positive steady

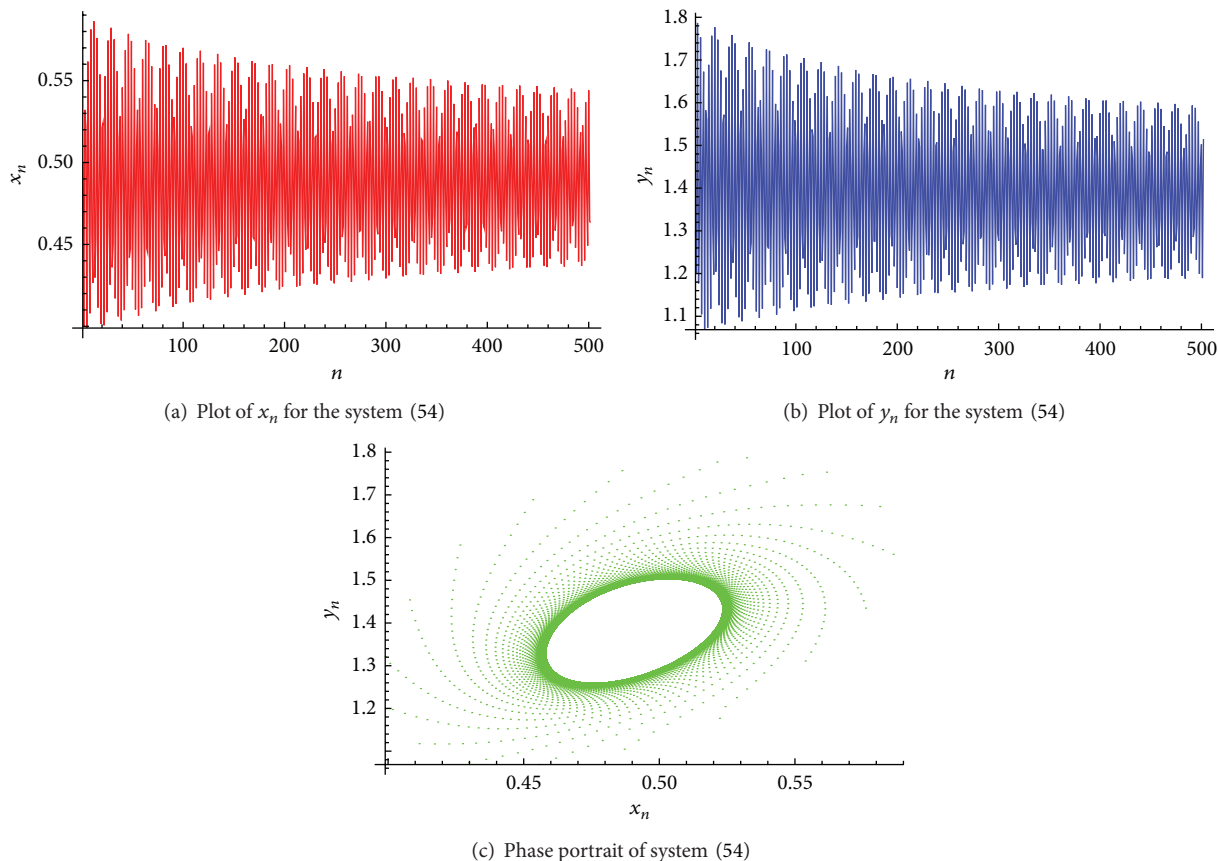


FIGURE 5: Plots for the system (54).

state of system (5). Under certain parametric conditions the boundedness and persistence of positive solutions are proved. Moreover, we have shown that unique positive equilibrium point of system (5) is locally as well as globally asymptotically stable. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which parametric conditions lead to these long-term behaviors. By constructing a discrete Lyapunov function, we have obtained the global asymptotic stability of the positive equilibrium of (5). Furthermore, rate of convergence of positive solutions of (5) which converge to its unique positive equilibrium point is demonstrated. Finally, some illustrative examples are provided to support our theoretical discussion.

**Conflict of Interests**

The authors declare that they have no conflict of interests regarding the publication of this paper.

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