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Research Article

Lyapunov Functions for a Class of Discrete SIRS Epidemic Models with Nonlinear Incidence Rate and Varying Population Sizes

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We investigate the dynamical behaviors of a class of discrete SIRS epidemic models with nonlinear incidence rate and varying population sizes. The model is required to possess different death rates for the susceptible, infectious, recovered, and constant recruitment into the susceptible class, infectious class, and recovered class, respectively. By using the inductive method, the positivity and boundedness of all solutions are obtained. Furthermore, by constructing new discrete type Lyapunov functions, the sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium are established.

1. Introduction

As well known in the theoretical study of epidemic models, the susceptible-infected-recovered (SIR) compartmental epidemic models are a kind of very important epidemic models and in recent years have been widely investigated. According to the assumptions of Kermack and McKendrick [1], the population of size $N(t)$ at time t is divided into three distinct classes: the susceptible class of size $S(t)$, the infectious class of size $I(t)$, and the recovered class $R(t)$ at time t. When a susceptible individual acquires the disease by contacting with an infectious individual, the susceptible individual moves into the infectious compartment and, subsequently, as a result of some measures such as medication or isolation the infector takes into the recovered class. If the recovered individuals retain their immunity permanently, then he/her remains in the recovered compartment. The model based on these assumptions is known as the SIR epidemic model. Furthermore, if the immunity is not permanent, that is, the recovered individual may lose his/her immunity after a period of time, then he/her returns to the susceptible class. Thus, we obtain the SIRS epidemic model.

Usually, there are two kinds of epidemic dynamical models: the continuous-time models described by differential equations and the discrete-time models described by difference equations. In this paper, we will focus our attention on discrete-time epidemic dynamical models. For an epidemic model, which is continuous-time model or discrete-time model, we all know that an important subject is to determine the global stability of the disease-free equilibrium and endemic equilibrium. Particularly, we expect to compute basic reproduction number $\mathscr{R}_0$ of the model and also to obtain the fact that the disease-free equilibrium is globally stable when $\mathscr{R}_0 \leq 1$, as well as the endemic equilibrium exists and is globally stable when $\mathscr{R}_0 > 1$.

Until now, the discrete-time SIR and SIRS epidemic models have been extensively studied in many articles; for example, see [2–22] and the reference therein. Many important results have been established. These results focus on the computation of the basic reproduction number, the local and global stability of the disease-free equilibrium and endemic equilibrium, the permanence, persistence, and extinction of the disease, and so forth. Particularly, in [2, 3], the authors studied a class of discrete-time SIRS epidemic models with time delays derived from corresponding continuous-time models by applying Mickens' nonstandard finite difference scheme. The sufficient conditions on the global asymptotic stability of the disease-free equilibrium and the permanence of the disease are established. In [4], the authors studied a discrete-time SIRS epidemic model with bilinear incidence rate derived from corresponding continuous-time model by applying backward Euler finite difference scheme. The sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium are established. In [5], the authors discussed a class of discrete-time SIRS epidemic models with general nonlinear incidence rate derived from corresponding continuous-time model by applying forward Euler scheme. The sufficient conditions for the existence and local stability of the disease-free equilibrium and endemic equilibrium are obtained. In [6], the authors discussed a class of discrete-time SIRS epidemic models with standard incidence rate discretized from corresponding continuous-time model by applying forward Euler scheme. The sufficient condition for the global stability of the endemic equilibrium is established.

However, we can see from the above literatures that the studies on the global stability for discrete-time SIRS epidemic models are not perfect. The necessary and sufficient conditions for the global stability of the disease-free equilibrium when basic reproduction number $\mathscr{R}_0 \le 1$, as well as the global stability of the endemic equilibrium when $\mathscr{R}_0 > 1$, are established only for bilinear incidence rate (see [4]). Therefore, motivated by the above works, as an extension of the results given in [4], in this paper, we consider the following discrete-time SIRS epidemic model with nonlinear incidence rate and varying population sizes derived from corresponding continuous-time model by applying backward Euler scheme:

$$\begin{aligned} S_{n+1} - S_n &= aA - \beta S_{n+1} g(I_{n+1}) - d_1 S_{n+1} + \delta R_{n+1}, \\ I_{n+1} - I_n &= bA + \beta S_{n+1} g(I_{n+1}) - (d_2 + \gamma) I_{n+1}, \\ R_{n+1} - R_n &= cA + \gamma I_{n+1} - (d_3 + \delta) R_{n+1}. \end{aligned} \tag{1}$$

By constructing new discrete type Lyapunov functions and using the theory of stability of difference equations, we will establish the global asymptotic stability of equilibria only under basic hypothesis (H) (see Section 2). That is, the disease-free equilibrium is globally asymptotically stable if and only if basic reproduction number $\mathscr{R}_0 \le 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

The organization of this paper is as follows. In the second section, we give a model description and further obtain the results on the positivity and boundedness of solutions of model (1). In the third section, we discuss the existence and global asymptotic stability of equilibria of model (1) for case $b = 0$. In the fourth section, we will study the global asymptotic stability of endemic equilibrium of model (1) for case $b > 0$. Lastly, in the fifth section, we will give a conclusion.

2. Preliminaries

In model (1), S_n, I_n, and R_n denote the numbers of susceptible, infected, and recovered individuals at nth period, respectively. A is the recruitment rate of the total population, parameters $a > 0$, $b \ge 0$, and $c \ge 0$ are the fraction constants of input to susceptible class S_n, infected class I_n, and recovered class R_n, respectively, and satisfying $a + b + c = 1$. d_1, d_2, and d_3 represent the death rate of susceptible, infected, and recovered individuals, respectively. Particularly, death rate d_2 includes the natural death rate and the disease-related death rate of the infected individuals. δ is the rate at which recovered individuals lose immunity and return to the susceptible class. γ is the natural recovery rate of the infective individuals. β is the proportionality constant, and the transmission of the infection is governed by a nonlinear incidence rate $\beta S g(I)$. In this paper, we always assume that $A, d_1, d_2, d_3, \delta, \beta, \gamma$ are positive constants.

The initial condition for model (1) is given by

$$S_0 > 0, \qquad I_0 > 0, \qquad R_0 > 0. \tag{2}$$

Throughout this paper, we always assume that

(H) $g(I)$ is continuous and monotonically increasing on $[0, +\infty)$, $g(0) = 0$, $I/g(I)$ is also monotonically increasing on $(0, +\infty)$, and $g'(0)$ exists with $g'(0) > 0$.

Remark 1. Hypothesis (H) is basic for model (1). Particularly, $g(I) = I/(1 + \omega I)$ with constant $\omega \ge 0$; then assumption (H_1) naturally holds. Furthermore, if function $g(I)$ satisfies that second-order derivative $g''(I)$ exists and $g''(I) \le 0$ for all $I \in [0, \infty)$, then we easily prove that $I/g(I)$ is monotone increasing on $I \in (0, +\infty)$.

On the positivity and boundedness of all solutions of model (1) with initial condition (2), we have the following results.

Lemma 2. *For any solution (S_n, I_n, R_n) of model* (1) *with initial condition* (2)*, it holds that*

$$S_n > 0, \quad I_n > 0, \quad R_n > 0 \quad \forall n \ge 0. \tag{3}$$

Proof. Model (1) is equivalent to the following form:

$$\begin{aligned} S_{n+1} &= \frac{aA + \delta R_{n+1} + S_n}{1 + \beta g(I_{n+1}) + d_1}, \\ I_{n+1} &= \frac{bA + \beta S_{n+1} g(I_{n+1}) + I_n}{1 + d_2 + \gamma}, \\ R_{n+1} &= \frac{cA + \gamma I_{n+1} + R_n}{1 + d_3 + \delta}. \end{aligned} \tag{4}$$

When $n = 0$, we have

$$\begin{aligned} S_1 &= \frac{aA + \delta R_1 + S_0}{1 + \beta g(I_1) + d_1}, \\ I_1 &= \frac{bA + \beta S_1 g(I_1) + I_0}{1 + d_2 + \gamma}, \\ R_1 &= \frac{cA + \gamma I_1 + R_0}{1 + d_3 + \delta}. \end{aligned} \tag{5}$$

Let $C_1 = 1 + d_3 + \delta$ and let $C_2 = 1 + d_2 + \gamma$; then obviously, $C_1C_2 - \delta\gamma > 0$. Substituting S_1, R_1 into I_1, we obtain that I_1 satisfies the following equation:

$$I_1 = \frac{\beta g(I_1) h + \delta\gamma\beta g(I_1) I_1 + C_1 (bA + I_0)(1 + d_1)}{(1 + \beta g(I_1) + d_1) C_1 C_2}, \quad (6)$$

where $h = C_1 A(a + b) + \delta(cA + R_0) + C_1(S_0 + I_0)$. From this, we further obtain that I_1 satisfies the following equation:

$$\begin{aligned} H(I_1) &\triangleq (C_1C_2 - \delta\gamma)\beta g(I_1) + C_1C_2(1 + d_1) \\ &\quad - \frac{C_1(bA + I_0)(1 + d_1)}{I_1} - h\beta\frac{g(I_1)}{I_1} \\ &= 0. \end{aligned} \quad (7)$$

From hypothesis (H), we obtain that $H(I_1)$ is monotonically increasing on $(0, +\infty)$, and, obviously,

$$\lim_{I_1 \to 0^+} H(I_1) = -\infty. \quad (8)$$

If $\lim_{I_1 \to \infty} g(I_1) < \infty$, then we have

$$\begin{aligned} \lim_{I_1 \to \infty} H(I_1) &= C_1C_2(1 + d_1) + (C_1C_2 - \delta\gamma) \\ &\quad \times \beta \lim_{I_1 \to \infty} g(I_1) > 0 \end{aligned} \quad (9)$$

and if $\lim_{I_1 \to \infty} g(I_1) = +\infty$, then $\lim_{I_1 \to \infty} H(I_1) = +\infty$. Hence, there is a unique positive solution $x^* > 0$ such that $H(x^*) = 0$. Therefore, we have $I_1 = x^* > 0$. Further, from (5) we also obtain $S_1 > 0$ and $R_1 > 0$.

When $n = 1$, in a similar way, we can obtain $S_2 > 0$, $I_2 > 0$ and $R_2 > 0$. By the induction, we finally obtain that $S_n > 0$, $I_n > 0$ and $R_n > 0$ for all $n \geq 0$. This completes the proof. □

Lemma 3. *For any solution* (S_n, I_n, R_n) *of model* (1) *with initial condition* (2), *it holds that*

$$\limsup_{n \to \infty} (S_n + I_n + R_n) \leq \frac{A}{d}, \quad (10)$$

where $d = \min\{d_1, d_2, d_3\}$.

Proof. Let $N_n = S_n + I_n + R_n$; then from model (1) we have

$$N_n = A + N_{n-1} - d_1S_n - d_2I_n - d_3R_n \leq A + N_{n-1} - dN_n. \quad (11)$$

Hence,

$$N_n \leq \frac{A + N_{n-1}}{1 + d}, \quad n = 1, 2, \ldots. \quad (12)$$

By using iteration method, we obtain

$$\begin{aligned} N_n &\leq \frac{A + N_{n-1}}{1 + d} \\ &\leq \frac{A}{1 + d} + \frac{A}{(1 + d)^2} + \frac{A}{(1 + d)^3} + \cdots + \frac{A}{(1 + d)^n} \\ &\quad + \frac{N_0}{(1 + d)^n} \\ &\leq \frac{A}{d}\left[1 - \frac{1}{(1 + d)^n}\right] + \frac{A}{(1 + d)^n} N_0. \end{aligned} \quad (13)$$

Therefore, it holds that

$$\limsup_{n \to +\infty} N_n \leq \frac{A}{d}. \quad (14)$$

This completes the proof. □

3. Case $b = 0$

If $b = 0$, we have $a + c = 1$, and $a > 0$, $c \geq 0$; then model (1) becomes into the following form:

$$\begin{aligned} S_{n+1} - S_n &= aA - \beta S_{n+1} g(I_{n+1}) - d_1 S_{n+1} + \delta R_{n+1}, \\ I_{n+1} - I_n &= \beta S_{n+1} g(I_{n+1}) - (d_2 + \gamma) I_{n+1}, \\ R_{n+1} - R_n &= cA + \gamma I_{n+1} - (d_3 + \delta) R_{n+1}. \end{aligned} \quad (15)$$

Particularly, when $b = c = 0$, then $a = 1$ and model (1) will become into the following well-known form:

$$\begin{aligned} S_{n+1} - S_n &= A - \beta S_{n+1} g(I_{n+1}) - d_1 S_{n+1} + \delta R_{n+1}, \\ I_{n+1} - I_n &= \beta S_{n+1} g(I_{n+1}) - (d_2 + \gamma) I_{n+1}, \\ R_{n+1} - R_n &= \gamma I_{n+1} - (d_3 + \delta) R_{n+1}. \end{aligned} \quad (16)$$

For model (15), under hypotheses (H), the basic reproduction number, that is an average number of secondary infectious cases produced by an infectious individual during his or her effective infectious period when introduced into an entirely susceptible population, can be defined by

$$\mathcal{R}_0 = \frac{\beta A(\delta + ad_3) g'(0)}{d_1(d_2 + \gamma)(\delta + d_3)}. \quad (17)$$

Here, β is the disease transmission rate, $1/(d_2 + \gamma)$ is the average infection period, and

$$\lim_{I \to 0^+} \frac{\beta A(\delta + ad_3)}{d_1(\delta + d_3)} \frac{g(I)}{I} = \frac{\beta A(\delta + ad_3)}{d_1(\delta + d_3)} g'(0) \quad (18)$$

implies that $(\beta A(\delta+ad_3)/d_1(\delta+d_3))g'(0)$ denotes the number of new cases infected per unit time by one infective individual which is introduced into the susceptible compartment in the case that all the members of the population are susceptible. Particularly, for model (16) we have the basic reproduction number as follows:

$$\mathcal{R}_0 = \frac{\beta A g'(0)}{d_1(d_2 + \gamma)}. \quad (19)$$

On the existence of equilibria of model (15), we have the following result.

Theorem 4. *(1) If* $\mathcal{R}_0 \leq 1$, *then model* (15) *only has a unique disease-free equilibrium* $E_0 = (S^0, 0, R^0)$, *where* $S^0 = A(\delta + ad_3)/d_1(\delta + d_3)$ *and* $R^0 = cA/(\delta + d_3)$.

(2) If $\mathcal{R}_0 > 1$, *then model* (15) *has a unique endemic equilibrium* $E^* = (S^*, I^*, R^*)$, *except for the disease-free equilibrium* E_0.

Proof. We know that an equilibrium $E = (S, I, R)$ of model (15) satisfies

$$\begin{aligned} aA - \beta S g(I) - d_1 S + \delta R &= 0, \\ \beta S g(I) - (d_2 + \gamma) I &= 0, \\ cA + \gamma I - (d_3 + \delta) R &= 0. \end{aligned} \tag{20}$$

Firstly, when $I = 0$, we have

$$\begin{aligned} aA - d_1 S + \delta R &= 0, \\ cA - (d_3 + \delta) R &= 0, \end{aligned} \tag{21}$$

from which we obtain the disease-free equilibrium $E_0 = (S^0, 0, R^0)$, where $S^0 = A(\delta + ad_3)/d_1(\delta + d_3)$ and $R^0 = cA/(\delta + d_3)$.

Secondly, when $I > 0$, from the second and third equations of (20), we obtain

$$R = \frac{cA + \gamma I}{\delta + d_3}, \qquad S = \frac{(d_2 + \gamma) I}{\beta g(I)}. \tag{22}$$

Substituting R, S into the first equation of (20), we have

$$\frac{A(\delta + ad_3)}{\delta + d_3} - \frac{d_3(d_2 + \gamma) + \delta d_2}{\delta + d_3} I - \frac{d_1(d_2 + \gamma) I}{\beta g(I)} = 0. \tag{23}$$

Denote

$$H(I) = \frac{A(\delta + ad_3)}{\delta + d_3} - \frac{d_3(d_2 + \gamma) + \delta d_2}{\delta + d_3} I - \frac{d_1(d_2 + \gamma) I}{\beta g(I)}. \tag{24}$$

By hypothesis (H), $H(I)$ is monotonically decreasing on $(0, +\infty)$ satisfying

$$\begin{aligned} \lim_{I \to 0^+} H(I) &= \frac{A(\delta + ad_3)}{\delta + d_3} - \frac{d_1(d_2 + \gamma)}{\beta g'(0)} \\ &= \frac{A(\delta + ad_3)}{\delta + d_3}\left(1 - \frac{1}{\mathscr{R}_0}\right), \end{aligned} \tag{25}$$

and we also have

$$\lim_{I \to +\infty} H(I) = -\infty. \tag{26}$$

When $\mathscr{R}_0 \le 1$, we have $\lim_{I \to 0^+} H(I) \le 0$. Then, there is not any $I^* > 0$ such that $H(I^*) = 0$. Therefore, model (15) only has a unique disease-free equilibrium E_0.

When $\mathscr{R}_0 > 1$, we have $\lim_{I \to 0^+} H(I) > 0$. Then, there exists a unique $I^* > 0$ such that $H(I^*) = 0$. Furthermore, we have $S^* = (d_2 + \gamma) I^* / \beta g(I^*) > 0$ and $R^* = (cA + \gamma I^*)/(\delta + d_3) > 0$. This implies that model (15) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$. This completes the proof. □

Remark 5. Particularly, for model (16), the disease-free equilibrium given in Theorem 4 will become into $E_0 = (A/d_1, 0, 0)$.

Now, we study the stability of equilibria of model (15). On the global stability of the disease-free equilibrium E_0, we have the following result.

Theorem 6. *Disease-free equilibrium E_0 of model* (15) *is globally asymptotically stable if and only if $\mathscr{R}_0 \le 1$.*

Proof. The necessity is obvious; we only need to prove the sufficiency. Model (15) can be rewritten as the following form:

$$\begin{aligned} S_{n+1} - S_n &= -(\beta g(I_{n+1}) + d_1)(S_{n+1} - S^0) + \delta(R_{n+1} - R^0) \\ &\quad - \beta S^0 g(I_{n+1}), \\ I_{n+1} - I_n &= \beta g(I_{n+1})(S_{n+1} - S^0) - (d_2 + \gamma) I_{n+1} \\ &\quad + \beta S^0 g(I_{n+1}), \\ R_{n+1} - R_n &= \gamma I_{n+1} - (d_3 + \delta)(R_{n+1} - R^0). \end{aligned} \tag{27}$$

We consider the following Lyapunov function:

$$\begin{aligned} W_n &= \frac{1}{2}(S_n - S^0 + I_n + R_n - R^0)^2 + \frac{k_1}{2}(S_n - S^0)^2 \\ &\quad + (k_2 + k_3) I_n + \frac{k_4}{2}(R_n - R^0)^2, \end{aligned} \tag{28}$$

where

$$\begin{aligned} k_1 &= \frac{d_1 + d_3}{\delta}, \qquad k_2 = k_1 S_0, \\ k_3 &= \frac{d_1 + d_2}{\beta g'(0)}, \qquad k_4 = \frac{d_2 + d_3 + \alpha}{\gamma}. \end{aligned} \tag{29}$$

Calculating the difference of W_n along (27), we have

$$\begin{aligned} W_{n+1} - W_n &= \frac{k_1}{2}\left[(S_{n+1} - S^0)^2 - (S_n - S^0)^2\right] \\ &\quad + (k_2 + k_3)(I_{n+1} - I_n) \\ &\quad + \frac{k_4}{2}\left[(R_{n+1} - R^0)^2 - (R_n - R^0)^2\right] \\ &\quad + \frac{1}{2}\left[(S_{n+1} - S^0 + I_{n+1} + R_{n+1} - R^0)^2 \right. \\ &\qquad \left. - (S_n - S^0 + I_n + R_n - R^0)^2\right] \\ &= \frac{k_1}{2}(S_{n+1} - S_n)(S_n - S_{n+1} + 2(S_{n+1} - S^0)) \\ &\quad + (k_2 + k_3)(I_{n+1} - I_n) \\ &\quad + \frac{k_4}{2}(R_{n+1} - R_n)(R_n - R_{n+1} + 2(R_{n+1} - R^0)) \\ &\quad + \frac{1}{2}(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\ &\quad \times (S_n - S_{n+1} + 2(S_{n+1} - S^0) + I_n - I_{n+1} \\ &\qquad + 2I_{n+1} + R_n - R_{n+1} + 2(R_{n+1} - R^0)) \end{aligned}$$

$$\begin{aligned}
&\leq k_1\left(S_{n+1}-S_n\right)\left(S_{n+1}-S^0\right)\\
&\quad+\left(k_2+k_3\right)\left(I_{n+1}-I_n\right)\\
&\quad+k_4\left(R_{n+1}-R_n\right)\left(R_{n+1}-R^0\right)\\
&\quad+\left(S_{n+1}-S_n+I_{n+1}-I_n+R_{n+1}-R_n\right)\\
&\quad\times\left(S_{n+1}-S^0+I_{n+1}+R_{n+1}-R^0\right)\\
&=k_1\left[-\left(\beta g\left(I_{n+1}\right)+d_1\right)\left(S_{n+1}-S^0\right)\right.\\
&\qquad\left.+\delta\left(R_{n+1}-R^0\right)-\beta S^0 g\left(I_{n+1}\right)\right]\left(S_{n+1}-S^0\right)\\
&\quad+k_2\left[\beta g\left(I_{n+1}\right)\left(S_{n+1}-S^0\right)-\left(d_2+\gamma\right)I_{n+1}\right.\\
&\qquad\left.+\beta S^0 g\left(I_{n+1}\right)\right]\\
&\quad+k_3\left[\beta S_{n+1}g\left(I_{n+1}\right)-\left(d_2+\gamma\right)I_{n+1}\right]\\
&\quad+k_4\left[\gamma I_{n+1}-\left(d_3+\delta\right)\left(R_{n+1}-R^0\right)\right]\\
&\quad\times\left(R_{n+1}-R^0\right)\\
&\quad+\left(-d_1S_{n+1}-d_2I_{n+1}-d_3R_{n+1}\right)\\
&\quad\times\left(S_{n+1}-S^0+I_{n+1}+R_{n+1}-R^0\right).
\end{aligned}\tag{30}$$

Since $R_0=\beta Ag'(0)(\delta+ad_3)/d_1(d_2+\gamma)(\delta+d_3)\leq 1$, we have $\beta S^0 g'(0)\leq d_2+\gamma$. Hence,

$$\begin{aligned}
W_{n+1}-W_n&\leq k_1\left[-\left(\beta g\left(I_{n+1}\right)+d_1\right)\left(S_{n+1}-S^0\right)\right.\\
&\qquad\left.+\delta\left(R_{n+1}-R^0\right)-\beta S^0 g\left(I_{n+1}\right)\right]\left(S_{n+1}-S^0\right)\\
&\quad+k_2\left[\beta g\left(I_{n+1}\right)\left(S_{n+1}-S^0\right)-\beta S^0 g'(0)I_{n+1}\right.\\
&\qquad\left.+\beta S^0 g\left(I_{n+1}\right)\right]\\
&\quad+k_3\left[\beta S_{n+1}g\left(I_{n+1}\right)-\beta S^0 g'(0)I_{n+1}\right]\\
&\quad+k_4\left[\gamma I_{n+1}-\left(d_3+\delta\right)\left(R_{n+1}-R^0\right)\right]\\
&\quad\times\left(R_{n+1}-R^0\right)\\
&\quad+\left[-d_1\left(S_{n+1}-S^0\right)-d_2I_{n+1}-d_3\left(R_{n+1}-R^0\right)\right]\\
&\quad\times\left(S_{n+1}-S^0+I_{n+1}+R_{n+1}-R^0\right)\\
&=-\left[k_1\left(\beta g\left(I_{n+1}\right)+d_1\right)+d_1\right]\left(S_{n+1}-S^0\right)^2\\
&\quad-d_2I_{n+1}^2-\left[k_4\left(d_3+\delta\right)+d_3\right]\left(R_{n+1}-R^0\right)^2\\
&\quad+k_2I_{n+1}\beta S_0\left[\frac{g\left(I_{n+1}\right)}{I_{n+1}}-g'(0)\right]\\
&\quad+k_3\beta S_{n+1}I_{n+1}\left[\frac{g\left(I_{n+1}\right)}{I_{n+1}}-g'(0)\right]\\
&\quad+k_3\beta I_{n+1}g'(0)\left(S_{n+1}-S^0\right)\\
&\leq-\left[k_1\left(\beta g\left(I_{n+1}\right)+d_1\right)+d_1\right]\left(S_{n+1}-S^0\right)^2\\
&\quad-d_2I_{n+1}^2-\left[k_4\left(d_3+\delta\right)+d_3\right]\left(R_{n+1}-R^0\right)^2\\
&\quad+\beta I_{n+1}\left(k_2S^0+k_3S_{n+1}\right)\left(\frac{g\left(I_{n+1}\right)}{I_{n+1}}-g'(0)\right).
\end{aligned}\tag{31}$$

Under hypothesis (H), we have for any $n\geq 0$

$$\frac{g\left(I_{n+1}\right)}{I_{n+1}}\leq\lim_{I\to 0^+}\frac{g(I)}{I}=g'(0).\tag{32}$$

Hence,

$$\begin{aligned}
W_{n+1}-W_n\leq&-\left[k_1\left(\beta g\left(I_{n+1}\right)+d_1\right)+d_1\right]\left(S_{n+1}-S^0\right)^2\\
&-d_2I_{n+1}^2-\left[k_4\left(d_3+\delta\right)+d_3\right]\left(R_{n+1}-R^0\right)^2.
\end{aligned}\tag{33}$$

This implies that

$$W_{n+1}-W_n<0\quad\forall\left(S_n,I_n,R_n\right)\neq\left(S^0,0,R^0\right).\tag{34}$$

By Lyapunov's theorems on the global asymptotical stability for difference equations, we directly obtained that the disease-free equilibrium E_0 is globally asymptotically stable. This completes the proof. □

On the global stability of the endemic equilibrium E^*, we have the following result.

Theorem 7. *Endemic equilibrium E^* of model* (15) *is globally asymptotically stable if and only if $\mathscr{R}_0>1$.*

Proof. The necessity is obvious, we only need to prove the sufficiency. Model (15) can be rewritten as the following form:

$$\begin{aligned}
S_{n+1}-S_n&=-\left(\beta g\left(I_{n+1}\right)+d_1\right)\left(S_{n+1}-S^*\right)+\delta\left(R_{n+1}-R^*\right)\\
&\quad-\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right),\\
I_{n+1}-I_n&=\beta g\left(I_{n+1}\right)\left(S_{n+1}-S^*\right)-\left(d_2+\gamma\right)\left(I_{n+1}-I^*\right)\\
&\quad+\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right),\\
R_{n+1}-R_n&=\gamma\left(I_{n+1}-I^*\right)-\left(d_3+\delta\right)\left(R_{n+1}-R^*\right).
\end{aligned}\tag{35}$$

We also have

$$\begin{aligned} I_{n+1} - I_n &= \beta S_{n+1} g(I_{n+1}) - (d_2+\gamma) I_{n+1} \\ &= I_{n+1}\left[\beta S_{n+1}\frac{g(I_{n+1})}{I_{n+1}} - (d_2+\gamma)\right] \\ &= I_{n+1}\left[\beta S_{n+1}\frac{g(I_{n+1})}{I_{n+1}} - \beta S^*\frac{g(I^*)}{I^*}\right] \\ &= I_{n+1}\left[\beta S_{n+1}\frac{g(I_{n+1})}{I_{n+1}} - \beta S^*\frac{g(I^*)}{I^*} + \beta S_{n+1}\frac{g(I^*)}{I^*}\right. \\ &\quad \left. -\beta S_{n+1}\frac{g(I^*)}{I^*}\right] \\ &= I_{n+1}\left[\beta S_{n+1}\left(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}\right)\right. \\ &\quad \left. +\beta\frac{g(I^*)}{I^*}(S_{n+1}-S^*)\right]. \end{aligned} \tag{36}$$

We consider the following Lyapunov function:

$$\begin{aligned} V_n &= \frac{1}{2}(S_n - S^* + I_n - I^* + R_n - R^*)^2 + \frac{k_1}{2}(S_n - S^*)^2 \\ &\quad + k_2\int_{I^*}^{I_n}\frac{g(\tau)-g(I^*)}{g(\tau)}d\tau + \frac{k_3}{2}(R_n - R^*)^2 \\ &\quad + k_4\left(\frac{I_n}{I^*} - 1 - \ln\frac{I_n}{I^*}\right), \end{aligned} \tag{37}$$

where

$$\begin{aligned} k_1 &= \frac{d_1+d_3}{\delta}, \qquad k_2 = k_1 S^*, \\ k_3 &= \frac{d_2+d_3+\alpha}{\gamma}, \qquad k_4 = \frac{(d_1+d_2)(I^*)^2}{\beta g(I^*)}. \end{aligned} \tag{38}$$

Calculating the difference of W_n along (35), we have

$$\begin{aligned} V_{n+1} - V_n &= \frac{k_1}{2}\left[(S_{n+1}-S^*)^2 - (S_n - S^*)^2\right] \\ &\quad + k_2\int_{I_n}^{I_{n+1}}\frac{g(\tau)-g(I^*)}{g(\tau)}d\tau \\ &\quad + \frac{k_3}{2}\left[(R_{n+1}-R^*)^2 - (R_n-R^*)^2\right] \\ &\quad + k_4\left(\frac{I_{n+1}-I_n}{I^*} - \ln\frac{I_{n+1}}{I_n}\right) \\ &\quad + \frac{1}{2}\left[(S_{n+1}-S^*+I_{n+1}-I^*+R_{n+1}-R^*)^2\right. \\ &\quad \left. - (S_n - S^* + I_n - I^* + R_n - R^*)^2\right] \\ &\le k_1(S_{n+1}-S_n)(S_{n+1}-S^*) \\ &\quad + k_2(I_{n+1}-I_n)\left[\frac{g(I_{n+1})-g(I^*)}{g(I_{n+1})}\right] \\ &\quad + k_3(R_{n+1}-R_n)(R_{n+1}-R^*) \\ &\quad + k_4(I_{n+1}-I_n)\left[\frac{I_{n+1}-I^*}{I^* I_{n+1}}\right] \\ &\quad + (S_{n+1}-S_n+I_{n+1}-I_n+R_{n+1}-R_n) \\ &\quad \times (S_{n+1}-S^*+I_{n+1}-I^*+R_{n+1}-R^*) \\ &= k_1\left[-(\beta g(I_{n+1})+d_1)(S_{n+1}-S^*)\right. \\ &\quad \left. +\delta(R_{n+1}-R^*) - \beta S^*(g(I_{n+1})-g(I^*))\right] \\ &\quad \times (S_{n+1}-S^*) \\ &\quad + k_2\left[\beta g(I_{n+1})(S_{n+1}-S^*)\right. \\ &\quad - (d_2+\gamma)(I_{n+1}-I^*) \\ &\quad \left. +\beta S^*(g(I_{n+1})-g(I^*))\right] \\ &\quad \times\left[\frac{g(I_{n+1})-g(I^*)}{g(I_{n+1})}\right] \\ &\quad + k_3\left[\gamma(I_{n+1}-I^*) - (d_3+\delta)(R_{n+1}-R^*)\right] \\ &\quad \times (R_{n+1}-R^*) \\ &\quad + k_4 I_{n+1}\left[\beta S_{n+1}\left(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}\right)\right. \\ &\quad \left. +\beta\frac{g(I^*)}{I^*}(S_{n+1}-S^*)\right]\left[\frac{I_{n+1}-I^*}{I^* I_{n+1}}\right] \\ &\quad + (-d_1 S_{n+1} - d_2 I_{n+1} - d_3 R_{n+1}) \\ &\quad \times (S_{n+1}-S^*+I_{n+1}-I^*+R_{n+1}-R^*) \\ &\le -\left[k_1(\beta g(I_{n+1})+d_1)+d_1\right](S_{n+1}-S^*)^2 \\ &\quad - d_2(I_{n+1}-I^*)^2 \\ &\quad - \left[k_3(d_3+\delta)+d_3\right](R_{n+1}-R^*)^2 \\ &\quad + k_2\frac{g(I_{n+1})-g(I^*)}{g(I_{n+1})} \\ &\quad \times\left[\beta S^*(g(I_{n+1})-g(I^*))\right. \\ &\quad \left. -(d_2+\gamma)(I_{n+1}-I^*)\right] \\ &\quad + \frac{k_4}{I^*}\beta S_{n+1}(I_{n+1}-I^*)\left(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}\right). \end{aligned} \tag{39}$$

Further, from hypothesis (H) and $d_2+\gamma = \beta S^*(g(I^*)/I^*)$, we have

$$\begin{aligned}
& k_2 \frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})} \\
&\quad \times [\beta S^* (g(I_{n+1}) - g(I^*)) - (d_2+\gamma)(I_{n+1} - I^*)] \\
&= k_2 \frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})} [\beta S^* g(I_{n+1}) - (d_2+\gamma) I_{n+1}] \\
&= \frac{k_2 I_{n+1}}{g(I_{n+1})} (g(I_{n+1}) - g(I^*)) \\
&\quad \times \left[\beta S^* \frac{g(I_{n+1})}{I_{n+1}} - (d_2+\gamma)\right] \\
&= \frac{k_2 I_{n+1}}{g(I_{n+1})} (g(I_{n+1}) - g(I^*)) \\
&\quad \times \left[\beta S^* \frac{g(I_{n+1})}{I_{n+1}} - \beta S^* \frac{g(I^*)}{I^*}\right] \\
&= \frac{k_2 \beta S^* I_{n+1}}{g(I_{n+1})} (g(I_{n+1}) - g(I^*)) \left[\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}\right] \\
&\le 0, \\
& \frac{k_4}{I^*} \beta S_{n+1} (I_{n+1} - I^*) \left(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}\right) \le 0.
\end{aligned} \tag{40}$$

Hence,

$$\begin{aligned}
V_{n+1} - V_n \le & -[k_1(\beta g(I_{n+1}) + d_1) + d_1](S_{n+1} - S^*)^2 \\
& - d_2 (I_{n+1} - I^*)^2 \\
& - [k_3(d_3+\delta) + d_3](R_{n+1} - R^*)^2.
\end{aligned} \tag{41}$$

This implies that

$$V_{n+1} - V_n < 0 \quad \forall (S_n, I_n, R_n) \ne (S^*, I^*, R^*). \tag{42}$$

By Lyapunov's theorems on the globally asymptotical stability for difference equations, we directly obtained that the endemic equilibrium E^* is globally asymptotically stable. This completes the proof. □

Remark 8. From the above discussion we immediately see that the basic reproduction number $\mathscr{R}_0$ can completely determine the global asymptotic stability of model (15).

As a consequence of Theorems 6 and 7, for model (16) we have the following corollary.

Corollary 9. *For model* (16) *one has the following.*

(1) *Disease-free equilibrium E_0 is globally asymptotically stable if and only if $\mathscr{R}_0 \le 1$.*

(2) *Endemic equilibrium E^* is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.*

Remark 10. From Corollary 9, we see that the corresponding results on the global asymptotic stability obtained in [4] for discrete-time SIRS epidemic models with bilinear incidence rate are extended to the models with nonlinear incidence rate. Furthermore, comparing with Lyapunov functions established in [4], we see that, in order to study the global asymptotic stability of model (15), a new Lyapunov function is constructed in this paper.

4. Case $b > 0$

We firstly discuss the existence of equilibria of model (1). It is easy to see that if $b > 0$, model (1) has no disease-free equilibrium. We have the following result about the existence of endemic equilibrium of model (1).

Theorem 11. *Model* (1) *always has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$.*

Proof. From model (1) we know that the endemic equilibrium $E^* = (S^*, I^*, R^*)$ satisfies

$$\begin{aligned}
aA - \beta S g(I) - d_1 S + \delta R &= 0, \\
bA + \beta S g(I) - (d_2+\gamma) I &= 0, \\
cA + \gamma I - (d_3+\delta) R &= 0.
\end{aligned} \tag{43}$$

By the second and third equations of (43), we can obtain

$$R = \frac{cA + \gamma I}{\delta + d_3}, \qquad S = \frac{(d_2+\gamma) I - bA}{\beta g(I)}. \tag{44}$$

Substituting (44) into the first equation of (43), we have

$$\begin{aligned}
& -[d_3(d_2+\gamma) + \delta d_2] I + A[\delta + (a+b) d_3] \\
& - \frac{d_1 (d_2+\gamma)(\delta+d_3) I}{\beta g(I)} + \frac{d_1 bA (\delta + d_3)}{\beta g(I)} = 0.
\end{aligned} \tag{45}$$

Denote

$$\begin{aligned}
\varphi(I) &= -[d_3(d_2+\gamma) + \delta d_2] I + A[\delta + (a+b) d_3], \\
\psi(I) &= \frac{d_1 (d_2+\gamma)(\delta+d_3) I}{\beta g(I)} - \frac{d_1 bA (\delta + d_3)}{\beta g(I)}.
\end{aligned} \tag{46}$$

Now, we consider equation $\varphi(I) = \psi(I)$, which is equivalent to (45). By hypothesis (H), $\varphi(I)$ is monotonically decreasing on $(0, +\infty)$ and $\psi(I)$ is monotonically increasing on $(0, +\infty)$. Let $d = \min\{d_1, d_2, d_3\}$; then we have $0 < bA/(d_2+\gamma) < A/d$. By calculating, we obtain

$$\begin{aligned}
\varphi\left(\frac{bA}{d_2+\gamma}\right) &= -[d_3(d_2+\gamma) + \delta d_2] \frac{bA}{d_2+\gamma} \\
&\quad + A[\delta + (a+b) d_3] \\
&= \frac{A[\delta(1-b) d_2 + \delta\gamma + a d_3 (d_2+\gamma)]}{d_2+\gamma} \\
&> 0,
\end{aligned}$$

$$\begin{aligned}\varphi\left(\frac{A}{d}\right) &= -\left[d_3\left(d_2+\gamma\right)+\delta d_2\right]\frac{A}{d}+A\left[\delta+(a+b)\,d_3\right]\\ &= -\frac{A}{d}\left[d_3\left(d_2+\gamma\right)+\delta d_2-d\left(\delta+(1-c)\,d_3\right)\right]\\ &= -\frac{A}{d}\left[d_3\left(\gamma+dc\right)+\left(\delta+d_3\right)\left(d_2-d\right)\right]\\ &< 0,\\ &\psi\left(\frac{bA}{d_2+\gamma}\right)=0,\qquad \psi\left(\frac{A}{d}\right)>0.\end{aligned} \tag{47}$$

Hence, there exists a unique $I^* \in (bA/(d_2+\gamma), A/d)$ such that $\varphi(I^*) = \psi(I^*)$. Furthermore, we have $S^* = ((d_2+\gamma)I^* - bA)/\beta g(I^*) > 0$ and $R^* = (cA+\gamma I^*)/(\delta+d_3) > 0$. This implies that model (1) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$. □

Now, we study the global stability of endemic equilibrium E^*; we have the following result.

Theorem 12. *Endemic equilibrium E^* of model* (1) *is always globally asymptotically stable.*

Proof. Model (1) becomes into the following form:

$$\begin{aligned}S_{n+1}-S_n &= -\left(\beta g\left(I_{n+1}\right)+d_1\right)\left(S_{n+1}-S^*\right)+\delta\left(R_{n+1}-R^*\right)\\ &\quad-\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right),\\ I_{n+1}-I_n &= \beta g\left(I_{n+1}\right)\left(S_{n+1}-S^*\right)-\left(d_2+\gamma\right)\left(I_{n+1}-I^*\right)\\ &\quad+\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right),\\ R_{n+1}-R_n &= \gamma\left(I_{n+1}-I^*\right)-\left(d_3+\delta\right)\left(R_{n+1}-R^*\right).\end{aligned} \tag{48}$$

We also have

$$\begin{aligned}I_{n+1}-I_n &= bA+\beta S_{n+1}g\left(I_{n+1}\right)-\left(d_2+\gamma\right)I_{n+1}\\ &= I_{n+1}\left[\frac{bA}{I_{n+1}}+\beta S_{n+1}\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\left(d_2+\gamma\right)\right]\\ &= I_{n+1}\left[\frac{bA}{I_{n+1}}+\beta S_{n+1}\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\frac{bA+\beta S^* g\left(I^*\right)}{I^*}\right]\\ &= I_{n+1}\left[\frac{bA}{I_{n+1}}-\frac{bA}{I^*}+\beta S_{n+1}\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\beta S^*\frac{g\left(I^*\right)}{I^*}\right.\\ &\quad\left.+\beta S_{n+1}\frac{g\left(I^*\right)}{I^*}-\beta S_{n+1}\frac{g\left(I^*\right)}{I^*}\right]\\ &= I_{n+1}\left[bA\left(\frac{1}{I_{n+1}}-\frac{1}{I^*}\right)\right.\\ &\quad+\beta S_{n+1}\left(\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\frac{g\left(I^*\right)}{I^*}\right)\\ &\quad\left.+\beta\frac{g\left(I^*\right)}{I^*}\left(S_{n+1}-S^*\right)\right].\end{aligned} \tag{49}$$

We consider the following Lyapunov function:

$$\begin{aligned}U_n &= \frac{1}{2}\left(S_n-S^*+I_n-I^*+R_n-R^*\right)^2+\frac{k_1}{2}\left(S_n-S^*\right)^2\\ &\quad+k_2\int_{I^*}^{I_n}\frac{g(\tau)-g\left(I^*\right)}{g(\tau)}d\tau+\frac{k_3}{2}\left(R_n-R^*\right)^2\\ &\quad+k_4\left(\frac{I_n}{I^*}-1-\ln\frac{I_n}{I^*}\right),\end{aligned} \tag{50}$$

where

$$\begin{aligned}k_1 &= \frac{d_1+d_3}{\delta},\qquad k_2=k_1S^*,\\ k_3 &= \frac{d_2+d_3}{\gamma},\qquad k_4=\frac{\left(d_1+d_2\right)\left(I^*\right)^2}{\beta g\left(I^*\right)}.\end{aligned} \tag{51}$$

Calculating the difference of U_n along (48), we have

$$\begin{aligned}U_{n+1}-U_n &\le k_1\left[-\left(\beta g\left(I_{n+1}\right)+d_1\right)\left(S_{n+1}-S^*\right)\right.\\ &\quad+\delta\left(R_{n+1}-R^*\right)\\ &\quad\left.-\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right)\right]\left(S_{n+1}-S^*\right)\\ &\quad+k_2\left[\beta g\left(I_{n+1}\right)\left(S_{n+1}-S^*\right)\right.\\ &\quad-\left(d_2+\gamma\right)\left(I_{n+1}-I^*\right)\\ &\quad\left.+\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right)\right]\\ &\quad\times\left(\frac{g\left(I_{n+1}\right)-g\left(I^*\right)}{g\left(I_{n+1}\right)}\right)\\ &\quad+k_3\left[\gamma\left(I_{n+1}-I^*\right)-\left(d_3+\delta\right)\left(R_{n+1}-R^*\right)\right]\\ &\quad\times\left(R_{n+1}-R^*\right)\\ &\quad+k_4I_{n+1}\left[bA\left(\frac{1}{I_{n+1}}-\frac{1}{I^*}\right)\right.\\ &\quad+\beta S_{n+1}\left(\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\frac{g\left(I^*\right)}{I^*}\right)\\ &\quad\left.+\beta\frac{g\left(I^*\right)}{I^*}\left(S_{n+1}-S^*\right)\right]\\ &\quad\times\left(\frac{I_{n+1}-I^*}{I^*I_{n+1}}\right)\\ &\quad+\left(-d_1S_{n+1}-d_2I_{n+1}-d_3R_{n+1}\right)\\ &\quad\times\left(S_{n+1}-S^*+I_{n+1}-I^*+R_{n+1}-R^*\right)\end{aligned}$$

$$
\begin{aligned}
&= -\left[k_1\left(\beta g\left(I_{n+1}\right)+d_1\right)+d_1\right]\left(S_{n+1}-S^*\right)^2 \\
&\quad - d_2\left(I_{n+1}-I^*\right)^2 \\
&\quad - \left[k_3\left(d_3+\delta\right)+d_3\right]\left(R_{n+1}-R^*\right)^2 \\
&\quad + k_2\frac{g\left(I_{n+1}\right)-g\left(I^*\right)}{g\left(I_{n+1}\right)}\left[\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right)\right. \\
&\qquad \left. -\left(d_2+\gamma\right)\left(I_{n+1}-I^*\right)\right] \\
&\quad + \frac{k_4}{I^*}bA\left(I_{n+1}-I^*\right)\left(\frac{1}{I_{n+1}}-\frac{1}{I^*}\right) \\
&\quad + \frac{k_4}{I^*}\beta S_{n+1}\left(I_{n+1}-I^*\right)\left(\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\frac{g\left(I^*\right)}{I^*}\right).
\end{aligned}
\tag{52}
$$

From hypothesis (H) and $d_2+\gamma=\beta S^*(g(I^*)/I^*)$, we have

$$
\begin{aligned}
&k_2\frac{g\left(I_{n+1}\right)-g\left(I^*\right)}{g\left(I_{n+1}\right)} \\
&\quad \times\left[\beta S^*\left(g\left(I_{n+1}\right)-g\left(I^*\right)\right)-\left(d_2+\gamma\right)\left(I_{n+1}-I^*\right)\right]\leq 0, \\
&\frac{k_4}{I^*}\beta S_{n+1}\left(I_{n+1}-I^*\right)\left(\frac{g\left(I_{n+1}\right)}{I_{n+1}}-\frac{g\left(I^*\right)}{I^*}\right)\leq 0,
\end{aligned}
\tag{53}
$$

and it is easy to see that

$$
\frac{k_4}{I^*}bA\left(I_{n+1}-I^*\right)\left(\frac{1}{I_{n+1}}-\frac{1}{I^*}\right)\leq 0. \tag{54}
$$

Hence,

$$
\begin{aligned}
U_{n+1}-U_n \leq &-\left[k_1\left(\beta g\left(I_{n+1}\right)+d_1\right)+d_1\right]\left(S_{n+1}-S^*\right)^2 \\
&- d_2\left(I_{n+1}-I^*\right)^2 \\
&- \left[k_3\left(d_3+\delta\right)+d_3\right]\left(R_{n+1}-R^*\right)^2.
\end{aligned}
\tag{55}
$$

This implies that

$$
U_{n+1}-U_n<0 \quad \forall\left(S_n,I_n,R_n\right)\neq\left(S^*,I^*,R^*\right). \tag{56}
$$

By Lyapunov's theorems on the global asymptotical stability of difference equations, we directly obtained that the endemic equilibrium E^* is globally asymptotically stable. This completes the proof. □

5. Conclusion

From the main results obtained in this paper, we see that the results on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for a discrete-time SIRS epidemic model with bilinear incidence rate obtained in [4] are directly extended. By constructing new discrete type Lyapunov functions we established the sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for a class of discrete-time SIRS epidemic models with general nonlinear incidence rate $\beta Sg(I)$ and different death rates d_1, d_2, and d_3. That is, the disease-free equilibrium is globally asymptotically stable if and only if basic reproduction number $\mathcal{R}_0\leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathcal{R}_0>1$.

An interesting and important open problem is whether the results obtained in this paper can be extended to the following discrete-time SIRS epidemic models with general nonlinear incidence rate:

$$
\begin{aligned}
S_{n+1}-S_n &= aA-\beta f\left(S_{n+1}\right)g\left(I_{n+1}\right)-d_1S_{n+1}+\delta R_{n+1}, \\
I_{n+1}-I_n &= bA+\beta f\left(S_{n+1}\right)g\left(I_{n+1}\right)-\left(d_2+\gamma\right)I_{n+1}, \\
R_{n+1}-R_n &= cA+\gamma I_{n+1}-\left(d_3+\delta\right)R_{n+1}
\end{aligned}
\tag{57}
$$

and with distributed delay

$$
\begin{aligned}
S_{n+1}-S_n &= aA-\beta S_{n+1}\sum_{j=0}^{m}f\left(j\right)g\left(I_{n-j}\right)-d_1S_{n+1}+\delta R_{n+1}, \\
I_{n+1}-I_n &= bA+\beta S_{n+1}\sum_{j=0}^{m}f\left(j\right)g\left(I_{n-j}\right)-\left(d_2+\gamma\right)I_{n+1}, \\
R_{n+1}-R_n &= cA+\gamma I_{n+1}-\left(d_3+\delta\right)R_{n+1}.
\end{aligned}
\tag{58}
$$

That is, only under the assumption which functions $g(I)$ and $I/g(I)$ are monotonically increasing with respect to I, whether we also can obtain that the disease-free equilibrium is globally asymptotically stable if basic reproduction number $\mathcal{R}_0\leq 1$, and the endemic equilibrium is globally asymptotically stable if $\mathcal{R}_0>1$.

In addition, in this paper, functions $g(I)$ and $I/g(I)$ in model (1) are assumed to be monotonically increasing with respect to I. Obviously, these conditions are rather strong and not easily satisfied in many practical applications. Therefore, an interesting and important open problem is whether the results obtained in this paper can be extended to model (1) with function $g(I)$ or $I/g(I)$ is not monotonically increasing with respect to I.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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